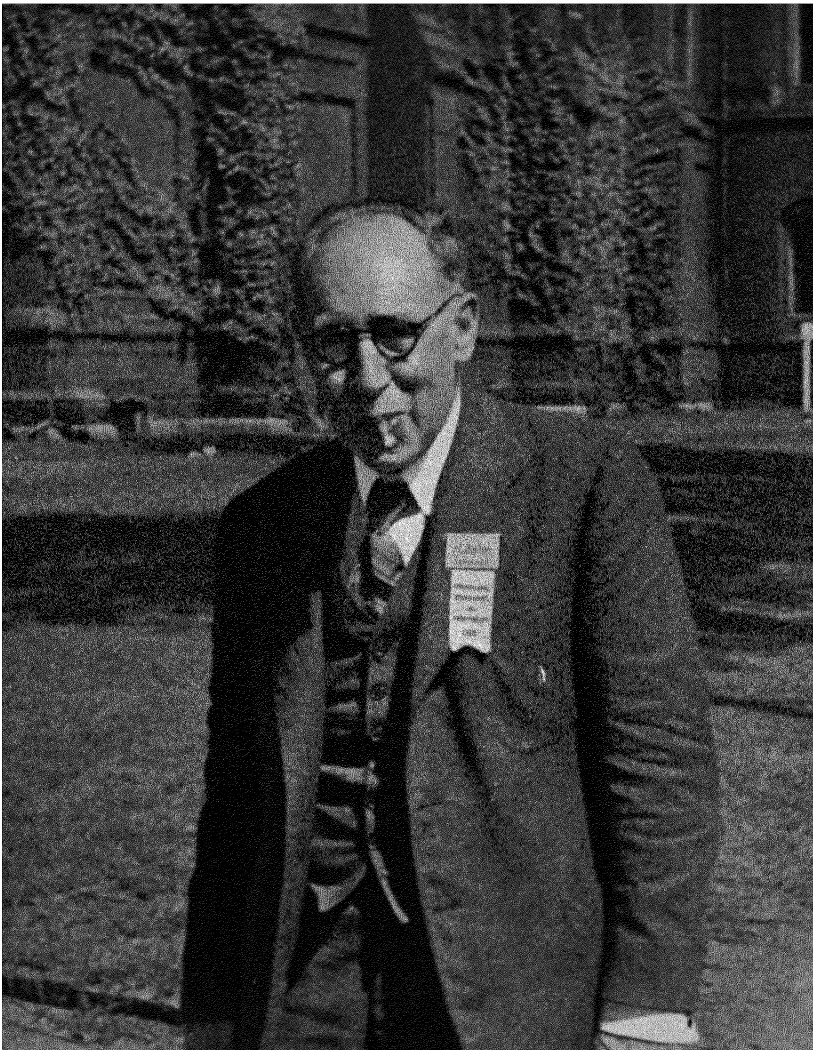
HARALD BOHR

COLLECTED MATHEMATICAL WORKS

Danish Mathematical Society
Institute of Mathematics, University of Copenhagen
Blegdamsvej 15, Copenhagen Ø. Denmark.



Harald Bohr at the International Congress of Mathematicians,
Cambridge, Massachusetts, U. S. A., 1950.

HARALD BOHR

COLLECTED MATHEMATICAL WORKS

IN THREE VOLUMES

III

ALMOST PERIODIC FUNCTIONS
CONTINUED

LINEAR CONGRUENCES . DIOPHANTINE APPROXIMATIONS
FUNCTION THEORY . ADDITION OF CONVEX CURVES
OTHER PAPERS . ENCYCLOPÆDIA ARTICLE
SUPPLEMENTS

DANSK MATEMATISK FORENING
KØBENHAVN
1952

Published with the support of

The Rask-Ørsted Foundation

and

The Carlsberg Foundation.

Edited by

Erling Følner *and* Børge Jessen.

Printed in Denmark.

Printing:

FR BAGGES KGL. HOFBOGTRYKKERI, COPENHAGEN

Offset printing:

S. L. MØLLERS BOGTRYKKERI, COPENHAGEN

TABLE OF CONTENTS
VOLUME III

ALMOST PERIODIC FUNCTIONS

CONTINUED

LINEAR CONGRUENCES · DIOPHANTINE APPROXIMATIONS

FUNCTION THEORY

ADDITION OF CONVEX CURVES

OTHER PAPERS

ENCYCLOPÆDIA ARTICLE

SUPPLEMENTS

ALMOST PERIODIC FUNCTIONS
(CONTINUED)

ON SOME TYPES OF FUNCTIONAL SPACES.

A Contribution to the Theory of Almost Periodic Functions.

By

HARALD BOHR and ERLING FØLNER

in Copenhagen.

With

AN APPENDIX.

By

ERLING FØLNER.

Preface.

In the present paper we shall study some types of functional spaces, the S^p-spaces, the W^p-spaces and the B^p-spaces ($p \geqq 1$) as well as the almost periodic subspaces of these spaces which were met with when generalising the theory of the almost periodic functions. The spaces of S-type, W-type and B-type will be treated separately.

In a later paper Følner will study the »ensemble« of all the types of spaces mentioned. In the investigation of this ensemble certain methods of constructing examples, developed in the present common paper, will be employed, though in a modified and generalised form as in different respects more properties have to be demanded of the constructed functions. In order to avoid repetitions and to make the latter paper more perspicuous, these generalisations of the examples will be treated by Følner in an appendix to the present paper.

Contents.

Introduction.

Throughout the paper we operate with (complex) LEBESGUE measurable functions of a real variable; therefore in the following by the word »function« we shall always mean a LEBESGUE measurable function. By the p-integral of a function $f(x)$ from a to b we shall mean $\int_a^b |f(x)|^p dx$, whether this integral is finite or infinite. We call a function $f(x)$, defined on the whole x-axis, p-integrable if the p-integral of $f(x)$ extended over any finite interval is finite.

The different types of generalised almost periodic functions, the S^p-a. p., W^p-a. p. and B^p-a. p. functions, can on the one hand be interpreted as generalisations of the ordinary almost periodic functions and on the other as generalisations of the p-integrable periodic functions.

An ordinary almost periodic function (in the following often shortly denoted as an o. a p. function) is a continuous complex function $f(x)$, defined for $-\infty < x < \infty$, which, corresponding to every $\varepsilon > 0$, has a relatively dense set of translation numbers $\tau = \tau(\varepsilon)$. A set is called relatively dense if there exists a length L such that any interval $\alpha < x < \alpha + L$ of this length contains at least one number of the set, and a number τ is called a translation number belonging to ε if it satisfies the inequality $|f(x + \tau) - f(x)| \leqq \varepsilon$ for all x.

The main theorem in the theory of the almost periodic functions states that *an almost periodic function can also be characterised as a function which may be approximated, uniformly for all x, by trigonometric polynomials, i. e. sums of the form*

$$\sum_{n=1}^{N} a_n e^{i \lambda_n x}$$

where the a_n are arbitrary complex numbers and the λ_n are real numbers.

It is this last property of the almost periodic functions which is used at their generalisation, the uniform convergence being only replaced by other limit notions. These limit notions are introduced by means of a distance notion, to two arbitrary functions a distance is ascribed, and a sequence of functions $f_n(x)$ is called convergent to the function $f(x)$, if the distance of $f_n(x)$ and $f(x)$ tends to zero for $n \to \infty$. Incidentally we remark that the uniform convergence for all x originates from the (ordinary) distance

$$D_0[f(x), g(x)] = \underset{-\infty < x < \infty}{\text{u. b.}} |f(x) - g(x)|.$$

Concerning the p-integrable periodic functions with a fixed period $b-a>0$ a similar main theorem is valid as for the almost periodic functions. $f(x)$ and $g(x)$ being two arbitrary periodic functions with the given period, we define the p-distance for $p \geqq 1$ by

$$D_p[f(x), g(x)] = \sqrt[p]{\frac{1}{b-a}\int_a^b |f(x)-g(x)|^p\, dx},$$

and we call a sequence $f_n(x)$ of periodic functions with the given period p-convergent to $f(x)$, if $D_p[f(x), f_n(x)] \to 0$ for $n \to \infty$. Then the main theorem states that *to any p-integrable periodic function $f(x)$ with the period $b-a$ a sequence of trigonometric polynomials with the period $b-a$ can be found which p-converges to $f(x)$.* (The converse theorem is here obvious.)

A comprehensive treatment of the generalised almost periodic functions was given in a paper by BESICOVITCH and BOHR: Almost Periodicity and General Trigonometric Series, Acta mathematica, vol. 57. We shall use some facts deduced in that paper, in each case quoting in detail the results we shall employ. On the one hand we shall use some simple relations for the different distances — they will be quoted in this introduction — and on the other hand certain properties of the generalised almost periodic functions which will be cited in Chapter I. In the following the paper in question will be quoted as I.

While the periodic functions may be considered as functions given in a finite interval, the period interval, in the theory of the almost periodic functions we principally have to operate with (i. e. in some way or other to take mean values over) the infinite interval $-\infty < x < \infty$. Desiring to transfer the p-distance mentioned above from a finite to an infinite interval we may choose among several different possibilities each of which presents its special peculiarity and its special interest. Within the set of all (measurable) functions we introduce for every $p \geqq 1$ three such distances which we denote, after STEPANOFF, WEYL and BESICOVITCH, by

$$D_{S_L^p}[f(x), g(x)], \quad D_{W^p}[f(x), g(x)] \quad \text{and} \quad D_{B^p}[f(x), g(x)].$$

STEPANOFF's distance is given by

$$D_{S_L^p}[f(x), g(x)] = \underset{-\infty<x<\infty}{\text{u. b.}} \sqrt[p]{\frac{1}{L}\int_x^{x+L} |f(\xi)-g(\xi)|^p\, d\xi}.$$

Here L is a fixed positive number; its value is unessential (L may for instance be chosen equal to 1) since given the two positive numbers L_1 and L_2 there exist two positive numbers k_1 and k_2, depending only on L_1 and L_2 and not on $f(x)$ and $g(x)$, such that (I, p. 221)

$$k_1 D_{S^p_{L_1}}[f(x), g(x)] \leqq D_{S^p_{L_2}}[f(x), g(x)] \leqq k_2 D_{S^p_{L_1}}[f(x), g(x)].$$

On account of these latter inequalities the distances $D_{S^p_L}$ corresponding to different L are said to be equivalent.

Concerning the Besicovitch distance the mean value is at once extended over the whole interval $-\infty < x < \infty$, viz.

$$D_{B^p}[f(x), g(x)] = \overline{\lim_{T\to\infty}} \sqrt[p]{\frac{1}{2T}\int_{-T}^{T} |f(x) - g(x)|^p\, dx}.$$

Finally the Weyl distance is an »intermediate thing« between the two distances cited above. Like Stepanoff, Weyl considers a fixed length L which he however lets increase to ∞, viz.

$$D_{W^p}[f(x), g(x)] = \lim_{L\to\infty} \underset{-\infty<x<\infty}{\text{u. b.}} \sqrt[p]{\frac{1}{L}\int_{x}^{x+L} |f(\xi) - g(\xi)|^p\, d\xi} = \lim_{L\to\infty} D_{S^p_L}[f(x), g(x)].$$

It is easy to prove that the limit always exists for $L \to \infty$.

As immediately seen, all these distances are generalisations of the distance D_p; for, if $f(x)$ and $g(x)$ are periodic functions with the period h, we have

$$D_p[f(x), g(x)] = D_{S^p_h}[f(x), g(x)] = D_{W^p}[f(x), g(x)] = D_{B^p}[f(x), g(x)].$$

Instead of S^p_1 we simply write S^p, and similarly we omit p, if $p = 1$, in the symbols S^p_L, W^p and B^p. Frequently it is convenient to use a symbol which may represent an arbitrary one of the symbols S^p_L, W^p and B^p; in this case we use the symbol G or, if we want to emphasise the exponent p, the symbol G^p. We observe that the common symbol for S_L, W and B is G^1.

A sequence of functions $f_n(x)$ is called G-convergent to the function $f(x)$, if $D_G[f(x), f_n(x)] \to 0$ for $n \to \infty$, and we write

$$f_n(x) \xrightarrow{G} f(x).$$[1]

[1] $f_n(x) \xrightarrow{S^p_{L_1}} f(x)$ means the same as $f_n(x) \xrightarrow{S^p_{L_2}} f(x)$, the distances $D_{S^p_L}$ being equivalent for different values of L.

A function is called a G-a. p. function, if there exists a sequence of trigonometric polynomials which G-converges to the function. The set of G-a. p. functions is called the G-a. p. set.[1] For every $p \geqq 1$ we have thus introduced the three important sets:

the S^p-a. p. set, the W^p-a. p. set and the B^p-a. p. set.

Concerning particular G^p-a. p. functions, besides the o. a. p. functions and the p-integrable periodic functions, we mention the G^p-limit periodic functions; a function $f(x)$ is called G^p-limit periodic, if there exists a sequence of p-integrable periodic functions (generally without a common period) which G^p-converges to $f(x)$.

For every $p \geqq 1$ the inequalities

$$D_O[f(x), g(x)] \geqq D_{S_L^p}[f(x), g(x)] \geqq D_{W^p}[f(x), g(x)] \geqq D_{B^p}[f(x), g(x)]$$

are valid (I, p. 222); hence denoting the set of o. a. p. functions as the o. a. p. set, we have for each $p \geqq 1$:

the o. a. p. set $\subseteqq$ the S^p-a. p. set $\subseteqq$ the W^p-a. p. set $\subseteqq$ the B^p-a. p. set.

Further we have for $1 \leqq p_1 < p_2$ (I, p. 222)

$$D_{G^{p_1}}[f(x), g(x)] \leqq D_{G^{p_2}}[f(x), g(x)].$$

(This inequality is a consequence of Hölder's inequality quoted below.) Hence it holds for $1 \leqq p_1 < p_2$ that

the G^{p_1}-a. p. set $\supseteqq$ the G^{p_2}-a. p. set.

Just as the distance $D_O[f(x), g(x)]$ originates from an O-norm $D_O[f(x)] = D_O[f(x), 0]$, and the distance $D_p[f(x), g(x)]$ from a p-norm $D_p[f(x)] = D_p[f(x), 0]$, every one of our distances $D_G[f(x), g(x)]$ originates from a G-norm $D_G[f(x)] = D_G[f(x), 0]$. Obviously the G-norm satisfies, like the O-norm and the p-norm, the relation

(1) $$D_G[a f(x)] = |a| \cdot D_G[f(x)] \qquad (a \text{ a complex number});$$

further it satisfies the inequality (I, p. 222)

(2) $$D_G[f(x) + g(x)] \leqq D_G[f(x)] + D_G[g(x)]$$

[1] The $S_{L_1}^p$-a. p. set is identical with the $S_{L_2}^p$-a. p. set, as the distances $D_{S_L^p}$ are equivalent for different values of L, and it is called the S^p-a. p. set. The functions in the S^p-a. p. set are called S^p-a. p. functions.

which is equivalent to the **Triangle Rule**

$$D_G[f(x), g(x)] \leqq D_G[f(x), h(x)] + D_G[h(x), g(x)].$$

(This inequality is a consequence of MINKOWSKI's inequality quoted below.)

As a trigonometric polynomial is bounded, its G-norm is finite, and in consequence of the Triangle Rule the same is valid for any G-a. p. function. A function with finite G-norm is called a G-function. The set of all G-functions is called the G-set[1], and we have

$$\text{the } G\text{-a. p. set} \subseteqq \text{the } G\text{-set}.$$

It is important to observe that *the S^p-set and the W^p-set are identical for each* $p \geqq 1$; that the S^p-set $\subseteqq$ the W^p-set is an immediate consequence of the inequality $D_{S^p} \geqq D_{W^p}$, and the converse is involved by the equation $D_{W^p} = \lim\limits_{L \to \infty} D_{S_L^p}$ which shows that if D_{W^p} is finite then $D_{S_L^p}$ will be finite for sufficiently large L (and therefore for all L). We emphasise that the analogue is not valid for the a. p. sets; in fact the S^p-a. p. set is a proper subset of the W^p-a. p. set.

The G-sets satisfy similar relations as the G-a. p. sets, and on account of the same distance relations:

For every $p \geqq 1$ is

$$\text{the } S^p\text{-set } (W^p\text{-set}) \subseteqq B^p\text{-set},$$

and for $1 \leqq p_1 < p_2$ is

$$\text{the } G^{p_1}\text{-set} \supseteqq \text{the } G^{p_2}\text{-set}.$$

In our G-sets we shall have to consider the so-called G-**fundamental sequences**. A sequence of functions $f_n(x)$ from the G-set is called a G-fundamental sequence, if $D_G[f_n(x), f_m(x)] \to 0$ when n and m, independently of each other, tend to ∞. Further we shall use the notion G-**closed**. We call a set of functions G-closed, if each function which is the G-limit of a sequence of functions from the set belongs itself to the set. On account of the Triangle Rule the G-set and the G-a. p. set are obviously G-closed.

Leaving out of account that the G-distance between two different functions may be zero, the G-set is organised as a **linear metric space** because of (1) and (2). It is easily shown that the same holds for the G-a. p. set. Firstly, the product of a G-a. p. function $f(x)$ by a constant is again a G-a. p. function;

[1] The $S_{L_1}^p$-set is identical with the $S_{L_2}^p$-set, as the distances $D_{S_L^p}$ are equivalent for different values of L, and it is called the S^p-set. The functions in the S^p-set are called S^p-functions.

for, if $s_n(x)$ is a sequence of trigonometric polynomials G-converging to $f(x)$, the sequence $a \cdot s_n(x)$ of trigonometric polynomials will G-converge to $a \cdot f(x)$, since

$$D_G[af(x), as_n(x)] = |a| \cdot D_G[f(x), s_n(x)].$$

Secondly, the sum of two G-a. p. functions $f^{(1)}(x)$ and $f^{(2)}(x)$ is again a G-a. p. function; for, if $s_n^{(1)}(x)$ is a sequence of trigonometric polynomials G-converging to $f^{(1)}(x)$, and $s_n^{(2)}(x)$ is a sequence of trigonometric polynomials G-converging to $f^{(2)}(x)$, the sequence $s_n^{(1)}(x) + s_n^{(2)}(x)$ consisting of trigonometric polynomials will G-converge to $f^{(1)}(x) + f^{(2)}(x)$, as

$$D_G[f^{(1)}(x) + f^{(2)}(x), s_n^{(1)}(x) + s_n^{(2)}(x)] = D_G[(f^{(1)}(x) - s_n^{(1)}(x)) + (f^{(2)}(x) - s_n^{(2)}(x))]$$
$$\leqq D_G[f^{(1)}(x), s_n^{(1)}(x)] + D_G[f^{(2)}(x), s_n^{(2)}(x)].$$

In the proofs of theorems on G-a. p. functions it is often convenient, instead of, as above, using the definition itself, to employ the following simple theorem: *A G-a. p. function can also be characterised as a function which is the G-limit of o. a. p. functions (and not just of trigonometric polynomials).* The proof is immediate. In fact, a function which can be approximated by o. a. p. functions must belong to the G-a. p. set, as the o. a. p. set $\subseteqq$ G-a. p. set, and the G-a. p. set is G-closed.

To pass from the G-set to a proper linear metric space where the distance between two different points is > 0 (and not only $\geqq 0$), an equivalence relation ($\sim$) between the G-functions is introduced in the following obvious way:

$$f(x) \sim g(x) \quad \text{if} \quad D_G[f(x), g(x)] = 0.$$

Then the G-set falls into classes of equivalent functions. Each of these classes is called a G-point. Evidently two functions of the G-set belong to the same G-point, if and only if they differ from each other by a function of the G-norm 0. Such functions of the G-norm 0 are called G-zero functions. Now a distance (again denoted by D_G) is introduced in the following manner: Let $\mathfrak{A}$ and $\mathfrak{B}$ be two arbitrary G-points; then we define $D_G[\mathfrak{A}, \mathfrak{B}]$ by the equation

$$D_G[\mathfrak{A}, \mathfrak{B}] = D_G[f(x), g(x)],$$

where $f(x)$ and $g(x)$ are arbitrary representatives of $\mathfrak{A}$ and $\mathfrak{B}$; this definition is evidently unique. The multiplication of a G-point by a constant, and the addition of two G-points being defined by means of representatives, it is plain that (1) and (2) are still satisfied, if we consider G-points instead of G-functions. And moreover the G-distance between two different G-points is always > 0.

Thus the set of G-points is organised as a linear metric space by the distance D_G. We denote it as the G-space.

If one function in a G-point is G-a. p., all functions of the point are G-a. p. functions, and the point is called a G-a. p. point. The set of the G-a. p. points, organised by the distance D_G, forms a linear subspace of the G-space. It is called the G-a. p. space.

Now we have introduced all the spaces which we shall investigate in the following, viz. for every $p \geqq 1$:

$$\begin{aligned}
&\text{the } S^p\text{-a. p. space} \subseteqq \text{the } S^p\text{-space,}\\
&\text{the } W^p\text{-a. p. space} \subseteqq \text{the } W^p\text{-space,}\\
&\text{the } B^p\text{-a. p. space} \subseteqq \text{the } B^p\text{-space.}
\end{aligned}$$

If one function in a G-point is G-limit periodic, so are all functions of the point, and the point is called a G-limit periodic point.

If a G-point contains a periodic function, it is called a periodic G-point (but of course it is not true, that all the functions of a periodic G-point are periodic functions).

We say that a sequence $\mathfrak{A}_1, \mathfrak{A}_2, \ldots$ of G-points G-converges to the G-point $\mathfrak{A}$ if $D_G[\mathfrak{A}, \mathfrak{A}_n] \to 0$ or, what is equivalent, if an (arbitrary) sequence of representatives $f_1(x), f_2(x), \ldots$ of $\mathfrak{A}_1, \mathfrak{A}_2, \ldots$ is G-convergent to a representative $f(x)$ of $\mathfrak{A}$.

A sequence $\mathfrak{A}_1, \mathfrak{A}_2, \ldots$ of G-points is called a G-fundamental sequence if $D_G[\mathfrak{A}_n, \mathfrak{A}_m] \to 0$ for n and m tending to ∞ or, what is equivalent, if a sequence of representatives $f_1(x), f_2(x), \ldots$ of $\mathfrak{A}_1, \mathfrak{A}_2, \ldots$ is a G-fundamental sequence.

As well known, a metric space is called complete if every fundamental sequence of the space is convergent; otherwise it is called incomplete.

A subset of a metric space is called closed (relatively to the latter) if every point of the space which is the limit of points of the subset belongs itself to the subset. Evidently, *the G-a. p. space is closed* (relatively to the G-space).

Concerning the Stepanoff distance, it is easy to see that, for any $p \geqq 1$, a function is a S^p-zero function only in the trivial case when it is 0 »almost everywhere« (i. e. except in a set of measure 0); consequently, for every p, an S^p-point consists of essentially only one function. In the two other cases (the W^p and B^p) the set of zero functions is considerably more comprehensive, and most comprehensive for $p = 1$; thus, while it is only a mathematical subtlety to speak of S^p-points instead of S^p-functions, it is of decisive importance to distinguish between G-points and G-functions in case of the W^p- and B^p-spaces.

To deduce the relations for the different distances (about which we referred to I) two very important inequalities are used, Hölder's inequality and Minkowski's inequality. As we later on shall apply these inequalities repeatedly, we quote them here in the introduction.

Hölder's inequality. *Let p and q be two positive numbers satisfying the condition*

$$\frac{1}{p}+\frac{1}{q}=1,$$

and $f(x)$ and $g(x)$ two complex functions, defined in the interval (a, b); then we have

$$\frac{1}{b-a}\int_a^b |f(x)g(x)|\,dx \leqq \sqrt[p]{\frac{1}{b-a}\int_a^b |f(x)|^p\,dx}\cdot\sqrt[q]{\frac{1}{b-a}\int_a^b |g(x)|^q\,dx}.$$

As the inequality can be reduced by $\frac{1}{b-a}$, *the corresponding inequality for integrals* (instead of mean values) *is also valid.*

We emphasise a special case of Hölder's inequality (which is obtained by replacing $f(x)$, $g(x)$ and p by $|f(x)|^{p_1}$, 1 and $\frac{p_2}{p_1}$ respectively), viz:

For $1 \leqq p_1 < p_2$ *is*

$$\sqrt[p_1]{\frac{1}{b-a}\int_a^b |f(x)|^{p_1}\,dx} \leqq \sqrt[p_2]{\frac{1}{b-a}\int_a^b |f(x)|^{p_2}\,dx}.$$

Minkowski's inequality. *Let $f(x)$ and $g(x)$ be two complex functions, defined in the interval (a, b), and $p \geqq 1$; then the inequality*

$$\sqrt[p]{\frac{1}{b-a}\int_a^b |f(x)+g(x)|^p\,dx} \leqq \sqrt[p]{\frac{1}{b-a}\int_a^b |f(x)|^p\,dx}+\sqrt[p]{\frac{1}{b-a}\int_a^b |g(x)|^p\,dx}$$

holds. As before the corresponding integral inequality is also valid.

Obviously the inequality can also be written in the form

$$\sqrt[p]{\frac{1}{b-a}\int_a^b |f(x)+g(x)|^p\,dx} \geqq \sqrt[p]{\frac{1}{b-a}\int_a^b |f(x)|^p\,dx}-\sqrt[p]{\frac{1}{b-a}\int_a^b |g(x)|^p\,dx}.$$

Besides trivial facts concerning LEBESGUE integrals we shall have to use

Fatou's theorem. *Let $f(t, n)$ be a non-negative function, given for all t in a finite interval (a, b) and for all positive integral values of n. Then we have*

$$\int_a^b \varliminf_{n\to\infty} f(t, n)\, dt \leqq \varliminf_{n\to\infty} \int_a^b f(t, n)\, dt.$$

Before passing to a summary of the paper we will here gather different remarks of a general character which, like the inequalities and the theorem above, will be of importance later on.

To begin with we introduce the notion of minimum of two complex functions $f(x)$ and $g(x)$, defined for all x, viz.

$$\min\,[f(x), g(x)] = \begin{cases} f(x) \text{ for the } x \text{ satisfying } |f(x)| \leqq |g(x)| \\ g(x) \text{ for the } x \text{ satisfying } |g(x)| < |f(x)|. \end{cases}$$

The little »lack of beauty« that $\min\,[f(x), g(x)]$ is not symmetric in $f(x)$ and $g(x)$ is of no importance whatsoever.

The definition of $\min\,[f(x), g(x)]$ involves immediately the following inequalities

$$|\min\,[f(x), g(x)]| \leqq |f(x)|, \qquad |\min\,[f(x), g(x)]| \leqq |g(x)|$$

and

$$|\min\,[f(x), g(x)] - f(x)| \leqq |g(x) - f(x)|, \qquad |\min\,[f(x), g(x)] - g(x)| \leqq |f(x) - g(x)|.$$

A G-point considered as a set of functions is G-closed, since, $f_1(x), f_2(x), \ldots$ being a sequence of functions of a G-point with the G-limit $f(x)$, we have

$$D_G\,[f(x), f_1(x)] = D_G\,[f(x), f_n(x)] \to 0,$$

so that $D_G\,[f(x), f_1(x)] = 0$, i. e. $f(x)$ belongs to the G-point.

A G-point considered as a set of functions is closed with respect to the minimum-operation, since, $f_1(x)$ and $f_2(x)$ being two functions of a G-point, we have $|\min\,[f_1(x), f_2(x)] - f_1(x)| \leqq |f_1(x) - f_2(x)|$ and consequently

$$D_G\,[\min\,[f_1(x), f_2(x)], f_1(x)] \leqq D_G\,[f_1(x), f_2(x)] = 0,$$

so that $\min\,[f_1(x), f_2(x)]$ lies also in the G-point.

Frequently it is convenient to use the distance

$$D^*_{B^p}[f(x), g(x)] =$$

$$\max\left[\overline{\lim_{T\to\infty}}\sqrt[p]{\frac{1}{T}\int_0^T |f(x)-g(x)|^p\,dx},\ \overline{\lim_{T\to\infty}}\sqrt[p]{\frac{1}{T}\int_{-T}^0 |f(x)-g(x)|^p\,dx}\right]$$

instead of the distance $D_{B^p}[f(x), g(x)]$. The two distances are equivalent, as

$$\sqrt[p]{\frac{1}{2}}\cdot D^*_{B^p}[f(x), g(x)] \leqq D_{B^p}[f(x), g(x)] \leqq D^*_{B^p}[f(x), g(x)].$$

The new distance originates from the norm $D^*_{B^p}[f(x)] = D^*_{B^p}[f(x), 0]$ which satisfies (1) and (2).

Finally we give a summary of the content of the paper.

In Chapter I the needed properties of the generalised almost periodic functions are quoted, inter alia certain translation properties and the approximation by Bochner-Fejér polynomials are treated.

In Chapter II the completeness or incompleteness of the different spaces is investigated. It is shown, what has been known in the main, that the S^p-space and the S^p-a. p. space as well as the B^p-space and the B^p-a. p. space are complete for every $p \geqq 1$, and, what has not been known before, that the W^p-space and the W^p-a. p. space are incomplete for every $p \geqq 1$. It is of special importance that the B^2-a. p. space is complete, as this involves the validity of Besicovitch's Theorem which is the analogue of the famous theorem of Riesz-Fischer on 2-integrable periodic functions.

In Chapter III, which has the character of an insertion, three theorems are proved which will be applied in the following Chapters.

Chapter IV deals with the mutual relations of the S^p-spaces and the S^p-a. p. spaces, Chapter V with the mutual relations of the W^p-spaces and the W^p-a. p. spaces, and Chapter VI with the mutual relations of the B^p-spaces and the B^p-a. p. spaces. The investigations of the spaces of W-type and those of B-type are essentially similar in some respects, but show also characteristic differences.

The paper consist partly of theorems and partly of counter examples. Many of the examples are simple and more or less trivial. The especially strong and substantial examples are called main examples. Most of the functions

constructed in our examples are piecewise constant and change between the value 0 and values >0. Hence they have a graph like the function outlined in Fig. 1.

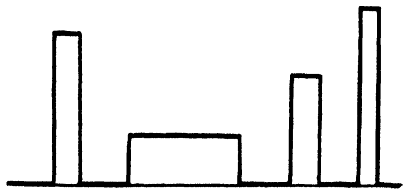


Fig. 1.

The graph thus consists of rectangles which stand with one side on the x-axis. These rectangles are called **towers**, and the function is given by indicating the size of the towers and their position on the x-axis. The size of a tower may be given by its »height« k and »breadth« b, but it is often given by prescribing the p-integral of the tower for two different values p_1 and p_2 $(\geqq 1)$; from these values k and b can immediately be calculated; for the p_1-integral of the tower is $I_1 = b\,k^{p_1}$ and the p_2-integral $I_2 = b k^{p_2}$, and hence

$$k = \left(\frac{I_2}{I_1}\right)^{\frac{1}{p_2-p_1}} \quad \text{and} \quad b = \left(\frac{I_1^{p_2}}{I_2^{p_1}}\right)^{\frac{1}{p_2-p_1}};$$

for an arbitrary p_3 $(\geqq 1)$ the p_3-integral becomes

$$I_3 = b k^{p_3} = I_1^{\frac{p_2-p_3}{p_2-p_1}} \cdot I_2^{\frac{p_3-p_1}{p_2-p_1}}.$$

When indicating a tower by the values of its p-integral for two different values of p, the integral corresponding to the smaller of these values is always chosen less than the other integral (i. e. the height k of the tower is always chosen >1); then the p-integral is a steadily increasing function of p which tends to ∞ for $p \to \infty$. The position of a tower is generally indicated by the number with which the center of the lowest side coïncides. The tower is said to **stand on** this number. Sometimes we speak about a tower as placed or standing on an interval. This means that the tower stands on the centre of the interval and does not protrude beyond the interval.

All our examples of G-a. p. functions are chosen among the **G-limit periodic functions**.

CHAPTER I.

The Generalised Almost Periodic Functions.

In this chapter some well known properties of the generalised almost periodic functions are quoted which will be applied in our later investigations. We also remind of the proofs of some of the theorems.

Already in the introduction we have mentioned that the product of a G-a. p. function by a constant and the sum of two G-a. p. functions are again G-a. p. functions. Moreover it is valid that the product of a G-a. p. function by an ordinary almost periodic function is again a G-a. p. function.

If $f(x)$ is G-a. p., the modulus $|f(x)|$ and the function

$$(f(x))_N = \begin{cases} f(x) & \text{for } |f(x)| \leqq N \\ N\dfrac{f(x)}{|f(x)|} & \text{for } |f(x)| \geqq N \end{cases}$$

which originates from the function $f(x)$ by »cutting it off« at the positive number N are again G-a. p. This is a corollary of the following theorem: *Let $f(x)$ be a G-a. p. function and $\Phi(z)$ be a function defined in the whole complex plane (or, in case of a real function $f(x)$, on the real axis) with a bounded difference quotient; then $\Phi(f(x))$ is G-a. p.* The proof is immediate: Let $f_n(x)$ be a sequence of o. a. p. functions which G-converges to $f(x)$; then $\Phi(f_n(x))$ too is a sequence of o. a. p. functions on account of the uniform continuity of $\Phi(z)$; further

$$\Phi(f_n(x)) \overset{G}{\rightarrow} \Phi(f(x))$$

since the inequality

$$|\Phi(f(x)) - \Phi(f_n(x))| \leqq K|f(x) - f_n(x)|$$

involves

$$D_G[\Phi(f(x)),\ \Phi(f_n(x))] \leqq K D_G[f(x),\ f_n(x)] \rightarrow 0.$$

If $f(x)$ is G-a. p., the function $(f(x))_N$ will G-converge to $f(x)$ for $N \rightarrow \infty$.

Proof. Let $\varepsilon > 0$ be given. We choose a trigonometrical polynomial $s(x)$ such that $D_G[f(x), s(x)] < \frac{\varepsilon}{2}$, and use the estimation

$$D_G[f(x),\ (f(x))_N] \leqq D_G[f(x),\ s(x)] + D_G[s(x),\ (f(x))_N].$$

For $N \geqq$ u. b. $|s(x)| = K$ we have $s(x) = (s(x))_N$, and hence on account of the inequality

$$|(f(x))_N - (g(x))_N| \leqq |f(x) - g(x)|$$

(the validity of which is seen by help of a simple geometrical consideration) we get

$$D_G[s(x), (f(x))_N] = D_G[(s(x))_N, (f(x))_N] \leqq D_G[s(x), f(x)].$$

Thus we have for $N \geqq K$

$$D_G[f(x), (f(x))_N] \leqq 2 D_G[f(x), s(x)] < \varepsilon.$$

Just as for the ordinary almost periodic functions and the p-integrable periodic functions there exists a theory of Fourier series for the G-a. p. functions. Each G-a. p. function $f(x)$ has a mean value

$$M\{f(x)\} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(x)\,dx = \lim_{T\to\infty} \frac{1}{T}\int_{0}^{T} f(x)\,dx = \lim_{T\to\infty} \frac{1}{T}\int_{-T}^{0} f(x)\,dx,$$

and the function

$$a(\lambda) = M\{f(x)\,e^{-i\lambda x}\}$$

of the real variable λ is different from 0 for at most an enumerable set of values of λ; these are called the Fourier exponents of $f(x)$ and denoted in one order or another by $\Lambda_1, \Lambda_2, \ldots$. The values of $a(\lambda)$, belonging to the Fourier exponents $\Lambda_1, \Lambda_2, \ldots$, are called the Fourier coefficients of $f(x)$ and denoted by $A_1, A_2, \ldots$ respectively. With the function $f(x)$ is associated the Fourier series $\Sigma A_n e^{i\Lambda_n x}$, and we write

$$f(x) \sim \Sigma A_n e^{i\Lambda_n x}.$$

Sometimes it is convenient to include certain »improper« terms $A_n e^{i\Lambda_n x}$ (at most an enumerable number) where $A_n = M\{f(x)e^{-i\Lambda_n x}\} = 0$. For such a term Λ_n is called an improper Fourier exponent and A_n $(=0)$ an improper Fourier coefficient belonging to the exponent Λ_n.

All the functions in a G-a. p. point have the same Fourier series which is called the Fourier series of the G-a. p. point. Corresponding to the uniqueness theorem of the o. a. p. functions the following uniqueness theorem is valid for the G-a. p. functions: *Two different G-a. p. points cannot have the same Fourier series.*

We emphasise that the Fourier series is formed in exactly the same way for all our types of generalised almost periodic functions. For a p-integrable periodic function it is easy to prove that it has the same Fourier series in the ordinary sense as in the G-a. p. sense.

Concerning the connection between a G-a. p. function and its Fourier series the usual rules on addition, and on multiplication by a constant are valid. If $f_m(x)$ is a sequence of G-a. p. functions G-converging to the G-a. p. function $f(x)$, the Fourier series of $f(x)$ can be obtained by a formal limit process from the Fourier series of $f_m(x)$.

The Bochner-Fejér method of summation, of great importance in the theory of the o. a. p. functions, can be transferred to the G-a. p. functions (I, § 12—13). Starting from the Fourier series of a G-a. p. function $f(x)$ *a sequence of trigonometric polynomials* $\sigma_q(x)$, *the Bochner-Fejér polynomials, can be found which G-converges to* $f(x)$. By means of kernels $K^q(t)$ which are non-negative trigonometric polynomials with the mean value 1 these Bochner-Féjer polynomials $\sigma_q(x)$ can be represented in the form

$$\sigma_q(x) = \underset{t}{M}\{f(x+t)\,K^q(t)\}.$$

To establish the approximation-properties of the Bochner-Fejér polynomials the following inequality which can be deduced from the representation above is of decisive importance:

$$D_G[\sigma_q(x)] \leqq D_G[f(x)].$$

It is essential for our later applications that this inequality holds even if G does not just denote the type of almost periodicity of the function $f(x)$. Certainly, in the proof of the inequality in question given in I, where G has an arbitrary fixed meaning, it was assumed that $f(x)$ was a. p. in the G-sense, but in fact it was only used in the proof that $f(x)$ was almost periodic in one sense or another and not just in the G-sense.

Concerning a W^p-a. p. function $f(x)$ the Bochner-Fejér sequence $\sigma_q(x)$ does not only W^p-converge to $f(x)$, but this W^p-convergence takes place with a certain »uniformity«; in fact, to every $\varepsilon > 0$ there can be determined an L_0 and a Q such that

$$D_{S_L^p}[f(x), \sigma_q(x)] \leqq \varepsilon \qquad \text{for } L \geqq L_0 \text{ and } q \geqq Q.$$

In the case of the W^2-a. p. functions, R. Schmidt (Math. Ann. Bd. 100) was the first to indicate approximating trigonometric polynomials with this property.

The Bochner-Fejér polynomials $\sigma_q(x)$ have the form

$$\sigma_q(x) = \sum_{n=1}^{N(q)} k_n^{(q)} A_n e^{i\Lambda_n x}$$

where the Λ_n are Fourier exponents and the A_n the corresponding Fourier coefficients of $f(x)$, and $N(q) \to \infty$ for $q \to \infty$. The factor $k_n^{(q)}$ satisfies the inequality $0 \leqq k_n^{(q)} \leqq 1$ and tends to 1 for fixed n and $q \to \infty$.

For a B^2-a. p. function $f(x)$ with the Fourier series $\Sigma A_n e^{i\Lambda_n x}$ the Parseval equation holds:

$$M\{|f(x)|^2\} = \Sigma |A_n|^2.$$

As cited in the introduction, for the B^2-a. p. functions the theorem of Besicovitch which is the analogue to Riesz-Fischer's Theorem on 2-integrable periodic functions is valid: *An arbitrary given trigonometric series* $\Sigma A_n e^{i\Lambda_n x}$ *is the Fourier series of a* B^2*-a. p. function if and only if* $\Sigma |A_n|^2$ *is convergent.* The proof of this theorem relies on the completeness of the B^2-a. p. space; we shall return to it in Chapter II.

Just as the o. a. p. functions, also the generalised almost periodic functions can (as shown in I) be characterised in two different ways, viz. on the one hand by their approximation by means of trigonometric polynomials, and on the other by translation properties. In the present paper the generalised almost periodic functions have been defined by their approximation properties. As to the S^p-a. p. and W^p-a. p. functions, however, also their translation properties will be needed for some of our investigations. In the two following theorems we shall state these translation properties which can easily be deduced from our definitions of the S^p-a. p. and W^p-a. p. functions.

Theorem 1. *An* S^p*-a. p. function* $f(x)$ *possesses, to every* $\varepsilon > 0$*, a relatively dense set of* S_L^p*-translation numbers (L arbitrary fixed), i. e. of numbers* τ *with the property*

$$D_{S_L^p}[f(x+\tau), f(x)] \leqq \varepsilon.$$

Proof. Let $\varphi(x)$ be an o. a. p. function such that

$$D_{S_L^p}[f(x), \varphi(x)] < \frac{\varepsilon}{3},$$

and let τ be an (ordinary) translation number of $\varphi(x)$ belonging to $\frac{\varepsilon}{3}$. Then we have

$$|\varphi(x+\tau) - \varphi(x)| \leqq \frac{\varepsilon}{3}$$

for all x and hence a fortiori

$$D_{S_L^p}[\varphi(x+\tau),\ \varphi(x)] \leqq \frac{\varepsilon}{3}\cdot$$

By means of the Triangle Rule we obtain

$$D_{S_L^p}[f(x+\tau),\ f(x)] \leqq$$
$$D_{S_L^p}[f(x+\tau),\ \varphi(x+\tau)] + D_{S_L^p}[\varphi(x+\tau),\ \varphi(x)] + D_{S_L^p}[\varphi(x),\ f(x)] \leqq \varepsilon.$$

Consequently τ is an S_L^p-translation number of $f(x)$ belonging to ε. As the τ's form a relatively dense set, the theorem is proved.

Theorem 2. *A W^p-a. p. function $f(x)$ has, to every $\varepsilon > 0$ and for L sufficiently large (i. e. $L \geqq L_0(\varepsilon)$), a relatively dense set of S_L^p-translation numbers.*

Proof. Let $\varphi(x)$ be an o. a. p. function so that

$$D_{W^p}[f(x),\ \varphi(x)] < \frac{\varepsilon}{3}\cdot$$

Since $D_{W^p} = \lim\limits_{L\to\infty} D_{S_L^p}$, we have for a sufficiently large L, i. e. for $L \geqq L_0(\varepsilon)$, that

$$D_{S_L^p}[f(x),\ \varphi(x)] < \frac{\varepsilon}{3}\cdot$$

It follows as in the proof of Theorem 1 that every (ordinary) translation number τ of $\varphi(x)$ belonging to $\frac{\varepsilon}{3}$ is an S_L^p-translation number of $f(x)$ belonging to ε for any $L \geqq L_0$.

In this paper, among the G-a. p. functions, we shall particularly consider the G-limit periodic functions, as all our G-almost periodic examples will be chosen among the latter functions. Therefore we finish this Chapter I by some remarks on G-limit periodic functions.

We begin by showing that a G-limit periodic function can also be characterised as a G-a. p. function whose Fourier exponents are rational multiples of one and the same real number.

1°. Let $f(x)$ be a G-a. p. function with Fourier exponents which are rational multiples of a number d. Since the exponents (in finite number) of any Bochner-Fejér polynomial $\sigma_q(x)$ are Fourier exponents of $f(x)$ and therefore integral multiples of a number d_q, it is evident that each $\sigma_q(x)$ is a periodic function $\left(\text{with period } \frac{2\pi}{d_q}\right)\cdot$ Hence $f(x)$ being the G-limit of the sequence $\sigma_q(x)$ is a G-limit periodic function.

2°. Let then $f(x)$ be a G-limit periodic function and $f_1(x), f_2(x), \ldots$ a sequence of p-integrable periodic functions, with periods $h_1, h_2, \ldots$, which G-converges to $f(x)$. We shall prove that all the Fourier exponents of $f(x)$ are rational multiples of a single number d.

We may assume that the Fourier series of $f(x)$ does not only consist of the constant term, since in this particular case the theorem is obviously valid. Then there exists a Fourier exponent $\Lambda_n \neq 0$. If A_n ($\neq 0$) denotes the Fourier coefficient of $f(x)$ belonging to this exponent Λ_n, and $A_n^{(m)}$ the (proper or improper) Fourier coefficient of $f_m(x)$ belonging to the exponent Λ_n, the coefficient $A_n^{(m)}$ tends to A_n for $m \to \infty$. Hence $A_n^{(m)} \neq 0$ for m sufficiently large, i. e. for $m \geqq m_0 = m_0(n)$. The exponent Λ_n thus being a proper Fourier exponent of $f_m(x)$ for $m \geqq m_0$, we have

$$\Lambda_n = \frac{2\pi}{h_m}\nu_m \ (\nu_m \text{ integral}) \quad \text{and thus} \quad h_m = \frac{2\pi}{\Lambda_n}\nu_m \quad \text{for } m \geqq m_0.$$

Consequently for $m \geqq m_0$ the periods h_m are integral multiples of the number $g = \frac{2\pi}{\Lambda_n}$, i. e.

$$h_m = \nu_m g \qquad\qquad \text{for } m \geqq m_0.$$

The Fourier exponents of $f_m(x)$ for $m \geqq m_0$ are thus to be found among the numbers $\frac{2\pi}{h_m}\mu = \frac{2\pi}{g}\cdot\frac{\mu}{\nu_m}$ (μ integral) so that they are all rational multiples of the number $d = \frac{2\pi}{g}$. Finally the same must be valid for the Fourier exponents of the function $f(x)$ itself, since the Fourier series of $f(x)$ can be obtained as the formal limit of the Fourier series of $f_m(x)$ for $m \to \infty$.

Remark. We saw in 1° that the Bochner-Fejér polynomials of a G-limit periodic function are periodic functions. We shall add a remark concerning the periods of the Bochner-Fejér polynomials of a G-limit periodic function $f(x)$ which, as in 2°, is given as the G-limit of a sequence of p-integrable periodic functions $f_1(x), f_2(x), \ldots$ with periods $h_1, h_2, \ldots$. In fact we shall show that any Bochner-Fejér polynomial of $f(x)$

$$\sigma(x) = a_0 + a_1 e^{i\Lambda_1 x} + a_2 e^{i\Lambda_2 x} + \cdots + a_N e^{i\Lambda_N x}$$

has the number h_m as a period for m sufficiently large. For, as we saw in 2°, for each $\Lambda_n \neq 0$ we have for $m \geqq m_0 = m_0(n)$

$$\Lambda_n = \frac{2\pi}{h_m} \nu_{n,m} \qquad (\nu_{n,m} \text{ integral});$$

hence for $m \geqq \max[m_0(1), m_0(2), \ldots, m_0(N)]$ each of the exponents of our $\sigma(x)$ is an integral multiple of $\frac{2\pi}{h_m}$, and h_m therefore a period of $\sigma(x)$.

For the generalised limit periodic functions the theorems 1 and 2 can be sharpened; we choose a formulation which is just adapted to our applications.

Theorem 1 a. *Let $f(x)$ be an S^p-a. p. function, and $f_1(x), f_2(x), \ldots$ a sequence of 1-integrable periodic functions, with the periods $h_1, h_2, \ldots$, which G^1-converges to $f(x)$. Let further $\varepsilon > 0$ be arbitrarily given. Then for fixed L, and m sufficiently large, i. e. for $m \geqq m_0(\varepsilon, L)$, all integral multiples of h_m are S_L^p-translation numbers of $f(x)$ belonging to ε.*

We observe at once that $f(x)$, as a G^1-limit periodic function, has a Fourier series of »limit periodic form« and is therefore not only S^p-a. p., but also S^p-limit periodic.

Proof. Let $\sigma(x)$ be a Bochner-Fejér polynomial of $f(x)$ for which

$$D_{S_L^p}[f(x), \sigma(x)] < \frac{\varepsilon}{2}.$$

In consequence of the remark above, $\sigma(x)$ has the period h_m for m sufficiently large. Then, for each such m, every integral multiple νh_m of h_m will be an S_L^p-translation number of $f(x)$ belonging to ε, since on account of the Triangle Rule

$$D_{S_L^p}[f(x + \nu h_m), f(x)] \leqq$$

$$D_{S_L^p}[f(x + \nu h_m), \sigma(x + \nu h_m)] + D_{S_L^p}[\sigma(x + \nu h_m), \sigma(x)] + D_{S_L^p}[\sigma(x), f(x)] =$$

$$2\, D_{S_L^p}[f(x), \sigma(x)] < \varepsilon.$$

Theorem 2 a. *Let $f(x)$ be a W^p-a. p. function, and $f_1(x), f_2(x), \ldots$ a sequence of 1-integrable periodic functions, with the periods $h_1, h_2, \ldots$, which G^1-converges to $f(x)$. Let further $\varepsilon > 0$ be arbitrarily given. Then for m and L sufficiently large, i. e. for $m \geqq m_0(\varepsilon)$ and $L \geqq L_0(\varepsilon)$, all integral multiples of h_m are S_L^p-translation numbers of $f(x)$ belonging to ε.*

We observe at once that $f(x)$, as a G^1-limit periodic function, has a Fourier series of limit periodic form and is therefore not only W^p-a. p., but also W^p-limit periodic.

Proof. Let $\sigma(x)$ be a Bochner-Fejér polynomial of $f(x)$ for which

$$D_{W^p}[f(x), \sigma(x)] < \frac{\varepsilon}{2}.$$

Since $D_{W^p} = \lim_{L\to\infty} D_{S^p_L}$, we have for L sufficiently large $(L \geqq L_0 = L_0(\varepsilon))$

$$D_{S^p_L}[f(x), \sigma(x)] < \frac{\varepsilon}{2}.$$

For m sufficiently large $(m \geqq m_0 = m_0(\varepsilon))$ the Bochner-Fejér polynomial $\sigma(x)$ has the period h_m, and as in the proof of Theorem 1 a we conclude that every integral multiple of h_m is an S^p_L-translation number of $f(x)$ belonging to ε.

CHAPTER II.

The Completeness or Incompleteness of the Different Spaces.

§ 1.

The Completeness of the S^p- and the S^p-a. p. Spaces.

In this paragraph we prove the following

Theorem. *The S^p-space and the S^p-a. p. space are complete for every $p \geqq 1$.*

Proof. It is sufficient to show that the S^p-space is complete, since this involves, the S^p-a. p. space being a closed subspace of the S^p-space, that every S^p-fundamental sequence of the S^p-a. p. space S^p-converges to a point of the S^p-space and therefore also to a point of the S^p-a. p. space. Thus we only have to show that every S^p-fundamental sequence of S^p-points is S^p-convergent or, what is equivalent, that every S^p-fundamental sequence of S^p-functions is S^p-convergent. Let then $f_1(x), f_2(x), \ldots$ be an S^p-fundamental sequence of S^p-functions, i. e. a sequence of S^p-functions for which $D_{S^p}[f_n(x), f_m(x)] \to 0$ when n and m tend to ∞. We shall prove that there exists a function $f(x)$ such that $D_{S^p}[f(x), f_m(x)] \to 0$ for $m \to \infty$. This function $f(x)$ will automatically be an S^p-function, as the S^p-set is S^p-closed.

We begin by determining an increasing sequence of positive integers $n_1 < n_2 < \cdots$ so that

$$D_{S^p}[f_n(x), f_m(x)] \leqq \frac{1}{2^\nu} \quad \text{for} \quad n \geqq n_\nu, \ m \geqq n_\nu \quad (\nu = 1, 2, \ldots).$$

Hence in particular

$$D_{S^p}[f_{n_\nu}(x), f_{n_{\nu+1}}(x)] \leqq \frac{1}{2^\nu} \qquad (\nu = 1, 2, \ldots).$$

Let

$$g_q(x) = \sum_{\nu=1}^{q} |f_{n_{\nu+1}}(x) - f_{n_\nu}(x)|,$$

then we have $0 \leqq g_1(x) \leqq g_2(x) \leqq \cdots$ so that $\lim_{q \to \infty} g_q(x)$ exists (as finite or infinite) for every x. Further we have for each q, on account of the Triangle Rule,

$$D_{S^p}[g_q(x)] \leqq D_{S^p}[|f_{n_2}(x) - f_{n_1}(x)|] + \cdots + D_{S^p}[|f_{n_{q+1}}(x) - f_{n_q}(x)|] =$$

$$D_{S^p}[f_{n_1}(x), f_{n_2}(x)] + \cdots + D_{S^p}[f_{n_q}(x), f_{n_{q+1}}(x)] \leqq$$

$$\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^q} < 1;$$

in particular

$$\int_{\mu}^{\mu+1} (g_q(x))^p \, dx < 1$$

is valid for every integer μ. Then we have for every integer μ

$$\int_{\mu}^{\mu+1} \lim_{q \to \infty} (g_q(x))^p \, dx = \lim_{q \to \infty} \int_{\mu}^{\mu+1} (g_q(x))^p \, dx \leqq 1.$$

Hence $\lim_{q \to \infty} g_q(x)$ is finite for almost all x. Thus the series

$$\sum_{\nu=1}^{\infty} (f_{n_{\nu+1}}(x) - f_{n_\nu}(x))$$

is absolutely convergent, in particular convergent, for almost all x, which shows that the sequence $f_{n_1}(x), f_{n_2}(x), \ldots$ is convergent (to a finite limit) for almost all x. We shall see that the function

$$f(x) = \lim_{\nu \to \infty} f_{n_\nu}(x)$$

fulfils our demands. Let $\varepsilon > 0$ be arbitrarily given. Then m_0 can be determined such that $D_{S^p}[f_n(x), f_m(x)] \leqq \varepsilon$ for $n \geqq m_0$ and $m \geqq m_0$; if further ν_0 is chosen so large that $n_{\nu_0} \geqq m_0$ we have $D_{S^p}[f_{n_\nu}(x), f_m(x)] \leqq \varepsilon$ for $\nu \geqq \nu_0$ and $m \geqq m_0$, and consequently

$$\int_x^{x+1} |f_{n_\nu}(\xi) - f_m(\xi)|^p \, d\xi \leqq \varepsilon^p \quad \text{for all} \quad x, \ \nu \geqq \nu_0, \ m \geqq m_0.$$

Since $|f_{n_\nu}(\xi) - f_m(\xi)| \to |f(\xi) - f_m(\xi)|$ for almost all ξ when $\nu \to \infty$, we get for every x and $m \geqq m_0$ by FATOU's Theorem

$$\int_x^{x+1} |f(\xi) - f_m(\xi)|^p \, d\xi \leqq \varliminf_{\nu \to \infty} \int_x^{x+1} |f_{n_\nu}(\xi) - f_m(\xi)|^p \, d\xi \leqq \varepsilon^p;$$

hence $D_{S^p}[f(x), f_m(x)] \leqq \varepsilon$ for $m \geqq m_0$, i. e. $D_{S^p}[f(x), f_m(x)] \to 0$ for $m \to \infty$.

This proof of the completeness of the S^p-space is an immediate transferring of a well-known proof of the theorem that a fundamental sequence of p-integrable periodic functions $f_1(x), f_2(x), \ldots$ with the period h is p-convergent. Besides, this last theorem can on its side easily be derived from the theorem above concerning S^p-functions. Indeed, such a sequence of periodic functions $f_n(x)$ is at the same time an S^p-fundamental sequence and will therefore S^p-converge to an S^p-function $f(x)$, and from this function $f(x)$ we can immediately find a function $g(x)$, periodic with the period h, which is the p-limit of our sequence $f_n(x)$. We can simply use the periodic function $g(x)$ which in the period interval $0 \leqq x < h$ coïncides with $f(x)$. In fact this function $g(x)$ is a p-integrable function with the period h, and

$$D_p[g(x), f_n(x)] = \sqrt[p]{\frac{1}{h}\int_0^h |g(x) - f_n(x)|^p \, dx} =$$

$$\sqrt[p]{\frac{1}{h}\int_0^h |f(x) - f_n(x)|^p \, dx} \leqq D_{S_h^p}[f(x), f_n(x)] \to 0$$

for $n \to \infty$.

§ 2.

The Completeness of the B^p- and the B^p-a. p. Spaces.

In this paragraph we prove the following

Theorem. *The B^p-space and the B^p-a. p. space are complete for every $p \geqq 1$.*

Proof. As the B^p-a. p. space is a closed subspace of the B^p-space, it is sufficient (just as in the S-case in § 1) to prove the theorem for the B^p-space. Thus we have to show that every B^p-fundamental sequence of B^p-points is B^p-convergent or, which is equivalent, that every B^p-fundamental sequence of B^p-functions is B^p-convergent. Let then $f_1(x), f_2(x), \ldots$ be a B^p-fundamental sequence of B^p-functions, i. e. a sequence of B^p-functions so that there exists a sequence of positive numbers ε_n tending to 0 for which the inequality

$$(D^*_{B^p}[f_n(x), f_{n+q}(x)])^p < \varepsilon_n$$

holds for all n and $q > 0$. (We prefer here to use the distance $D^*_{B^p}$ instead of the distance D_{B^p}). We shall prove that a function $f(x)$ can be found such that

$$D^*_{B^p}[f(x), f_n(x)] \to 0 \qquad \text{for } n \to \infty.$$

This function $f(x)$ will automatically be a B^p-function, as the B^p-set is B^p-closed. Our construction of $f(x)$ is principally the same as that which Besicovitch used in the proof of his theorem concerning the Fourier series of B^2-a. p. functions; the following arrangement of the proof is due to B. Jessen. We will construct a function $f(x)$ such that

$$(D^*_{B^p}[f(x), f_n(x)])^p \leqq 2\varepsilon_n \qquad \text{for all } n.$$

As the construction is analogous for $x > 0$ and $x < 0$, we confine ourselves to state it for $x > 0$. Starting from the assumptions

$$\overline{\lim_{T\to\infty}} \frac{1}{T}\int_0^T |f_n(x) - f_{n+q}(x)|^p\, dx < \varepsilon_n \qquad \text{for all } n \text{ and } q > 0, \tag{1}$$

the task is to construct $f(x)$ so that

$$\overline{\lim_{T\to\infty}} \frac{1}{T}\int_0^T |f(x) - f_n(x)|^p\, dx \leqq 2\varepsilon_n \qquad \text{for all } n. \tag{2}$$

The construction of $f(x)$ is indicated in Fig. 2, and we shall show that the occurring positive numbers $T_1 < T_2 < \cdots$ can be chosen so that

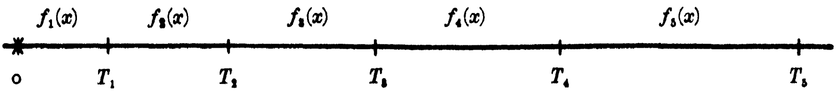


Fig. 2.

$$(3) \qquad \frac{1}{T}\int_0^T |f(x) - f_n(x)|^p\,dx < 2\,\varepsilon_n \qquad \text{for } T > T_n \text{ and all } n$$

which obviously involves (2). To this purpose we first set up a number of conditions, arranged in certain groups, for the numbers $T_1, T_2, \ldots$ which involve (3) (and thereby (2)); afterwards, by help of (1), we shall show that these conditions can be satisfied simultaneously.

Group 1. The inequality (3) is satisfied for $n = 1$, if

$$\frac{1}{T}\int_0^T |f_2(x) - f_1(x)|^p\,dx < \varepsilon_1 \qquad \text{for } T > T_1 \quad \boxed{T_1}$$

$$\frac{1}{T}\int_0^T |f_3(x) - f_1(x)|^p\,dx < \varepsilon_1 \qquad \text{for } T > T_2 \quad \boxed{T_2}$$

$$\frac{1}{T}\int_0^T |f_4(x) - f_1(x)|^p\,dx < \varepsilon_1 \qquad \text{for } T > T_3 \quad \boxed{T_3}$$

etc. etc.

and further

$$\frac{1}{T_2 - T_1}\int_{T_1}^{T_2} |f_2(x) - f_1(x)|^p\,dx < \varepsilon_1 \qquad \boxed{T_2(T_1)}$$

$$\frac{1}{T_3 - T_2}\int_{T_2}^{T_3} |f_3(x) - f_1(x)|^p\,dx < \varepsilon_1 \qquad \boxed{T_3(T_2)}$$

etc. etc.

For, if $T > T_1$ lies between T_m and T_{m+1}, we have

$$\int_0^T |f(x) - f_1(x)|^p\,dx = \int_0^{T_1} + \int_{T_1}^{T_2} + \cdots + \int_{T_{m-1}}^{T_m} + \int_{T_m}^{T} |f(x) - f_1(x)|^p\,dx <$$

$$0(T_1 - 0) + \varepsilon_1(T_2 - T_1) + \cdots + \varepsilon_1(T_m - T_{m-1}) + \varepsilon_1 T < 2\,\varepsilon_1 T.$$

Group 2. The inequality (3) is satisfied for $n = 2$, if

$$\frac{1}{T}\int_0^T |f_3(x) - f_2(x)|^p \, dx < \varepsilon_2 \qquad \text{for } T > T_2 \quad \boxed{T_2}$$

$$\frac{1}{T}\int_0^T |f_4(x) - f_2(x)|^p \, dx < \varepsilon_2 \qquad \text{for } T > T_3 \quad \boxed{T_3'}$$

etc. etc.

and further

$$\frac{1}{T_2}\int_0^{T_2} |f_1(x) - f_2(x)|^p \, dx < \varepsilon_2 \qquad \boxed{T_2'(T_1)}$$

$$\frac{1}{T_3 - T_2}\int_{T_2}^{T_3} |f_3(x) - f_2(x)|^p \, dx < \varepsilon_2 \qquad \boxed{T_3(T_2)}$$

$$\frac{1}{T_4 - T_3}\int_{T_3}^{T_4} |f_4(x) - f_2(x)|^p \, dx < \varepsilon_2 \qquad \boxed{T_4(T_3)}$$

etc. etc.

For, if $T > T_2$ lies between T_m and T_{m+1}, we have

$$\int_0^T |f(x) - f_2(x)|^p \, dx = \int_0^{T_2} + \int_{T_2}^{T_3} + \cdots + \int_{T_{m-1}}^{T_m} + \int_{T_m}^{T} |f(x) - f_2(x)|^p \, dx <$$

$$\varepsilon_2 (T_2 - 0) + \varepsilon_2 (T_3 - T_2) + \cdots + \varepsilon_2 (T_m - T_{m-1}) + \varepsilon_2 T < 2\, \varepsilon_2 T.$$

Group 3. Correspondingly it is seen that the inequality (3) is satisfied for $n = 3$, if

$$\frac{1}{T}\int_0^T |f_4(x) - f_3(x)|^p \, dx < \varepsilon_3 \qquad \text{for } T > T_3 \quad \boxed{T_3}$$

$$\frac{1}{T}\int_0^T |f_5(x) - f_3(x)|^p \, dx < \varepsilon_3 \qquad \text{for } T > T_4 \quad \boxed{T_4}$$

etc. etc.

and further

$$\frac{1}{T_3}\left[\int_0^{T_1} |f_1(x) - f_3(x)|^p \, dx + \int_{T_1}^{T_2} |f_2(x) - f_3(x)|^p \, dx\right] < \varepsilon_3 \qquad \boxed{T_3(T_1, T_2)}$$

$$\frac{1}{T_4 - T_3} \int_{T_3}^{T_4} |f_4(x) - f_3(x)|^p \, dx < \varepsilon_3 \qquad \boxed{T_4(T_3)}$$

$$\frac{1}{T_5 - T_4} \int_{T_4}^{T_5} |f_5(x) - f_4(x)|^p \, dx < \varepsilon_3 \qquad \boxed{T_5(T_4)}$$

etc. etc.

etc.

After each condition the T_n concerned are indicated in a rectangle. A composed indication like $T_2(T_1)$ means that the condition be understood as a claim to T_2 after T_1 having been chosen. We observe that, in consequence of (1), every condition is satisfied for all sufficiently large values of the number T_n in question. Since we have only a finite number of conditions for every T_n, and since the composed conditions have the form $T_n(\ldots)$ where the T's in the bracket have lower indices than n, it is obvious that the numbers $T_1, T_2, \ldots$ can be chosen successively so that all the conditions are satisfied.

We finish the paragraph by showing how the theorem of Besicovitch concerning B^2-a. p. functions can be deduced from the completeness of the B^2-a. p. space. From the Parseval equation for a B^2-a. p. function it results immediately that a necessary condition for a trigonometric series $\sum_1^\infty A_n e^{i\Lambda_n x}$ to be the Fourier series of a B^2-a. p. function is that $\sum_1^\infty |A_n|^2$ is convergent. Besicovitch's Theorem states that this condition is also sufficient.

Let then $\sum_1^\infty A_n e^{i\Lambda_n x}$ be a trigonometric series for which $\sum_1^\infty |A_n|^2$ is convergent. We shall prove that the series is the Fourier series of a B^2-a. p. function. We consider the sum of the first n terms of the series

$$s_n(x) = A_1 e^{i\Lambda_1 x} + A_2 e^{i\Lambda_2 x} + \cdots + A_n e^{i\Lambda_n x}.$$

From the Parseval equation for an o. a. p. function in the (trivial) case where it is a trigonometric polynomial we have

$$D_{B^2}[s_n(x), s_{n+q}(x)] = D_{B^2}[A_{n+1} e^{i\Lambda_{n+1} x} + \cdots + A_{n+q} e^{i\Lambda_{n+q} x}] = \sqrt{\sum_{\nu=n+1}^{n+q} |A_\nu|^2};$$

therefore, $\Sigma|A_n|^2$ being convergent, the sequence $s_n(x)$ is a B^2-fundamental sequence and thus (on account of the completeness of the B^2-a. p. space) B^2-converges to a B^2-a. p. function $f(x)$. The given series $\sum_1^\infty A_n e^{i\Lambda_n x}$ must be the Fourier series of this function $f(x)$, since the Fourier series of $f(x)$ can be obtained as the formal limit of the Fourier series of $s_n(x)$ (i. e. $s_n(x)$ itself) for $n \to \infty$.

Incidentally the proof shows that the Fourier series of a B^2-a. p. function B^2-converges to the function.

§ 3.

The Incompleteness of the W^p- and the W^p-a. p. Spaces. Main Example 1.

In this last paragraph we finally prove the following

Theorem. *The W^p-space and the W^p-a. p. space are incomplete for every $p \geqq 1$.*

As the W^p-a. p. space is a closed subspace of the W^p-space it is sufficient to show that the W^p-a. p. space is incomplete, since a W^p-fundamental sequence of W^p-a. p. points which is not W^p-convergent to any W^p-a. p. point is neither W^p-convergent to any W^p-point. Thus we have to prove that for every $p \geqq 1$ there exists a W^p-fundamental sequence of W^p a. p. points which is not W^p-convergent, or, in other terms, that there exists a W^p-fundamental sequence of W^p-a. p. functions which is not W^p-convergent. We give a single example which can be used for all p by constructing *a sequence $F_1(x)$, $F_2(x)$, ... of W^p-a. p. functions which is a W^p-fundamental sequence for every $p \geqq 1$, but which is not W^p-convergent for any p.* In order to show that the sequence is not W^p-convergent for any p, it is sufficient to show that the sequence is not W^p-convergent for $p = 1$; for a sequence W^p-converging to $F(x)$ for some p or other would also W-converge to $F(x)$, since $D_W[F(x), F_n(x)] \leqq D_{W^p}[F(x), F_n(x)]$.

Main example 1. Let $m_1, m_2, \ldots$ be a sequence of integers $\geqq 2$, and let

$$h_1 = m_1, \quad h_2 = m_1 m_2, \quad h_3 = m_1 m_2 m_3, \ldots.$$

For $n = 1, 2, \ldots$ we put

$$f_n(x) = \begin{cases} 1 \text{ for } \nu h_n - \frac{1}{2} \leqq x \leqq \nu h_n + \frac{1}{2} \quad (\nu = 0, \pm 1, \pm 2, \ldots) \\ 0 \text{ for all other } x. \end{cases}$$

The function $f_1(x)$ thus consists of towers of breadth 1 and height 1 placed on all the numbers $\equiv 0 \pmod{h_1}$, the function $f_2(x)$ of towers of the same kind placed on all the numbers $\equiv 0 \pmod{h_2}$, etc. The function $f_n(x)$ is periodic with the period h_n.

Further we put

$$F_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x)$$

(see Fig. 3 where $m_1 = m_2 = m_3 = 2$ and $n = 3$).

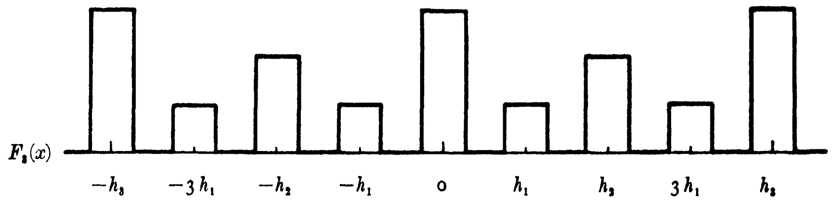


Fig. 3.

$F_1(x)$ thus consists of towers of breadth 1 and height 1 placed on all the numbers $\equiv 0 \pmod{h_1}$.

$F_2(x)$ consists partly of towers of breadth 1 and height 1 placed on all the numbers $\equiv 0 \pmod{h_1}$ but $\not\equiv 0 \pmod{h_2}$, and partly of towers of breadth 1 and height 2 placed on all numbers $\equiv 0 \pmod{h_2}$.

$F_3(x)$ consists partly of towers of breadth 1 and height 1 placed on all numbers $\equiv 0 \pmod{h_1}$ but $\not\equiv 0 \pmod{h_2}$, partly of towers of breadth 1 and height 2 placed on all the numbers $\equiv 0 \pmod{h_2}$ but $\not\equiv 0 \pmod{h_3}$, and finally of towers of breadth 1 and height 3 placed on all numbers $\equiv 0 \pmod{h_3}$.

The function $F_n(x)$ is not only a W^p-a. p. function for every p, but moreover a bounded periodic function with the period h_n.

We begin by showing that $F_1(x), F_2(x), \ldots$ is a W^p-fundamental sequence for every $p \geqq 1$, i. e. that to any $\varepsilon > 0$ there exists an $N = N(\varepsilon, p)$ such that $D_{W^p}[F_n(x), F_{n+q}(x)] < \varepsilon$ for $n \geqq N$ and $q > 0$. Since $(F_{n+q}(x) - F_n(x))^p$ is periodic (with the period h_{n+q}), we have

$$D_{W^p}[F_n(x), F_{n+q}(x)] = \sqrt[p]{M\{(F_{n+q}(x) - F_n(x))^p\}} =$$

$$\sqrt[p]{M\{(f_{n+1}(x) + f_{n+2}(x) + \cdots + f_{n+q}(x))^p\}}.$$

Hence in consequence of Minkowski's inequality

$$D_{W^p}[F_n(x), F_{n+q}(x)] \leqq \sqrt[p]{M\{(f_{n+1}(x))^p\}} + \sqrt[p]{M\{(f_{n+2}(x))^p\}} + \cdots + \sqrt[p]{M\{(f_{n+q}(x))^p\}} =$$

$$\sqrt[p]{\frac{1}{h_{n+1}}} + \sqrt[p]{\frac{1}{h_{n+2}}} + \cdots + \sqrt[p]{\frac{1}{h_{n+q}}} =$$

$$\sqrt[p]{\frac{1}{m_1 m_2 \ldots m_{n+1}}} + \sqrt[p]{\frac{1}{m_1 m_2 \ldots m_{n+2}}} + \cdots + \sqrt[p]{\frac{1}{m_1 m_2 \ldots m_{n+q}}} \leqq$$

$$\sqrt[p]{\frac{1}{2^{n+1}}} + \sqrt[p]{\frac{1}{2^{n+2}}} + \cdots + \sqrt[p]{\frac{1}{2^{n+q}}},$$

where the right-hand side is less than the remainder R_n after the nth term of the convergent geometrical series

$$\sum_1^\infty \frac{1}{2^{\frac{n}{p}}}$$

and hence is $< \varepsilon$ for $n \geqq N = N(\varepsilon, p)$.

Next, we shall prove that the sequence $F_n(x)$ is not W-convergent. Roughly speaking, the reason is that the periodic function $F_n(x)$ (of the increasing sequence $F_n(x)$) has arbitrarily high towers for n sufficiently large which prevents its W-distance from a fixed W-function from tending to 0. Indirectly, we assume that there exists a function $F(x)$ such that

$$F_n(x) \xrightarrow{W} F(x).$$

$F(x)$ being a W-function or, what is equivalent, an S-function, the norm $D_S[F(x)]$ is finite, i. e. a constant K can be found so that

$$\int_x^{x+1} |F(t)|\, dt < K \qquad \text{for all } x.$$

We choose a fixed $N > K$, and consider $F_n(x)$ for $n \geqq N$. For the distance $D_W[F(x), F_n(x)]$ we have

$$D_W[F(x), F_n(x)] = D_W[F(x) - F_n(x)] \geqq D_B[F(x) - F_n(x)] =$$

$$\overline{\lim_{T \to \infty}} \frac{1}{2T} \int_{-T}^{T} |F(x) - F_n(x)|\, dx \geqq \overline{\lim_{m \to \infty}} \frac{1}{(2m+1) h_N} \int_{-(m+\frac{1}{2}) h_N}^{(m+\frac{1}{2}) h_N} |F(x) - F_n(x)|\, dx.$$

Integrating only over a part of the interval $\left(-\left(m+\frac{1}{2}\right)h_N, \left(m+\frac{1}{2}\right)h_N\right)$, and dropping the non-negative contributions from the rest of the interval, we get

$$D_W[F(x), F_n(x)] \geqq \overline{\lim_{m\to\infty}} \frac{1}{(2m+1)h_N} \sum_{\nu=-m}^{m} \int_{\nu h_N-\frac{1}{2}}^{\nu h_N+\frac{1}{2}} |F(x)-F_n(x)|\, dx.$$

Now, since $n \geqq N$, we have $F_n(x) \geqq N$ in every one of the $2m+1$ intervals

$$\left(\nu h_N - \frac{1}{2}, \nu h_N + \frac{1}{2}\right),$$

and hence

$$\int_{\nu h_N-\frac{1}{2}}^{\nu h_N+\frac{1}{2}} |F(x)-F_n(x)|\, dx \geqq \int_{\nu h_N-\frac{1}{2}}^{\nu h_N+\frac{1}{2}} F_n(x)\, dx - \int_{\nu h_N-\frac{1}{2}}^{\nu h_N+\frac{1}{2}} |F(x)|\, dx > N-K.$$

Thus we finally get for $n \geqq N$

$$D_W[F(x), F_n(x)] \geqq \frac{1}{h_N}(N-K),$$

where the right-hand side is a (perhaps »very small«) positive constant independent of n, and this contradicts the assumption that

$$D_W[F(x), F_n(x)] \to 0 \qquad \text{for } n\to\infty.$$

As in main example 1, in all the main examples of the paper (as well as of the appendix) a sequence of functions $F_1(x)$, $F_2(x)$, ... of bounded periodic functions with the periods $h_1 = m_1$, $h_2 = m_1 m_2$, ... is considered where $m_1, m_2, \ldots$ are integers $\geqq 2$. In most of the main examples further claims are put to these numbers concerning the rapidity with which they tend to ∞. In main examples 1 and 2 (and main example IV of the appendix), however, no such claim is made to the numbers $m_1, m_2, \ldots$, and we might as well have chosen them all equal to 2; in order to get the greatest possible analogy between our main examples, we have preferred not to make such a specialisation.

CHAPTER III.

Two Theorems on G^p-Functions and a Theorem on Periodic G-Points.

We begin by stating two theorems on the behaviour of G^p-functions for fixed G and variable p (of course $p \geqq 1$), the first theorem dealing with G^p-a. p. functions, the other with G^p-zero functions.

Theorem 1. *If a function is G^1-a. p. and belongs to the G^{p_0}-set for a $p_0 > 1$, it is G^p-a. p. for $p < p_0$.*

A bounded function being a G^p-function for all p, the theorem has the following

Corollary. *A bounded G^1-a. p. function is G^p-a. p. for all p.*

We next turn to the G^p-zero functions. As regards the S^p-zero functions we have already mentioned the (trivial) fact that these functions for each p are just those functions which are equal to 0 almost everywhere. In reality, the following theorem on G^p-zero functions therefore only deals with the cases $G = W$ and $G = B$, but of course it also holds for $G = S$.

Theorem 2. *If a function is a G^1-zero function and belongs to the G^{p_0}-set for a $p_0 > 1$, it is a G^p-zero function for $p < p_0$.*

Evidently we have (of the same reasons as above) the following

Corollary. *A bounded G^1-zero function is a G^p-zero function for all p.*

The proofs of the two theorems are based on Hölder's inequality. Let p_1 be an arbitrary number, $1 < p_1 < p_0$, and $f(x)$ an arbitrary function. In Hölder's inequality we replace $f(x)$ and $g(x)$ by $|f(x)|^{\frac{1}{p}}$ and $|f(x)|^{\frac{p_0}{q}}$, where the two positive numbers p and q are determined so that $\frac{1}{p} + \frac{p_0}{q} = p_1$ and $\frac{1}{p} + \frac{1}{q} = 1$, i. e.

$$\frac{1}{p} = \frac{p_0 - p_1}{p_0 - 1}, \quad \frac{1}{q} = \frac{p_1 - 1}{p_0 - 1}.$$

We then obtain

$$\frac{1}{b-a}\int_a^b |f(x)|^{p_1}\,dx \leqq \left[\frac{1}{b-a}\int_a^b |f(x)|\,dx\right]^{\frac{p_0-p_1}{p_0-1}} \cdot \left[\frac{1}{b-a}\int_a^b |f(x)|^{p_0}\,dx\right]^{\frac{p_1-1}{p_0-1}}$$

and, letting the interval (a, b) vary and »passing to the limit« in accordance with the definitions of the different distances, we get for $G = S$, W or B the inequality

$$(1) \qquad (D_{G^{p_1}}[f(x)])^{p_1} \leqq (D_{G^1}[f(x)])^{\frac{p_0-p_1}{p_0-1}} \cdot (D_{G^{p_0}}[f(x)])^{\frac{p_1-1}{p_0-1}p_0}.$$

Proof of Theorem 1. Let $f(x)$ be a function satisfying the assumptions of Theorem 1, i. e. a G^1-a. p. function and a G^{p_0}-function for a $p_0 > 1$. Let $\sigma_q(x)$ be a Bochner-Fejér sequence of $f(x)$. Then $D_{G^1}[f(x), \sigma_q(x)] \to 0$ for $q \to \infty$ and (as mentioned in Chapter I)

$$D_{G^{p_0}}[\sigma_q(x)] \leqq D_{G^{p_0}}[f(x)].$$

For an arbitrary p_1 between 1 and p_0 we have because of (1)

$$(D_{G^{p_1}}[f(x), \sigma_q(x)])^{p_1} \leqq (D_{G^1}[f(x), \sigma_q(x)])^{\frac{p_0-p_1}{p_0-1}} \cdot (D_{G^{p_0}}[f(x), \sigma_q(x)])^{\frac{p_1-1}{p_0-1}p_0} \leqq$$

$$(D_{G^1}[f(x), \sigma_q(x)])^{\frac{p_0-p_1}{p_0-1}} \cdot (D_{G^{p_0}}[f(x)] + D_{G^{p_0}}[\sigma_q(x)])^{\frac{p_1-1}{p_0-1}p_0} \leqq$$

$$(D_{G^1}[f(x), \sigma_q(x)])^{\frac{p_0-p_1}{p_0-1}} \cdot (2\, D_{G^{p_0}}[f(x)])^{\frac{p_1-1}{p_0-1}p_0}$$

where the right-hand side tends to 0 for $q \to \infty$, since $D_{G^1}[f(x), \sigma_q(x)] \to 0$. Consequently $D_{G^{p_1}}[f(x), \sigma_q(x)] \to 0$ so that $f(x)$ is a G^{p_1}-a. p. function.

Proof of theorem 2. Let $f(x)$ be a function satisfying the assumptions of Theorem 2, i. e. a G^1-zero function and a G^{p_0}-function for a $p_0 > 1$. For an arbitrary p_1 between 1 and p_0 we have because of (1)

$$(D_{G^{p_1}}[f(x)])^{p_1} \leqq (D_{G^1}[f(x)])^{\frac{p_0-p_1}{p_0-1}} \cdot (D_{G^{p_0}}[f(x)])^{\frac{p_1-1}{p_0-1}p_0} = 0,$$

i. e. $D_{G^{p_1}}[f(x)] = 0$.

Remark. Using the theory of Fourier series (in particular the uniqueness theorem) we may consider Theorem 2 as a special case of Theorem 1. In fact, a G^p-zero function being the same as a G^p-a. p. function with the Fourier series 0, the function $f(x)$ of Theorem 2 is on account of Theorem 1 a G^p-a. p. function for $p < p_0$, and having the Fourier series 0 it is therefore a G^p-zero function for $p < p_0$.

Now we pass to a theorem of a somewhat different character which will be useful for us later on.

Theorem on the periodic points. *If $f(x)$ is a 1-integrable periodic function, and if (for a $p>1$) the G^1-point »around« $f(x)$ contains any G^p-function at all, then the function $f(x)$ itself is p-integrable.*

In other words: A 1-integrable periodic function $f(x)$ is a G^p-function for all those p for which there exist G^p-functions in the (periodic) G^1-point around $f(x)$.

Proof. Since $D_{B^p} \leqq D_{G^p}$, a G^p-function is also a B^p-function, and the G^1-point around $f(x)$ is contained in the B-point around $f(x)$. Therefore it is sufficient to prove the theorem for $G = B$. Hence we assume that there exists a B-zero function $j(x)$ such that $f(x) + j(x)$ is a B^p-function, and we have to prove that $f(x)$ is p-integrable. Let the period of $f(x)$ be $b - a$, and let $T = \nu(b-a)$ where ν is a positive integer. Using first the inequality

$$|f(x) + g(x)| \geqq |(f(x))_N \dot{+} (g(x))_N|$$

and afterwards Minkowski's inequality, we have for an arbitrary fixed $N > 0$

$$\sqrt[p]{\frac{1}{2T}\int_{-T}^{T}|f(x)+j(x)|^p\,dx} \geqq \sqrt[p]{\frac{1}{2T}\int_{-T}^{T}|(f(x))_N + (j(x))_N|^p\,dx} \geqq$$

$$\sqrt[p]{\frac{1}{2T}\int_{-T}^{T}|(f(x))_N|^p\,dx} - \sqrt[p]{\frac{1}{2T}\int_{-T}^{T}|(j(x))_N|^p\,dx}.$$

Now, as $|(j(x))_N|$ is $\leqq N$ as well as $\leqq |j(x)|$ we have $|(j(x))_N|^p \leqq N^{p-1}\,|j(x)|$. Hence

$$\sqrt[p]{\frac{1}{2T}\int_{-T}^{T}|f(x)+j(x)|^p\,dx} \geqq$$

$$\sqrt[p]{\frac{1}{b-a}\int_{a}^{b}|(f(x))_N|^p\,dx} - N^{1-\frac{1}{p}}\sqrt[p]{\frac{1}{2T}\int_{-T}^{T}|j(x)|\,dx}.$$

Letting $\nu \to \infty$, we get, since $j(x)$ is a B-zero function,

$$D_{B^p}[f(x)+j(x)] \geqq \sqrt[p]{\frac{1}{b-a}\int_{a}^{b}|(f(x))_N|^p\,dx}.$$

Finally, letting $N \to \infty$, we get the inequality

$$D_{B^p}[f(x) + j(x)] \geqq \sqrt[p]{\frac{1}{b-a}\int_a^b |f(x)|^p\, dx} = D_{B^p}[f(x)]$$

which shows in particular that $f(x)$ is p-integrable.

We add two remarks on the periodic G-points.

1°. A periodic G-point contains essentially only one periodic function, or precisely speaking: *Two periodic functions in a G-point are identical almost everywhere.* For they have the same Fourier series in almost periodic and therefore in periodic sense; consequently they have a common period, and further they are identical almost everywhere because of the uniqueness theorem on p-integrable periodic functions with a fixed period. A period of some periodic function in a periodic G-point is called a period of the G-point.

2°. Every periodic G-point with the period h has a Fourier series of the form

$$\sum_{-\infty}^{\infty} A_n e^{i\frac{2\pi}{h}nx},$$

where all the Fourier exponents are integral multiples of the number $\frac{2\pi}{h}$. We shall prove that the converse is also true, i. e. *that every G-point which has a Fourier series of the form*

$$\sum_{-\infty}^{\infty} A_n e^{i\frac{2\pi}{h}nx}$$

is a periodic G-point with the period h. Let $\sigma_q(x)$ be a Bochner-Fejér sequence of the Fourier series. All the Fourier exponents being integral multiples of the number $\frac{2\pi}{h}$, the Bochner-Fejér polynomials are periodic with the period h. The sequence $\sigma_q(x)$ being G-convergent is in particular a G-fundamental sequence. As all the $\sigma_q(x)$ are periodic with the period h, we have

$$D_p[\sigma_{q_1}(x), \sigma_{q_2}(x)] = D_G[\sigma_{q_1}(x), \sigma_{q_2}(x)] \quad (G = S_h^p,\ W^p,\ B^p).$$

Thus $\sigma_q(x)$ is also a p-fundamental sequence and therefore p-converges to a p-integrable periodic function $f(x)$ with the period h. Since, on account of

$$D_G[f(x), \sigma_q(x)] = D_p[f(x), \sigma_q(x)] \qquad (G = S^p_h, W^p, B^p),$$

the Bochner-Fejér sequence $\sigma_q(x)$ also G-converges to $f(x)$, the function $f(x)$ belongs to our G-a. p. point.

We remind in this connection of the fact (stated in Chapter I), that the G-limit periodic functions can be characterised as G-a. p. functions with Fourier series of the form

$$\Sigma A_n e^{i d r_n x}$$

where all the Fourier exponents are rational multiples of a number d. Evidently the same characterisation holds for the G-limit periodic points.

The theorem on the periodic points involves in particular that the upper bound P_1 for the p for which »the periodic representative« $f(x)$ of a periodic G^1-point is p-integrable is equal to the upper bound P_2 of the p for which the G^1-point contains G^p-functions. It may be of interest to show that this (more special) result can also be derived by help of Fourier series. Indirectly, we assume that the first upper bound P_1 is less than the other P_2. We choose p_1 so that $P_1 < p_1 < P_2$. Then there exists a G^{p_1}-function $g(x)$ in the G^1-point. Let now p_2 be chosen so that $P_1 < p_2 < p_1$. The function $g(x)$, lying in the periodic G^1-point, is G^1-a. p. and, being also a G^{p_1}-function, is simultaneously a G^{p_2}-a. p. function in consequence of Theorem 1. The Fourier series of the function $g(x)$ being that of the periodic G^1-point has the form

$$\Sigma A_n e^{i \frac{2\pi}{h} n x}.$$

The G^{p_2}-a. p. point around $g(x)$ having the same Fourier series is therefore, in consequence of Remark 2°, a periodic G^{p_2}-point and thus contains a p_2-integrable periodic function $h(x)$. The two periodic functions $f(x)$ and $h(x)$ both lying in our G^1-point must, in consequence of Remark 1°, be equal almost everywhere. Consequently $f(x)$ (as $h(x)$) is a p_2-integrable function, in contradiction to $p_2 > P_1$

CHAPTER IV.

The Mutual Relations of the S^p-Spaces and the S^p-a. p. Spaces.

§ 1.

Introduction.

Since $D_{G^p} \geqq D_{G^1}$ for $p \geqq 1$, every G^p-function is also a G^1-function, and every G^p-zero function is also a G^1-zero function. Consequently every G^p-point is entirely contained in a G^1-point. In the S-case however, as mentioned above, the S^p-zero functions have an especially simple character, being the same for every p, namely the functions which are 0 almost everywhere. Consequently *every S^p-point is itself an S-point* (and not only contained in an S-point). We start from an S-point and will investigate its behaviour as regards the S^p-spaces and the S^p-a. p. spaces. We call an S-point **alive at the time** p_1 **as to the** S^p**-spaces**, if the S-point is an S^{p_1}-point. Otherwise it is said to be **dead at the time** p_1 as to the S^p-spaces. If we know, whether an S-point is alive or dead at the time p_1 as to the S^p-spaces, we say that we know the **behaviour** of the S-point at the time p_1 as to the S^p-spaces. If an S point is alive at one date, it is also alive at all the previous dates. The upper bound P of all p for which the S-point is alive is called the **lifetime** of the S-point as to the S^p-spaces. Beforehand, nothing can be said about the behaviour of the S-point at its moment of death (i. e. at the time P). If the S-point is S-a. p., we can, analogously, consider its lifetime as to the S^p-a. p. spaces and its behaviour as to the S^p-a. p. spaces in the moment of death. In consequence of Theorem 1, Chapter III *an S-a. p. point has the same lifetime as to the S^p-spaces and as to the S^p-a. p. spaces.* In the following two paragraphs we shall state all the possibilities which may occur.

§ 2.

S-Points which are not S-a. p. Points.

We consider an arbitrary S-point which is not S-a. p. and denote, as above, its lifetime as to the S^p-spaces by P. It will be proved by examples that the following possibilities (which are all those imaginable beforehand) may occur:

1. The lifetime $P = \infty$.
2. The lifetime P is arbitrary finite, $1 \leqq P < \infty$.
   - 2 a. The point is dead at the time P as to the S^p-spaces $(P > 1)$.
   - 2 b. The point is alive at the time P as to the S^p-spaces $(P \geqq 1)$

Example to 1.

We define $f(x)$ for $-\infty < x < \infty$ by

$$f(x) = \begin{cases} 1 & \text{for } 0 \leqq x \leqq 1 \\ 0 & \text{for all other } x. \end{cases}$$

Obviously, $f(x)$ being bounded is an S^p-function for every $p \geqq 1$. And that $f(x)$ is not S-a. p. is an immediate consequence of Theorem 1 of Chapter I, as $f(x)$ has no relatively dense set of S-translation-numbers belonging for instance to $\frac{1}{2}$, the equality $D_S[f(x+\tau), f(x)] = 1$ being valid for $|\tau| \geqq 1$.

Thus the S-point around $f(x)$ is not S-a. p. and has the lifetime $P = \infty$.

Example to 2 a.

P being an arbitrary number, $1 < P < \infty$, we define $f(x)$ for $-\infty < x < \infty$ by

$$f(x) = \begin{cases} \left(\frac{1}{x}\right)^{\frac{1}{P}} & \text{for } 0 < x \leqq 1 \\ 0 & \text{for all other } x. \end{cases}$$

The function $f(x)$ is an S^p-function for $p < P$ but not for $p = P$, since

$$(D_{S^p}[f(x)])^p = \int_0^1 \left(\frac{1}{x}\right)^{\frac{p}{P}} dx$$

and $\int_0^1 \left(\frac{1}{x}\right)^{\alpha} dx$ is convergent for $\alpha < 1$ and divergent for $\alpha = 1$. Further $f(x)$ is not S-a. p., as

$$D_S[f(x+\tau), f(x)] = \int_0^1 \left(\frac{1}{x}\right)^{\frac{1}{P}} dx \,(> 0) \quad \text{for} \quad |\tau| \geqq 1.$$

Thus the S-point around $f(x)$ is not S-a. p., has the lifetime P and is dead at the time P.

Example to 2 b.

P being an arbitrary number, $1 \leqq P < \infty$, we define $f(x)$ for $-\infty < x < \infty$ by

$$f(x) = \begin{cases} \left(\dfrac{1}{x(\log x)^2}\right)^{\frac{1}{P}} & \text{for } 0 < x \leqq a < 1 \\ 0 & \text{for all other } x. \end{cases}$$

The function $f(x)$ is an S^p-function for $p = P$, but not for $p > P$, since

$$(D_{S_a^p}[f(x)])^p = \frac{1}{a}\int_0^a \left(\frac{1}{x(\log x)^2}\right)^{\frac{p}{P}} dx$$

and $\int_0^a \left(\dfrac{1}{x(\log x)^2}\right)^{\alpha} dx$ is convergent for $\alpha = 1$ and divergent for $\alpha > 1$. Further $f(x)$ is not S-a. p. as

$$D_{S_a}[f(x+\tau), f(x)] = \frac{1}{a}\int_0^a \left(\frac{1}{x(\log x)^2}\right)^{\frac{1}{P}} dx \, (> 0) \quad \text{for } |\tau| \geqq a.$$

Thus the S-point around $f(x)$ is not S-a. p., has the lifetime P and is alive at the time P.

§ 3.

S-a. p. Points. Main Example 2.

Next we consider the S-a. p. points. As mentioned in § 1, each such point has the same lifetime as to the S^p-spaces and the S^p-a. p. spaces. We will show that the following possibilities (which are all those imaginable beforehand) may occur:

1. The lifetime $P = \infty$.
2. The lifetime P is arbitrary finite, $1 \leqq P < \infty$.
   - 2 a. The point is dead at the time P as to the S^p-spaces ($P > 1$).
   - 2 b. The point is alive at the time P as to the S^p-spaces.
     - 2 b α. The point is alive at the time P as to the S^p-a. p. spaces ($P \geqq 1$).
     - 2 b β. The point is dead at the time P as to the S^p-a. p. spaces ($P > 1$).

The case 2 b β, i. e. that of an S-a. p. point which is an S^P-point but not an S^P-a. p. point, is the only not trivial one.

Example to 1.

Let $f(x)$ be a bounded periodic function. Then the S-point around $f(x)$ has the demanded properties.

Example to 2 a.

Let P be arbitrarily given, $1 < P < \infty$. We consider the periodic function $f(x)$ with the period 1, given in the period interval $0 < x \leqq 1$ by $f(x) = \left(\frac{1}{x}\right)^{\frac{1}{P}}$. Then the S-point around $f(x)$ has the demanded properties.

Example to 2 bα.

Let P be arbitrarily given, $1 \leqq P < \infty$. We consider the periodic function $f(x)$ with the period $a < 1$ which is given in the period interval $0 < x \leqq a$ by $f(x) = \left(\frac{1}{x(\log x)^2}\right)^{\frac{1}{P}}$. Then the S-point around $f(x)$ has the demanded properties.

Example to 2 bβ. **Main example 2.**

P being an arbitrarily given number, $1 < P < \infty$, we shall indicate *a function $F(x)$ which is S-a. p. (even S-limit periodic) and an S^P-function, but not an S^P-a. p. function.* The S-point around $F(x)$ is then of the type desired.

Let $m_1, m_2, \ldots$ be arbitrary integers $\geqq 2$, and $(1 >)\ \varepsilon_1 > \varepsilon_2 > \cdots$ a decreasing sequence tending to 0. In this main example, by a tower of type n we shall understand a tower with the 1-integral ε_n and the P-integral 1. The breadth b_n of a tower of type n is then $\varepsilon_n^{\frac{P}{P-1}}$ so that $b_n \to 0$ for $n \to \infty$ and $b_n < 1$ for all n.

We put

$$h_1 = m_1, \ h_2 = m_1 m_2, \ h_3 = m_1 m_2 m_3, \ \ldots$$

and, as in main example 1, we construct a sequence $F_1(x), F_2(x), \ldots$ of bounded periodic functions with the periods $h_1, h_2, \ldots$. The construction appears from the following array (compare with main example 1).

$F_1(x)$: On all numbers $\equiv 0 \pmod{h_1}$ is placed a tower of type 1.

$F_2(x)$: On all numbers $\equiv 0 \pmod{h_1}$ but $\not\equiv 0 \pmod{h_2}$ is placed a tower of type 1.
. . . . $\equiv 0 \pmod{h_2}$ . . . . 2.

$F_3(x)$: On all numbers $\equiv 0 \pmod{h_1}$ but $\not\equiv 0 \pmod{h_2}$ is placed a tower of type 1.
. . . . $\equiv 0 \pmod{h_2}$ but $\not\equiv 0 \pmod{h_3}$ . . . . 2.
. . . . $\equiv 0 \pmod{h_3}$ . . . . 3.

$\vdots$

$F_n(x)$: On all numbers $\equiv 0 \pmod{h_1}$ but $\not\equiv 0 \pmod{h_2}$ is placed a tower of type 1.
. . . . $\equiv 0 \pmod{h_2}$ but $\not\equiv 0 \pmod{h_3}$ . . . . 2.
. . . . $\equiv 0 \pmod{h_3}$ but $\not\equiv 0 \pmod{h_4}$ . . . . 3.
$\vdots$ $\vdots$
. . . . $\equiv 0 \pmod{h_{n-1}}$ but $\not\equiv 0 \pmod{h_n}$ . . . . $n-1$.
. . . . $\equiv 0 \pmod{h_n}$ . . . . n.

$\vdots$

(See Fig. 4 where $m_1 = m_2 = m_3 = 2$ and $n = 3$). Since $b_n < 1 \leqq \frac{h_1}{2} < h_1$, two towers never overlap.

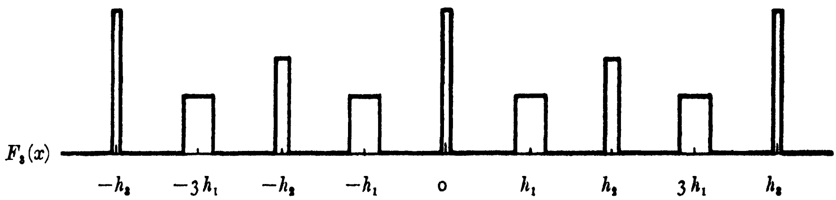


Fig. 4.

The function $F_n(x)$ is obviously a bounded periodic function with the period h_n. We shall show that $F_n(x)$ S-converges to a function $F(x)$ whose construction appears from the following array.

$F(x)$: On all numbers $\equiv 0 \pmod{h_1}$ but $\not\equiv 0 \pmod{h_2}$ is placed a tower of type 1.
. . . . $\equiv 0 \pmod{h_2}$ but $\not\equiv 0 \pmod{h_3}$ . . . . 2.
. . . . $\equiv 0 \pmod{h_3}$ but $\not\equiv 0 \pmod{h_4}$ . . . . 3.
$\vdots$ $\vdots$
. . . . $\equiv 0 \pmod{h_n}$ but $\not\equiv 0 \pmod{h_{n+1}}$ . . . . n.
$\vdots$ $\vdots$

The function $F(x)$ differs from $F_n(x)$ only as regards the towers on the numbers $\equiv 0 \pmod{h_{n+1}}$. As the breadth of the tower placed on 0 in $F_n(x)$ tends to 0 for $n \to \infty$, while in $F(x)$ there is no tower on 0, it is plain that

$$F(x) = \lim_{n\to\infty} F_n(x) \qquad \text{for all } x \neq 0.$$

We shall prove that

$$F_n(x) \xrightarrow{S} F(x).$$

On the numbers $\equiv 0 \pmod{h_{n+1}}$, in $F_n(x)$ there are standing towers of the type n, while in $F(x)$, with exception of the number 0, there are standing towers of the types $n+1, n+2, \ldots$. Hence, denoting by $n+q(m)$ $(q(m) \geqq 1)$ the type of the tower in $F(x)$ placed on a number $m \equiv 0 \pmod{h_{n+1}}$ but $\neq 0$, we have for $m \equiv 0 \pmod{h_{n+1}}$

$$\int_{m-\frac{h_1}{2}}^{m+\frac{h_1}{2}} |F(x) - F_n(x)|\, dx \leqq \int_{m-\frac{h_1}{2}}^{m+\frac{h_1}{2}} F(x)\, dx + \int_{m-\frac{h_1}{2}}^{m+\frac{h_1}{2}} F_n(x)\, dx =$$

$$\begin{cases} \varepsilon_n \text{ for } m = 0 \\ \varepsilon_{n+q(m)} + \varepsilon_n \text{ for } m \equiv 0 \pmod{h_{n+1}} \text{ but } \neq 0, \end{cases}$$

while for $m \equiv 0 \pmod{h_1}$ but $\not\equiv 0 \pmod{h_{n+1}}$

$$\int_{m-\frac{h_1}{2}}^{m+\frac{h_1}{2}} |F(x) - F_n(x)|\, dx = 0.$$

Thus we have

$$\int_{m-\frac{h_1}{2}}^{m+\frac{h_1}{2}} |F(x) - F_n(x)|\, dx < 2\,\varepsilon_n \quad \text{for all} \quad m \equiv 0 \pmod{h_1}.$$

Since an arbitrary interval of the length h_1 is contained in an interval

$$m - \frac{h_1}{2} \leqq x \leqq m + \frac{3\,h_1}{2} \quad (m \equiv 0 \pmod{h_1}),$$

we get

$$\int_{x-\frac{h_1}{2}}^{x+\frac{h_1}{2}} |F(t) - F_n(t)|\, dt < 4\,\varepsilon_n \quad \text{for all } x,$$

so that $D_{S_{h_1}}[F(x), F_n(x)] \leqq \frac{4\,\varepsilon_n}{h_1}$ which tends to 0 for $n \to \infty$. Hence $F_n(x) \xrightarrow{S} F(x)$, and $F(x)$ is therefore an S-a. p. function.

The function $F(x)$ is obviously an S^P-function, as all the towers of $F(x)$ have the P-integral 1 and therefore

$$\int_{m-\frac{h_1}{2}}^{m+\frac{h_1}{2}} (F(x))^P\, dx \leqq 1 \qquad \text{for all } m \equiv 0 \pmod{h_1}$$

and hence

$$\int_{x-\frac{h_1}{2}}^{x+\frac{h_1}{2}} (F(t))^P dt \leqq 2 \qquad \text{for all } x,$$

so that

$$D_{S^P_{h_1}}[F(x)] \leqq \sqrt[P]{\frac{2}{h_1}}.$$

Finally $F(x)$ is n o t a n S^P-a. p. f u n c t i o n. Otherwise by Theorem I a, Chapter I, the function $F(x)$ being the S-limit function of the periodic functions $F_n(x)$ with the periods h_n, the number h_n should be a »fine« $S^P_{h_1}$-translation number of $F(x)$ for »large« n. This, however, is impossible. In fact, by the translation h_n the interval $-\frac{h_1}{2} \leqq x \leqq \frac{h_1}{2}$ containing 0 is translated into the interval $-\frac{h_1}{2} + h_n \leqq x \leqq \frac{h_1}{2} + h_n$ containing h_n, and in the first interval $F(x)$ has no tower while in the second it has a tower with the P-integral I, so that

$$D_{S^P_{h_1}}[F(x + h_n),\ F(x)] \geqq \sqrt[P]{\frac{1}{h_1}}$$

for every h_n.

Besides, in order to prove that $F(x)$ is not S^P-a. p., we could have confined ourselves to apply the general Theorem I of Chapter I instead of the Theorem I a (dealing with limit periodic functions). In fact, for every τ with a modulus $\geqq \frac{h_1}{2}$ we have

$$D_{S^P_{h_1}}[F(x + \tau),\ F(x)] \geqq \sqrt[P]{\frac{1}{h_1}},$$

since the interval $-\frac{3}{4}h_1 \leqq x \leqq \frac{3}{4}h_1$ of length $\frac{3}{2}h_1$ in which $F(x)$ has no tower will be translated by τ into an interval containing at least one of the towers of $F(x)$.

We remark, that the function $F(x)$ constructed above is of similar character as a type of examples of o. a. p. functions treated by TOEPLITZ (Mathematische Annalen, Bd. 98).

CHAPTER V.

The Mutual Relations of the W^p-Spaces and the W^p-a. p. Spaces.

§ 1.

Introduction.

In this Chapter we shall study the mutual relations of the W^p-spaces and the W^p-a. p. spaces, p ranging over all values $\geqq 1$. Of all the W^p-points $(p \geqq 1)$ the W-points are the most comprehensive, and every W^p-point is contained in a W-point. We therefore consider an arbitrary W-point and shall investigate how this W-point behaves as to the W^p-spaces and the W^p-a. p. spaces. First we consider the W^p-points $(p > 1)$ contained in our W-point, but subsequently also the single functions of the W-point. In our characterisation of the W^p-points and of the functions in the W-point only the W^p-spaces and the W^p-a. p. spaces are applied (and not the other types of spaces). Before carrying out our investigations we must have some knowledge about the W-zero functions.

§ 2.

W-zero Functions.

Let $f(x)$ be a W-zero function. We denote the upper bound of the p for which $f(x)$ is a W^p-function by P. In consequence of Theorem 2, Chapter III the function $f(x)$ is a W^p-zero function for $p<P$ so that P can also be defined as the upper bound of those p for which $f(x)$ is a W^p-zero function. We will show that the following possibilities (which are all those imaginable beforehand) may be realised:

1. $P=\infty$.
2. P arbitrary finite, $1 \leqq P < \infty$.
   - 2 a. $f(x)$ is not a W^P-function $(P>1)$.
   - 2 b. $f(x)$ is a W^P-function.
     - 2 b α. $f(x)$ is a W^P-zero function $(P \geqq 1)$.
     - 2 b β. $f(x)$ is not a W^P-zero function $(P>1)$.

Example to 1.

A quite obvious example is $f(x) = 0$ for all x. Another example is a function which is bounded and tends to 0 for $x \to \pm\infty$.

Example to 2 a.

$$f(x) = \begin{cases} \left(\frac{1}{x}\right)^{\frac{1}{p}} & \text{for } 0 < x \leqq 1 \\ 0 & \text{for all other } x. \end{cases}$$

Example to 2 b α.

$$f(x) = \begin{cases} \left(\frac{1}{x(\log x)^2}\right)^{\frac{1}{p}} & \text{for } 0 < x \leqq a < 1 \\ 0 & \text{for all other } x. \end{cases}$$

Example to 2 b β.

We construct a function $f(x)$ in the following way: Let $\varepsilon_1, \varepsilon_2, \ldots$ be a sequence of positive numbers $\leqq 1$ which tends to 0. On the number n ($n = 1, 2, \ldots$) a tower with the 1-integral ε_n and the P-integral 1 is placed. As the breadths of the towers are $\leqq 1$, they do not overlap. $f(x)$ is a W-zero function as

$$\int_x^{x+1} f(t)\, dt \to 0 \quad \text{for} \quad x \to \infty.$$

Further $f(x)$ is a W^p-function, but not a W^p-zero function as

$$\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} (f(x))^p\, dx = 1 \quad \text{for} \quad n = 1, 2, \ldots.$$

There exists an infinite number of such functions which do not differ by W^p-zero functions (i. e. do not belong to the same W^p-point), for instance the functions $a \cdot f(x)$, where a is an arbitrary complex number $\neq 0$ and $f(x)$ the function constructed above.

It may be observed that a function $f(x)$ which is a W-zero function and a W^p- but not W^p-zero function can never be a W^p-a. p. function. In fact, if it was W^p-a. p., it would, as it has the Fourier series 0, be a W^p-zero function.

§ 3.

W-Points in General.

In this paragraph we shall state the laws for the W^p-points ($p > 1$) and the functions in a W-point. A single proof belonging to this investigation will be postponed to § 6 because of its particular character. In § 4 and § 5 examples are given which serve as existence proofs for the different types of W-points.

We consider an arbitrary W-point. We call the point alive at the time p_1 as to the W^p-spaces, if it contains at least one W^{p_1}-point, or, what is equivalent, if it contains at least one W^{p_1}-function; otherwise the W-point is said to be dead at the time p_1. If the W-point is W-a. p., we define in an analogous way the meaning of the point being alive or dead at the time p_1 as to the W^p-a. p. spaces. If we know, whether the W-point is alive or dead at the time p_1 as to the W^p-spaces (W^p-a. p. spaces) we say that we know its behaviour at the time p_1 as to the W^p-spaces (W^p-a. p. spaces). If the W-point is alive at one date, it is alive at all the previous dates. By the lifetime P of the W-point as to the W^p-spaces we understand the upper bound of those p for which the W-point is alive as to the W^p-spaces. If the W-point is W-a. p. in an analogous way its lifetime as to the W^p-a. p. spaces is defined. In consequence of Theorem 1, Chapter III *a W-a. p. point has the same lifetime as to the W^p-spaces and as to the W^p-a. p. spaces.* Beforehand, we cannot say anything about the behaviour of the W-point at the moment of death as to the W^p-spaces and eventually the W^p-a. p. spaces.

At first we study the W^p-points in our W-point. Let $p > 1$ be arbitrarily given. The set of all the W^p-functions in the W-point, if such functions exist, divides into a set of W^p-points. These W^p-points are called the p-descendants of the W-point. In consequence of example 2 bβ of § 2, if there is one p-descendant, there will be an infinite number of them, since the sum of any function $a f(x)$, $a \neq 0$, of this example with p instead of P (or rather the W^p-point around this function) and a fixed p-descendant is again a p-descendant. Let p_1 and p_2 be two numbers, $1 < p_1 < p_2$. We consider a p_1-descendant of the W-point. The set of W^{p_2}-functions in the p_1-descendant, if such functions exist, divides into a set of W^{p_2}-points which are called the p_2-descendants of the p_1-descendant. They are at the same time p_2-descendants of the W-point. We will prove *that only one of the p_1-descendants of the W-point can have p_2-descendants for any $p_2 > p_1$*, so that

all the p_2-descendants of the W-point (if existing) are p_2-descendants of one and the same p_1-descendant. In fact, the difference of two functions, each taken from its p_2-descendant, is a W-zero function and a W^{p_2}-function, and hence, in consequence of Theorem 2, Chapter III, a W^p-zero function for $p < p_2$, in particular for $p = p_1$. This »generating« W^{p_1}-point is called the p_1-generator; all the other p_1-descendants die at the time p_1 at the moment they are »born« (i. e. come into existence as points) and are therefore called the stillborn brothers of the p_1-descendant. The p_1-generator is defined for $1 < p_1 < P$.

If the W-point from which we are starting is a W-a. p. point the p_1-generator ($1 < p_1 < P$) *will be W^{p_1}-a. p.* In fact, the p_2-descendants of the p_1-generator ($p_2 > p_1$) consist of W^{p_2}-functions which are simultaneously W-a. p. functions; thus, in consequence of Theorem 1, Chapter III these functions are W^{p_1}-a. p. and, lying in the p_2-descendants of the p_1-generator, they also lie in the p_1-generator itself, which is therefore W^{p_1}-a. p. *In a W-a. p. point at most one of the p-descendants can be W^p-a. p.* (so that for $p < P$ the p-generator is the only W^p-a. p. p-descendant). For a W^p-a. p. point in the W-a. p. point has the same Fourier series as the W-a. p. point itself, and, in consequence of the uniqueness theorem, there exists only one W^p-a. p. point with a given Fourier series.

In the preceding we have used »biological« phrases. We will also give another methaphor of the situation. We speak of a W-rocket, the »components« of which at the time p are the p-descendants of the W-point; the p-generator is called the p-nucleus and the still-born brothers of the p-generator are called the p-sparks of the rocket (see Fig. 5, which suggests the »evolution« of the W-point »in the course of time«, i. e. for increasing p).

In connection with the figure we remind of certain facts given above: If the W-point is W-a. p., the p-generator is W^p-a. p. for every p, $1 < p < P$, whereas no one of its still-born brothers is. Further, if the W-a. p. point is alive at the moment of death P as to the W^p-spaces, so that there exist P-descendants, at most one of them is W^P-a. p. As we shall see in § 5, some of those W-a. p. points have a W^P-a. p. P-descendant, whereas others have not.

Next we consider the single functions in the W-point. A function $f(x)$ is called alive at the time p_1 as to the W^p-spaces, if $f(x)$ is a W^{p_1}-function; otherwise $f(x)$ is said to be dead at the time p_1. If the W-point is W-a. p., we define in an analogous way what is to be understood by $f(x)$ being alive or dead at the time p_1 as to the W^p-a. p. spaces. The upper bound of the p for

which $f(x)$ is alive as to the W^p-spaces is called the **lifetime** of $f(x)$ as to the W^p-spaces. If the W-point is W-a. p., the lifetime of $f(x)$ as to the W^p-a. p. spaces is defined in the analogous way. In this case, in consequence of Theorem 1, Chapter III, *the function $f(x)$ has the same lifetime as to the W^p-spaces and as to the W^p-a. p. spaces.*

We start our investigations about the functions in a given W-point by mentioning, without proof, that in every W-point there exists a **through**

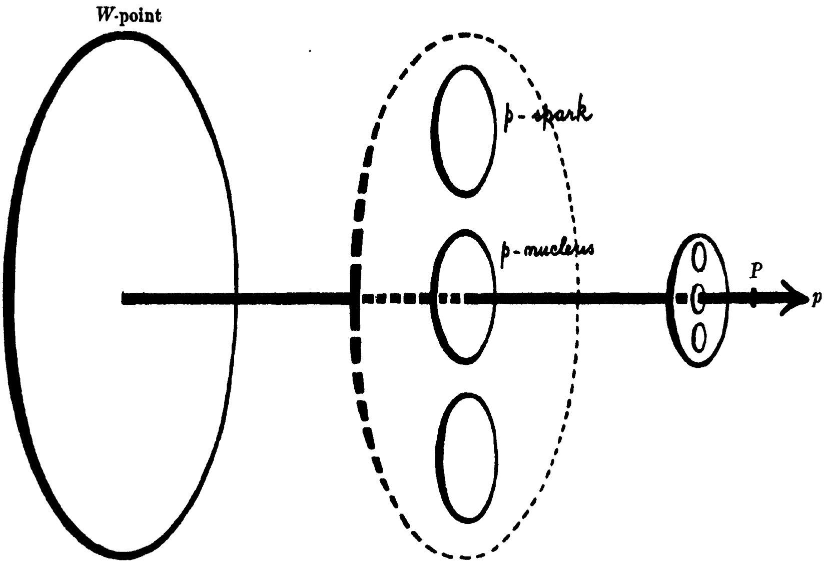


Fig. 5.

function as to the W^p- and the W^p-a. p. spaces, i. e. a function which is a W^p-function for just those p for which the W-point contains W^p-functions and a W^p-a. p. function for just those p for which the W-point contains W^p-a. p. functions. If the W-point is alive at the time P as to the W^p-a. p. spaces, or is not alive at the time P as to the W^p-a. p. spaces but is alive as to the W^p-spaces, it is obvious that there exists a through function; in fact an arbitrary one of the W^p-a. p., respectively W^p-functions contained in the W-point will be a through function. The problem is to show that there exists a through function also in the case where (if $P < \infty$) the W-point is dead at the time P as to the W^p-spaces. In order to prove this, it is of course sufficient to show that there exists a through function as to the W^p-spaces, since such a function, if the

W-point be a W-a. p. point, will at the same time be a through function as to the W^p-a. p. spaces. We postpone the proof to § 6. Taking the existence of a through function for granted, we shall now give a complete account of the functions lying in our arbitrarily given W-point, whose lifetime P ($1 \leqq P \leqq \infty$) and behaviour at the moment of death P (if $P < \infty$) as to the W^p-spaces and the W^p-a. p. spaces are assumed to be known. (What possibilities may occur for a W-point in this respect will, as mentioned in § 1, be discussed in § 4 and § 5). By the investigation of the functions in our W-point we distinguish between the W-point not being W-a. p. or being W-a. p.

I. The W-point is not W-a. p. Denoting the lifetime of a function $f(x)$ in the W-point as to the W^p-spaces by P_1, there are the following possibilities. *The lifetime* P_1 *may be an arbitrary number,* $1 \leqq P_1 \leqq P$, and for any fixed choice of P_1 there are, if $P_1 < \infty$, the *two possibilities:*

1. $f(x)$ is dead as to the W^p-spaces at the time P_1.
2. $f(x)$ is alive as to the W^p-spaces at the time P_1,

with exception, however, of the case $P_1 = 1$ where of course only 2. can occur. and the case $P_1 = P$ where 2. can only occur if the W-point is alive as to the W^p-spaces at the time P.

Proof. Let $g(x)$ be a through function in the W-point as to the W^p-spaces. In the special cases where $P_1 = P = \infty$, or $P_1 = P < \infty$ and moreover the given W-point and the desired function $f(x)$ have the same behaviour as to the W^p-spaces at their common moment of death $P = P_1$, we may as $f(x)$ simply use the function $g(x)$ itself. In all other cases we obtain, on account of the linearity of the W^p-sets, a function $f(x)$ of the type wanted by adding to $g(x)$ a W-zero function of lifetime P_1 which in case of 1. is not a W^{P_1}-function, and in case of 2. is a W^{P_1}-function.

We observe that for $1 < P_1 < P$ all the functions in the P_1-sparks are of the type 2.

II. The W-point is W-a. p. *The lifetime* P_1 *of a function* $f(x)$ *in the* W*-a. p. point as to the* W^p*- and the* W^p*-a. p. spaces may be an arbitrary number in the interval* $1 \leqq P_1 \leqq P$, and for every fixed choice of P_1 there are, if $P_1 < \infty$, the following *three possibilities:*

1. $f(x)$ is dead as to the W^p-spaces at the time P_1,
2. $f(x)$ is alive as to the W^p-spaces, but dead as to the W^p-a. p. spaces at the time P_1,
3. $f(x)$ is alive as to the W^p-a. p. spaces at the time P_1,

with exception, however, of the case $P_1 = 1$ where of course only 3. can occur, and the case $P_1 = P$ where 3. can only occur if the W-a. p. point is alive as to the W^p-a. p. spaces at the time P, and 2. can only occur if the W-a. p. point is alive as to the W^p-spaces at the time P.

Proof. Let $g(x)$ be a through function in the W-a. p. point as to the W^p- and the W^p-a. p. spaces. If $P_1 = P = \infty$, or $P_1 = P < \infty$ and moreover the given W-a. p. point and the desired function $f(x)$ have the same behaviour as to the W^p-spaces and the W^p-a. p. spaces at their common moment of death $P = P_1$, we may as $f(x)$ simply use the through function $g(x)$ itself. In all other cases we obtain, on account of the linearity of the W^p-sets, a function $f(x)$ of the type wanted by adding to $g(x)$ a suitable W-zero function: We get a function $f(x)$ of the type 1., by adding to $g(x)$ a W-zero function with the lifetime P_1 as to the W^p-spaces which is not a W^{P_1}-function. Similarly we get a function $f(x)$ of the type 2. by adding to $g(x)$ a W-zero function which is a W^{P_1}-function but not a W^{P_1}-zero function (since, in consequence of the uniqueness theorem, two W^{P_1}-a. p. functions in our W-a. p. point must differ by a W^{P_1}-zero function). Finally we get a function of the type 3. by adding to $g(x)$ a W-zero function which has the lifetime P_1 as to the W^p-spaces and is a W^{P_1}-zero function.

We observe that for $1 < P_1 < P$ all the functions in the P_1-sparks are of type 2.

§ 4.

W-Points which are not W-a. p. Points.

In this paragraph we shall consider the W-points which are not W-a. p. points, and we shall investigate what possibilities may occur for such points concerning as well the lifetime P as the behaviour at the moment of death as to the W^p-spaces. We shall show *that all possibilities which are imaginable beforehand may occur*, viz.

1. $P = \infty$.
2. P arbitrary finite, $1 \leqq P < \infty$.
   2 a. The point is dead as to the W^p-spaces at the time P $(P > 1)$.
   2 b. The point is alive as to the W^p-spaces at the time P $(P \geqq 1)$.

A (trivial) example to 1. with the lifetime ∞ is first given. Next, in order to get examples of W-points which are not W-a. p. and have an arbitrarily given finite lifetime P and a given behaviour at the moment of death as to the W^p-spaces, we add the W-point of the first example to a periodic W-point with the lifetime P and the desired behaviour at the moment of death as to the W^p-spaces. (In consequence of the theorem of Chapter III on the periodic G-points, the point »behaves« entirely as the periodic function contained in it). By this addition, the almost periodicity of the periodic W-point is destroyed, whereas its lifetime and behaviour at the moment of death as to the W^p-spaces are preserved.

Example to 1.

Let

$$f(x) = \begin{cases} 1 \text{ for } & 0 \leqq x < \infty \\ -1 \text{ for } & -\infty < x < 0. \end{cases}$$

The function $f(x)$ being bounded is obviously a W^p-function for every p; further $f(x)$ is not W-a. p., as

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(x)\,dx = 1 \quad \text{while} \quad \lim_{T \to \infty} \frac{1}{T} \int_{-T}^0 f(x)\,dx = -1 \; (\neq 1).$$

The W-point around $f(x)$ is thus not W-a. p. and has the lifetime $P = \infty$ as to the W^p-spaces.

Example to 2 a.

Let P be an arbitrary number, $1 < P < \infty$. Let $f(x)$ be the function of example 1, and $h(x)$ a periodic function which is p-integrable for $p < P$ but not P-integrable. Denote by $\mathfrak{A}$ the W-point around $f(x)$ and by $\mathfrak{B}$ the W-point around $h(x)$. Then the point $\mathfrak{C} = \mathfrak{A} + \mathfrak{B}$ will not be W-a. p., will have the lifetime P as to the W^p-spaces and be dead at the time P. That $\mathfrak{C}$ is not W-a. p. results from the linearity of the W-a. p. space, $\mathfrak{B}$ being W-a. p. and $\mathfrak{A}$ not being W-a. p. Further the point $\mathfrak{C}$ contains the function $f(x) + h(x)$ which is a

W^p-function for $p<P$. Finally no W^P-function lies in the point $\mathfrak{C}$; in fact the functions in $\mathfrak{C}$ can be obtained by adding to $f(x)$ all the functions in $\mathfrak{B}$, and $f(x)$ is a W^P-function, whereas, in consequence of the theorem on the periodic points, no function in $\mathfrak{B}$ is a W^P-function.

Example to 2 b.

Let P be an arbitrary number, $1 \leqq P < \infty$. Let $f(x)$ be the function of example 1, and $h(x)$ a periodic function which is P-integrable but not p-integrable for $p>P$. Denote by $\mathfrak{A}$ the W-point around $f(x)$ and by $\mathfrak{B}$ the W-point around $h(x)$. Then the W-point $\mathfrak{C}=\mathfrak{A}+\mathfrak{B}$ will not be W-a. p., will have the lifetime P as to the W^p-spaces and be alive at the time P. The proof is quite analogous to that of example 2 a: From the linearity of the W-a. p. space it results that $\mathfrak{C}$ is not W-a. p., the point $\mathfrak{B}$ being W-a. p. and $\mathfrak{A}$ not being W-a. p. Further the point $\mathfrak{C}$ contains the function $f(x)+h(x)$ which is a W^P-function. Finally for $p>P$ no W^p-function lies in the point $\mathfrak{C}$, as this latter point consists just of the functions obtained by adding $f(x)$ to all the functions in $\mathfrak{B}$, and $f(x)$ is a W^p-function, whereas, by the theorem on the periodic points, no function in $\mathfrak{B}$ is a W^p-function.

§ 5.

W-a. p. Points. Main Example 3.

In this paragraph we consider an arbitrary W-a. p. point, whose lifetime as to the W^p- and the W^p-a. p. spaces is denoted by P, and shall show *that also here all possibilities which are imaginable beforehand may occur*, viz.

1. $P=\infty$.
2. P arbitrary finite, $1 \leqq P < \infty$.
   - 2 a. The point is dead as to the W^p-spaces at the time P $(P>1)$.
   - 2 b. The point is alive as to the W^p-spaces at the time P.
     - 2 b α. The point is alive as to the W^p-a. p. spaces at the time P $(P \geqq 1)$.
     - 2 b β. The point is dead as to the W^p-a. p. spaces at the time P $(P>1)$.

Example to 1.

The W-point around a bounded periodic function.

Example to 2 a.

Let P be an arbitrary number, $1 < P < \infty$, and $h(x)$ a periodic function which is p-integrable for $p < P$ but not P-integrable. Then the W-point around $h(x)$ has the desired properties. Firstly, it is obviously W-a. p., $h(x)$ being W-a. p. Secondly, it contains a W^p-a. p. function for $p < P$, viz. $h(x)$. And thirdly, by the theorem on the periodic points, it does not contain any W^P-function.

Example to 2 b α.

Let P be an arbitrary number, $1 \leqq P < \infty$, and $h(x)$ a periodic function which is P-integrable, but not p-integrable for $p > P$. Then the W-point around $h(x)$ has the wanted properties. Firstly, the W-point is W-a. p., $h(x)$ being W-a. p. Secondly, it contains a W^P-a. p. function, viz. $h(x)$. And thirdly, by the theorem on the periodic points, it does not contain any W^p-function for $p > P$.

The case 2 b β remains. A rather complicated construction is necessary in order to show that this case can be realized.

Example to 2 b β. **Main example 3.**

Let P be arbitrarily given, $1 < P < \infty$. We wish to construct *a function $F(x)$ which is a W^p-a. p. function for $p < P$ and a W^P-function, but such that the W-point around $F(x)$ does not contain any W^P-a. p. function.* Then the W-point around $F(x)$ will be an example to 2 b β.

Let $m_1, m_2, \ldots$ be a sequence of odd numbers $\geqq 3$, increasing so strongly that the product

$$\left(1-\frac{1}{m_1}\right)\left(1-\frac{1}{m_2}\right)\left(1-\frac{1}{m_3}\right)\cdots$$

is convergent (> 0). As usually we put

$$h_1 = m_1, \quad h_2 = m_1 m_2, \quad h_3 = m_1 m_2 m_3, \ldots .$$

Further let $\varepsilon_1, \varepsilon_2, \ldots$ be a sequence of numbers such that $1 > \varepsilon_1 > \varepsilon_2 \cdots \to 0$. By a tower of type n we understand a tower with the 1-integral ε_n and the P-integral 1. Since $\varepsilon_n < 1$, the breadth of a tower of type n is less than 1. In the following, the letters ν, μ, η denote integers. We construct a sequence of functions $F_1(x), F_2(x), \ldots$ in the following way:

$F_1(x)$: In every interval $\nu h_1 \leqq x < (\nu + 1) h_1$ a tower of type 1 is placed on the central one of the subintervals $\mu \leqq x < \mu + 1$.

$F_2(x)$: Is obtained from $F_1(x)$ by »filling up« the central[1] one of the subintervals $\mu h_1 \leqq x < (\mu + 1) h_1$ of every interval $\nu h_2 \leqq x < (\nu + 1) h_2$ by towers of type 2, i. e. by placing a tower of type 2 on every subinterval $\eta \leqq x < \eta + 1$ of the mentioned central interval where no tower of $F_1(x)$ is standing.

$F_3(x)$: Is obtained from $F_2(x)$ by »filling up« the central one of the subintervals $\mu h_2 \leqq x < (\mu + 1) h_2$ of every interval $\nu h_3 \leqq x < (\nu + 1) h_3$ by towers of type 3, i. e. by placing a tower of type 3 on every subinterval $\eta \leqq x < \eta + 1$ of the mentioned central interval where no tower of $F_2(x)$ is standing.

$\vdots$

$F_{n+1}(x)$: Is obtained from $F_n(x)$ by »filling up« the central one of the subintervals $\mu h_n \leqq x < (\mu + 1) h_n$ of every interval $\nu h_{n+1} \leqq x < (\nu + 1) h_{n+1}$ by towers of type $n + 1$, i. e. by placing a tower of type $n + 1$ on every subinterval $\eta \leqq x < \eta + 1$ of the mentioned central interval where no tower of $F_n(x)$ is standing.

$\vdots$

(see Fig. 6, where $m_1 = m_2 = 3$ and $n = 2$).

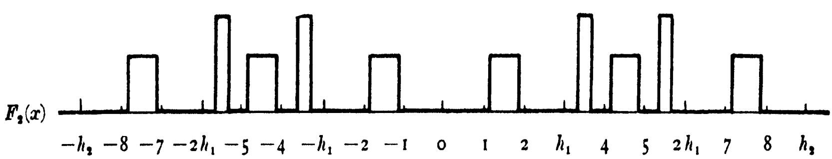


Fig. 6.

$F_n(x)$ is a bounded periodic function with the period h_n. Further $F_n(x) = F_{n+1}(x)$ for $-h_n \leqq x < h_n$ and moreover $F_n(x) = F_{n+1}(x) = F_{n+2}(x) = \cdots$ for the same x, since $h_n < h_{n+1} < h_{n+2} < \cdots$; consequently, as $h_n \to \infty$, a limit function

$$F(x) = \lim_{n \to \infty} F_n(x)$$

exists for $-\infty < x < \infty$, and

$$F(x) = F_n(x) \quad \text{for} \quad -h_n \leqq x < h_n.$$

[1] Here and in the following instead of the central one of the subintervals we could have chosen any one of the subintervals with exception of the first and the last, but in order to be able to use main example 3 to further purposes in the appendix we have made the specialisation mentioned.

The function $F_n(x)$, differing from $F_{n+q}(x)$ only by towers of the types $n+1$, $n+2, \ldots, n+q$, differs from $F(x)$ only by towers of types $n+1$, $n+2, \ldots$; hence (as $\varepsilon_{n+1} > \varepsilon_{n+2} > \cdots$)

$$\int_m^{m+1} (F(x) - F_n(x))\, dx \leqq \varepsilon_{n+1} \quad \text{for each integer } m$$

and thus

$$\int_x^{x+1} (F(t) - F_n(t))\, dt \leqq 2\,\varepsilon_{n+1} \quad \text{for all } x,$$

i. e.

$$D_S[F(x), F_n(x)] \leqq 2\,\varepsilon_{n+1}.$$

Since $\varepsilon_n \to 0$, we have $F_n(x) \xrightarrow{S} F(x)$; thus the function $F(x)$ is an S-limit periodic function, in particular an S-a. p. function. All our towers having the P-integral 1, obviously $F(x)$ is simultaneously an S^P-function (or, what is equivalent, a W^P-function) and therefore, in consequence of Theorem 1, Chapter III, also an S^p-a. p. function for $p<P$. Hence our function is not only a W^P-function and a W^p-a. p. function for $p<P$, as desired, but *moreover an S^p-a. p. function for $p<P$.*

We have to prove that the W-point around $F(x)$ does not contain any W^P-a. p. function. As a preparation we prove that the function $F(x)$ itself is not a W^P-a. p. function.

We begin by some preliminary remarks:

By d_i $(i = 1, 2, \ldots)$ we denote the relative density of all the »places« $\eta \leqq x < \eta + 1$ on which there are standing no towers in $F_i(x)$, exactly speaking, the ratio $d_i = \frac{e_i}{h_i}$ between the number e_i of the »empty« places in a period interval $\mu h_i \leqq x < (\mu+1) h_i$ of $F_i(x)$ and the total number h_i of places in such an interval. It is easy to see that

$$d_{i+1} = \left(1 - \frac{1}{m_{i+1}}\right) d_i.$$

In fact, when passing over from $F_i(x)$ to $F_{i+1}(x)$ we fill out just one of the m_{i+1} period intervals $\mu h_i \leqq x < (\mu+1) h_i$ of $F_i(x)$ of which a period interval $\nu h_{i+1} \leqq x < (\nu+1) h_{i+1}$ of $F_{i+1}(x)$ consists, so that $e_{i+1} = (m_{i+1} - 1) e_i$ and hence

$$d_{i+1} = \frac{e_{i+1}}{h_{i+1}} = \frac{(m_{i+1}-1)\, e_i}{m_{i+1}\, h_i} = \left(1 - \frac{1}{m_{i+1}}\right) \frac{e_i}{h_i} = \left(1 - \frac{1}{m_{i+1}}\right) d_i.$$

Especially we get (by induction), as $d_1 = \frac{e_1}{h_1} = \left(1 - \frac{1}{m_1}\right)$,

$$d_n = \left(1 - \frac{1}{m_1}\right)\left(1 - \frac{1}{m_2}\right)\cdots\left(1 - \frac{1}{m_n}\right) \qquad (n = 1, 2, \ldots).$$

We emphasise that, on account of the convergence of the infinite product

$$\left(1 - \frac{1}{m_1}\right)\left(1 - \frac{1}{m_2}\right)\cdots,$$

the relative density d_n of the empty places in the function $F_n(x)$ keeps greater than a positive constant when $n \to \infty$, so that after each construction of one of our functions $F_n(x)$ an »essential« part of the x-axis is kept free from towers.

We can now easily show that our function $F(x)$ is not W^P-a. p. To this purpose we consider, for an arbitrary fixed n, among the m_{n+1} intervals $\mu h_n \leqq x < (\mu + 1) h_n$ of the interval $\nu h_{n+1} \leqq x < (\nu + 1) h_{n+1}$, the central one which we denote by $\alpha_\nu \leqq x < \alpha_\nu + h_n$. Then we have

$$(1) \qquad \sqrt[P]{\frac{1}{h_n}\int_{\alpha_\nu}^{\alpha_\nu + h_n} (F_{n+1}(x) - F_{n+1}(x + h_n))^P \, dx} = \sqrt[P]{\left(1 - \frac{1}{m_1}\right)\left(1 - \frac{1}{m_2}\right)\cdots\left(1 - \frac{1}{m_n}\right)}.$$

For, in the interval $\alpha_\nu + h_n \leqq x < \alpha_\nu + 2 h_n$ (to the right of the central interval), $F_{n+1}(x)$ has the same towers as $F_n(x)$, whereas, in consequence of the above, in the central interval $\alpha_\nu \leqq x < \alpha_\nu + h_n$ itself $F_{n+1}(x)$ has the same towers as $F_n(x)$ plus $h_n d_n = h_n \left(1 - \frac{1}{m_1}\right)\left(1 - \frac{1}{m_2}\right)\cdots\left(1 - \frac{1}{m_n}\right)$ towers of type $n + 1$ (see Fig. 7),

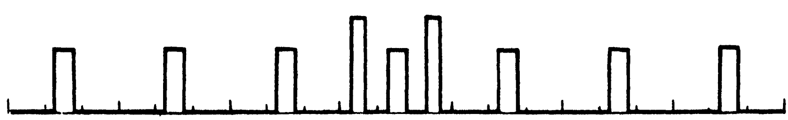


Fig. 7.

and all our towers have the P-integral 1. Now, however, $F(x) = F_{n+1}(x)$ for $-h_{n+1} \leqq x < h_{n+1}$. Hence we have ($\alpha_0 \leqq x < \alpha_0 + h_n$ denoting the central one of the intervals $\mu h_n \leqq x < (\mu + 1) h_n$ in $0 \leqq x < h_{n+1}$)

$$(2) \qquad \sqrt[P]{\frac{1}{h_n}\int_{\alpha_0}^{\alpha_0+h_n}(F(x)-F(x+h_n))^P\,dx}=\sqrt[P]{\left(1-\frac{1}{m_1}\right)\left(1-\frac{1}{m_2}\right)\cdots\left(1-\frac{1}{m_n}\right)}.$$

By help of (2) we conclude that $F(x)$ is not W^P-a. p. Otherwise, in consequence of Theorem 2 a, Chapter I, since $F_n(x)$ is a sequence of periodic functions with the periods h_n which S-converges to $F(x)$, we should have

$$D_{S_L^P}[F(x+h_n),\ F(x)] \leqq \text{say } \frac{1}{2}\sqrt[P]{\left(1-\frac{1}{m_1}\right)\left(1-\frac{1}{m_2}\right)\cdots}$$

for $L \geqq$ some L_0 and $n \geqq$ some N_0, and therefore, choosing n so large that $h_n \geqq L_0$ and $n \geqq N_0$,

$$D_{S_{h_n}^P}[F(x+h_n),\ F(x)] \leqq \frac{1}{2}\sqrt[P]{\left(1-\frac{1}{m_1}\right)\left(1-\frac{1}{m_2}\right)\cdots}$$

which contradicts the relation (2).

More generally, however, we have to show that in the whole W-point around $F(x)$ there lies no W^P-a. p. function, i. e. that a function $G(x) = F(x)+J(x)$ where $J(x)$ is a W-zero function can never be a W^P-a. p. function. Assuming, indirectly, that $G(x) = F(x) + J(x)$ be a W^P-a. p. function, the function $J(x)$, being a W-zero function and a W^P-function, would (on account of Theorem 2, Chapter III) be a W^p-zero function for $p<P$. It might be supposed that, in a similar way as above, we could arrive at a contradiction by considering, for sufficiently large n, only the »first« of the central intervals, $\alpha_0 \leqq x < \alpha_0 + h_n$, and by concluding from the fact that

$$\frac{1}{h_n}\int_{\alpha_0}^{\alpha_0+h_n}(F(x)-F(x+h_n))^P\,dx$$

is »large« (i. e. not vanishing) that

$$\frac{1}{h_n}\int_{\alpha_0}^{\alpha_0+h_n}(F(x)-F(x+h_n))^p\,dx$$

for $p<P$ and near to P would also be »large«. This would namely involve, as $D_{S_{h_n}^P}[G(x+h_n),\ G(x)]$ and therefore $D_{S_{h_n}^p}[G(x+h_n),\ G(x)]$ for $p<P$ should be

»small«, that $D_{S^p_{h_n}}[J(x+h_n), J(x)]$ and hence $2\,D_{S^p_{h_n}}[J(x)]$ would be »large«, in contradiction to $J(x)$ being a W^p-zero function for every $p < P$. However, this attempt of argumentation fails for the following reason: The larger n is chosen, the nearer to P we have to choose the number p, so that we must operate with a variable p, while on the other hand the carrying out of the idea indicated would claim a fixed $p<P$. In the real proof we are forced to consider all the central intervals $\alpha_\nu \leqq x < \alpha_\nu + h_n$ (for sufficiently large n), and not only the first one; this means a certain complication, as $F(x)$ is not equal to $F_{n+1}(x)$ in all the intervals $\alpha_\nu \leqq x < \alpha_\nu + 2h_n$ (as in the first one $\alpha_0 \leqq x < \alpha_0 + 2h_n$). But except that, the reasoning is still the same as in the above attempt. Besides, in this way it is as easy to prove *that even in the B-point and not only in the W-point around $F(x)$ there is no W^p-a. p. function*, and this we are therefore going to do.

Denoting, as before, the central one of the intervals $\mu h_n \leqq x < (\mu+1)h_n$ in $\nu h_{n+1} \leqq x < (\nu+1)h_{n+1}$ by $\alpha_\nu \leqq x < \alpha_\nu + h_n$ we can, to

$$\varepsilon = \text{say } \frac{1}{4}\left(1-\frac{1}{m_1}\right)\left(1-\frac{1}{m_2}\right)\cdots$$

and an arbitrarily fixed n, determine $P_0(n) < P$ such that the inequality

$$(3)\qquad \sqrt[p]{\frac{1}{h_n}\int_{\alpha_\nu}^{\alpha_\nu+h_n}(F_{n+1}(x)-F_{n+1}(x+h_n))^p\,dx} > \sqrt[p]{\left(1-\frac{1}{m_1}\right)\left(1-\frac{1}{m_2}\right)\cdots\left(1-\frac{1}{m_n}\right)} - \varepsilon$$

is valid for every ν and for $P_0(n) < p < P$. This results, through continuity reasons, from the relation (1) using that for $p \to P$ the p-integral of a tower of type $n+1$ tends to its P-integral $(=1)$.

The problem is to pass from $F_{n+1}(x)$ to $F(x)$, or more conveniently to $F_{n+q}(x)$ (which for a large q is identical with $F(x)$ in the large interval $-h_{n+q} \leqq x < h_{n+q}$). To this purpose, for an arbitrary $q>0$, we consider a period interval (of length h_{n+q}) of $F_{n+q}(x)$ consisting of $\frac{h_{n+q}}{h_{n+1}}$ $(= m_{n+2}m_{n+3}\ldots m_{n+q})$ period intervals of $F_{n+1}(x)$ and ask: What is the relative density $d^{(n+1)}_{n+q}$ of those of the latter intervals in which $F_{n+q}(x)$ is identical with $F_{n+1}(x)$, or exactly speaking, what is the ratio $d^{(n+1)}_{n+q}$ between the number of those of the intervals in which $F_{n+q}(x)$ is identical with $F_{n+1}(x)$ and the total number $\frac{h_{n+q}}{h_{n+1}}$ of these intervals?

Above, we have met a similar question, viz. that of determining the relative density d_n of the empty places in the function $F_n(x)$. By a similar reasoning we find that the answer to our present question is

$$d_{n+q}^{(n+1)} = \left(1 - \frac{1}{m_{n+2}}\right)\left(1 - \frac{1}{m_{n+3}}\right) \cdots \left(1 - \frac{1}{m_{n+q}}\right).$$

[It may however be observed that in order to get this expression for $d_{n+q}^{(n+1)}$ we need not carry out this similar reasoning as we can directly establish the relation $d_{n+q}^{(n+1)} = \dfrac{d_{n+q}}{d_{n+1}}$ from which (using the expression for d_i) we just get the expression for $d_{n+q}^{(n+1)}$ given above. The relation in question may be obtained in the following way: We consider the empty places of the function $F_{n+q}(x)$ in one of its period intervals. On the one hand, the number of those places is, per definition, e_{n+q}. On the other hand however, as such empty places only occur in the $d_{n+q}^{(n+1)} \cdot \dfrac{h_{n+q}}{h_{n+1}}$ intervals $\nu h_{n+1} \leqq x < (\nu + 1) h_{n+1}$ in which $F_{n+q}(x)$ is identical with $F_{n+1}(x)$, the number in question may also be expressed by $d_{n+q}^{(n+1)} \dfrac{h_{n+q}}{h_{n+1}} \cdot e_{n+1}$. Putting this last expression equal to e_{n+q} $\left(\text{and using that } \dfrac{e_i}{h_i} = d_i\right)$ we just get the relation $d_{n+q}^{(n+1)} = \dfrac{d_{n+q}}{d_{n+1}}$].

Since

$$F(x) = F_{n+q}(x) \quad \text{for} \quad -h_{n+q} \leqq x < h_{n+q}$$

the function $F_{n+1}(x)$ is identical with $F(x)$ in $2\, d_{n+q}^{(n+1)} \dfrac{h_{n+q}}{h_{n+1}}$ of the $2 \dfrac{h_{n+q}}{h_{n+1}}$ intervals $\nu h_{n+1} \leqq x < (\nu + 1) h_{n+1}$ of which the interval $-h_{n+q} \leqq x < h_{n+q}$ consists. By (3) in every one of these $2\, d_{n+q}^{(n+1)} \dfrac{h_{n+q}}{h_{n+1}}$ intervals $\nu h_{n+1} \leqq x < (\nu + 1) h_{n+1}$ there is lying an interval $\alpha_\nu \leqq x < \alpha_\nu + h_n$ such that

$$(4) \qquad \sqrt[p]{\frac{1}{h_n} \int_{\alpha_\nu}^{\alpha_\nu + h_n} (F(x) - F(x + h_n))^p \, dx} > \sqrt[p]{\left(1 - \frac{1}{m_1}\right)\left(1 - \frac{1}{m_2}\right) \cdots \left(1 - \frac{1}{m_n}\right)} - \varepsilon$$

holds for $P_0(n) < p < P$, where $P_0(n)$ is independent of q.

We shall show that this involves that no W^p-a. p. function can lie in the B-point around $F(x)$. Indirectly, we assume that such a function $G(x)$ exists. Then $F(x) - G(x) = J(x)$ is a B-zero function and a B^p-function and therefore, in consequence of Theorem 2, Chapter III, a B^p-zero function for $p < P$. Further, $F_n(x)$ being a sequence of periodic functions, with the periods h_n, which S-converges to $F(x)$ and hence B-converges to $G(x)$, the number h_n, in consequence of Theorem 2 a, Chapter I, is an S_L^P-translation number of our W^P-a. p. function $G(x)$ belonging to our

$$\varepsilon = \frac{1}{4}\left(1 - \frac{1}{m_1}\right)\left(1 - \frac{1}{m_2}\right)\cdots$$

for $L \geqq$ some L_0, $n \geqq$ some N_0, i. e.

$$D_{S_L^P}[G(x + h_n), G(x)] \leqq \varepsilon \quad \text{for} \quad n \geqq N_0, \ L \geqq L_0.$$

We choose N so large that $h_N \geqq L_0$ and $N \geqq N_0$. Then we have

$$D_{S_{h_N}^P}[G(x + h_N), G(x)] \leqq \varepsilon$$

and therefore a fortiori

$$(5) \qquad D_{S_{h_N}^p}[G(x + h_N), G(x)] \leqq \varepsilon \quad \text{for} \quad p < P.$$

Now we consider $F(x)$ in the interval $-h_{N+q} \leqq x < h_{N+q}$. In $2\, d_{N+q}^{(N+1)} \cdot \frac{h_{N+q}}{h_{N+1}}$ of the $2\,\frac{h_{N+q}}{h_{N+1}}$ intervals $\nu\, h_{N+1} \leqq x < (\nu + 1)\, h_{N+1}$ of which the interval $-h_{N+q} \leqq x < h_{N+q}$ consists, there is, as we have seen above (at (4)), an interval $\alpha_\nu \leqq x < \alpha_\nu + h_N$ such that

$$(6) \qquad \sqrt[p]{\frac{1}{h_N}\int_{\alpha_\nu}^{\alpha_\nu + h_N} (F(x) - F(x + h_N))^p\, dx} > \sqrt[p]{\left(1 - \frac{1}{m_1}\right)\left(1 - \frac{1}{m_2}\right)\cdots\left(1 - \frac{1}{m_N}\right)} - \varepsilon$$

for $P_0(N) < p < P$, where $P_0(N)$ is independent of q.

For every p satisfying $P_0(N) < p < P$ the inequalities (5) and (6) involve by help of Minkowski's inequality

$$\sqrt[p]{\frac{1}{h_N}\int\limits_{\alpha_\nu}^{\alpha_\nu+h_N} |J(x)-J(x+h_N)|^p\,dx} =$$

$$\sqrt[p]{\frac{1}{h_N}\int\limits_{\alpha_\nu}^{\alpha_\nu+h_N} |(F(x)-F(x+h_N))-(G(x)-G(x+h_N))|^p\,dx} \geqq$$

$$\sqrt[p]{\frac{1}{h_N}\int\limits_{\alpha_\nu}^{\alpha_\nu+h_N} (F(x)-F(x+h_N))^p\,dx} - \sqrt[p]{\frac{1}{h_N}\int\limits_{\alpha_\nu}^{\alpha_\nu+h_N} |G(x)-G(x+h_N)|^p\,dx} >$$

$$\sqrt[p]{\left(1-\frac{1}{m_1}\right)\left(1-\frac{1}{m_2}\right)\cdot\ \left(1-\frac{1}{m_N}\right)} - 2\varepsilon >$$

$$\left(1-\frac{1}{m_1}\right)\left(1-\frac{1}{m_2}\right)\cdot\ \ -2\varepsilon = \frac{1}{2}\left(1-\frac{1}{m_1}\right)\left(1-\frac{1}{m_2}\right)\cdots = k$$

where k is a constant (independent of q).

Finally, p being a fixed number, $P_0(N)<p<P$, we get (using the expression for the relative density $d_{n+q}^{(n+1)}$)

$$\frac{1}{2h_{N+q}}\int\limits_{-h_{N+q}}^{h_{N+q}} |J(x)-J(x+h_N)|^p\,dx \geqq \frac{1}{2h_{N+q}}\cdot 2\,d_{N+q}^{(N+1)}\frac{h_{N+q}}{h_{N+1}}\cdot k^p h_N =$$

$$\frac{h_N}{h_{N+1}}k^p d_{N+q}^{(N+1)} = \frac{k^p}{m_{N+1}}\left(1-\frac{1}{m_{N+2}}\right)\left(1-\frac{1}{m_{N+3}}\right)\cdots\left(1-\frac{1}{m_{N+q}}\right) >$$

$$\frac{k^p}{m_{N+1}}\left(1-\frac{1}{m_{N+2}}\right)\left(1-\frac{1}{m_{N+3}}\right)\cdots = k' > 0$$

where $k'=k'(N)$ is independent of q. Hence

$$(7)\quad D_{B^p}[J(x), J(x+h_N)] = \overline{\lim_{T\to\infty}} \sqrt[p]{\frac{1}{2T}\int\limits_{-T}^{T} |J(x)-J(x+h_N)|^p\,dx} \geqq \sqrt[p]{k'} > 0.$$

On the other hand, $J(x)$ being a B^p-zero function, we have $D_{B^p}[J(x), J(x+h_N)] \leqq 2\,D_{B^p}[J(x)] = 0$, i. e.

$$D_{B^p}[J(x), J(x+h_N)] = 0,$$

in contradiction to (7).

§ 6.

Through Functions.

Already in § 3 we used the following

Theorem. *Let $\mathfrak{A}$ be a W-point with the lifetime P, $1 < P \leqq \infty$, as to the W^p-spaces which is dead at the time P (if $P < \infty$). Then there exists a through function $f^*(x)$ in $\mathfrak{A}$ as to the W^p-spaces, i. e. a function which is a W^p-function for every $p < P$.*

We shall now prove this theorem. Roughly speaking, the method is as follows: In our W-point we choose functions $f_1(x)$, $f_2(x)$, ... whose lifetimes approach P more and more. Starting from these functions we shall arrive at a through function. The first idea might perhaps be to consider the function which is equal to $f_1(x)$ say for $|x| < 1$, to $f_2(x)$ for $1 \leqq |x| < 2$, to $f_3(x)$ for $2 \leqq |x| < 3$ etc. This function, however, is far from being a through function; it needs not even to be p-integrable for larger p than is $f_1(x)$. However, it proves possible to modify the functions $f_1(x)$, $f_2(x)$, ... in such a way that, applying the above method on the modified functions $f_1^*(x)$, $f_2^*(x)$, ..., we really get a through function $f^*(x)$. By our modification of the functions $f_1(x)$, $f_2(x)$, ... we wish to obtain, firstly, that the modified functions $f_1^*(x)$, $f_2^*(x)$, ... be p-integrable for all $p < P$, and secondly, that each function of the modified sequence $f_1^*(x)$, $f_2^*(x)$, ... differs »so little« from its successor that the »composed« function $f^*(x)$ differs »little« from each of the functions $f_1^*(x)$, $f_2^*(x)$, ... in the following sense: $f^*(x)$ is a W^p-function »almost as far« as each of these functions $f_n^*(x)$ (and is therefore a W^p-function for all $p < P$), and like these functions it belongs to the W-point.

Ve begin with the two following remarks.

Remark 1. *Let $f(x)$ be a 1-integrable function. Then we can always by adding a W-zero function obtain a function $g(x)$ which is p-integrable for every p, and such that $|g(x)| \leqq |f(x)|$ for all x.*

This may be done in the following way. For $n = 0, \pm 1, \pm 2, \ldots$ we determine N_n so large that

$$\int_n^{n+1} |f(x) - (f(x))_{N_n}| \, dx < \frac{1}{|n| + 1},$$

$(f(x))_{N_n}$ denoting, as usually, the function cut off at N_n, and define

$$g(x) = (f(x))_{N_n} \quad \text{for} \quad n \leqq x < n+1 \quad (n = 0, \pm 1, \pm 2, \ldots).$$

Then we have

$$D_W[f(x), g(x)] = 0,$$

since the mean value of $|f(x) - g(x)|$ over a sufficiently large interval is arbitrarily small wherever on the x-axis the integration starts. Further, the function $g(x)$, being bounded in every interval $n \leqq x < n+1$, is p-integrable for every p. Finally $|g(x)| \leqq |f(x)|$ since $|(f(x))_{N_n}| \leqq |f(x)|$.

Remark 2. As easily seen, for an S^p-function it does not always hold (as in the case of a p-integrable periodic function) that $D_{S^p}[f(x), (f(x))_N] \to 0$ for $N \to \infty$. *But, if $1 \leqq p_1 < p_2$ and if $f(x)$ is an S^{p_2}-function, it does hold that*

$$D_{S^{p_1}}[f(x), (f(x))_N] \to 0 \quad \text{for} \quad N \to \infty,$$

i. e. to a given $\varepsilon > 0$ the inequality

$$\sqrt[p_1]{\int_x^{x+1} |f(t) - (f(t))_N|^{p_1}\, dt} < \varepsilon$$

is valid for all x provided N is chosen sufficiently large.

This may be seen in the following way. We have for all t

$$|f(t) - (f(t))_N|^{p_1} \leqq \frac{1}{N^{p_2 - p_1}} \cdot |f(t)|^{p_2};$$

in fact, the inequality is obvious for those t for which $|f(t)| \leqq N$ as the left-hand side is 0, and for those t for which $|f(t)| \geqq N$ we have

$$|f(t) - (f(t))_N|^{p_1} \leqq |f(t)|^{p_1} \leqq \frac{1}{N^{p_2 - p_1}} \cdot |f(t)|^{p_2}.$$

Hence for all x

$$\int_x^{x+1} |f(t) - (f(t))_N|^{p_1}\, dt \leqq \frac{1}{N^{p_2 - p_1}} \int_x^{x+1} |f(t)|^{p_2}\, dt \leqq \frac{1}{N^{p_2 - p_1}} (D_{S^{p_2}}[f(x)])^{p_2}$$

which tends to 0 for $N \to \infty$.

Now, we pass to the proof of the theorem. Let $1 < p_1 < p_2 < \cdots \to P$. Then in the W-point $\mathfrak{A}$ there exists a W^{p_1}-function $f_1(x)$, a W^{p_2}-function $f_2(x), \ldots$, and,

in consequence of Remark 1, we may assume that all the functions $f_n(x)$ are p-integrable for every p. Let

$$f_2(x) - f_1(x) = j_1(x)$$
$$f_3(x) - f_2(x) = j_2(x)$$
$$f_4(x) - f_3(x) = j_3(x)$$
$$\vdots$$

Then $j_1(x), j_2(x), \ldots$ are all W-zero functions, and $j_1(x)$ is a W^{p_1}-function i. e. an S^{p_1}-function, $j_2(x)$ an S^{p_2}-function, ... Let further

$$j_1^*(x) = j_1(x) - (j_1(x))_{N_1}$$
$$j_2^*(x) = j_2(x) - (j_2(x))_{N_2}$$
$$j_3^*(x) = j_3(x) - (j_3(x))_{N_3}$$
$$\vdots$$

where $N_1, N_2, \ldots$ will be chosen below. All the functions $j_1^*(x), j_2^*(x), \ldots$ are W-zero functions, since $|j_1^*(x)| \leqq |j_1(x)|$, $|j_2^*(x)| \leqq |j_2(x)|$, ... For the same reason the function $j_1^*(x)$ is an S^{p_1}-function, $j_2^*(x)$ an S^{p_2}-function, ... Let $\sum_1^\infty \varepsilon_n$ be a convergent series of positive terms. In consequence of Remark 2 it is possible to choose $N_1, N_2, \ldots$ such that for all x

$$\int_x^{x+1} |j_1^*(t)|\, dt < \varepsilon_1$$

$$\sqrt[p_1]{\int_x^{x+1} |j_2^*(t)|^{p_1}\, dt} < \varepsilon_2$$

$$\sqrt[p_2]{\int_x^{x+1} |j_3^*(t)|^{p_2}\, dt} < \varepsilon_3$$

$$\vdots$$

We can now indicate the »modified« functions $f_1^*(x), f_2^*(x), \ldots$. They are successively determined by

$$f_1^*(x) = f_1(x)$$
$$f_2^*(x) = f_1^*(x) + j_1^*(x)$$
$$f_3^*(x) = f_2^*(x) + j_2^*(x)$$
$$\vdots$$

It may be observed that these functions $f_1^*(x), f_2^*(x), \ldots$ are constructed starting from $f_1(x)$ by help of the functions $j_1^*(x), j_2^*(x), \ldots$ in exactly the same way as $f_1(x), f_2(x), \ldots$ may be obtained starting from $f_1(x)$ by help of the functions $j_1(x), j_2(x), \ldots$. We find

$$\begin{aligned} & f_1^*(x) = f_1(x) \\ (1) \qquad & f_2^*(x) = f_2(x) - (j_1(x))_{N_1} \\ & f_3^*(x) = f_3(x) - (j_1(x))_{N_1} - (j_2(x))_{N_2} \\ & \vdots \end{aligned}$$

All the functions $j_1^*(x), j_2^*(x), \ldots$ being W-zero functions, the functions $f_1^*(x)$, $f_2^*(x), \ldots$ will (on account of their definitions) belong to $\mathfrak{A}$. Further, all the functions $(j_1(x))_{N_1}$, $(j_2(x))_{N_2}, \ldots$ being bounded, it results from the equations (1) that $f_1^*(x)$ is a W^{p_1}-function, $f_2^*(x)$ is a W^{p_2}-function, ..., and that they are all p-integrable for every p. From the way in which $N_1, N_2, \ldots$ are determined, we conclude that for all x and all $n = 1, 2, \ldots$ (putting $p_0 = 1$)

$$(2) \qquad \sqrt[p_{n-1}]{\int_x^{x+1} |f_m^*(t) - f_n^*(t)|^{p_{n-1}}\, dt} < \varepsilon_n + \varepsilon_{n+1} + \cdots \quad \text{for} \quad m > n,$$

since

$$\sqrt[p_{n-1}]{\int_x^{x+1} |f_m^*(t) - f_n^*(t)|^{p_{n-1}}\, dt} \leqq$$

$$\sqrt[p_{n-1}]{\int_x^{x+1} |f_{n+1}^*(t) - f_n^*(t)|^{p_{n-1}}\, dt} + \sqrt[p_{n-1}]{\int_x^{x+1} |f_{n+2}^*(t) - f_{n+1}^*(t)|^{p_{n-1}}\, dt} + \cdots +$$

$$+ \sqrt[p_{n-1}]{\int_x^{x+1} |f_m^*(t) - f_{m-1}^*(t)|^{p_{n-1}}\, dt} =$$

$$\sqrt[p_{n-1}]{\int_x^{x+1} |j_n^*(t)|^{p_{n-1}}\, dt} + \sqrt[p_{n-1}]{\int_x^{x+1} |j_{n+1}^*(t)|^{p_{n-1}}\, dt} + \cdots + \sqrt[p_{n-1}]{\int_x^{x+1} |j_{m-1}^*(t)|^{p_{n-1}}\, dt} \leqq$$

$$\sqrt[p_{n-1}]{\int_x^{x+1} |j_n^*(t)|^{p_{n-1}}\, dt} + \sqrt[p_n]{\int_x^{x+1} |j_{n+1}^*(t)|^{p_n}\, dt} + \cdots + \sqrt[p_{m-2}]{\int_x^{x+1} |j_{m-1}^*(t)|^{p_{m-2}}\, dt} <$$

$$\varepsilon_n + \varepsilon_{n+1} + \cdots + \varepsilon_{m-1} < \varepsilon_n + \varepsilon_{n+1} + \cdots.$$

Finally we define $f^*(x)$ by (see Fig. 8)

$$f^*(x) = f^*_{n+1}(x)$$

for

$$n \leqq |x| < n + 1 \qquad (n = 0, 1, 2, \ldots).$$

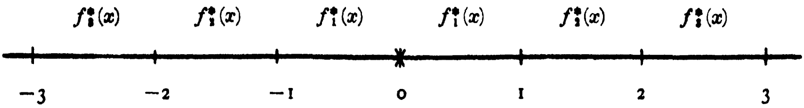


Fig. 8.

The function $f^*(x)$ is p-integrable for each p, all $f^*_n(x)$ being p-integrable. We consider the difference $f^*(x) - f^*_n(x)$ for an arbitrarily fixed n, and shall estimate

$$D_{W^{p_{n-1}}}[f^*(x), f^*_n(x)].$$

The inequality (2) tells us, how much $f^*(x)$ differs from $f^*_n(x)$ for all x outside the finite interval $-n \leqq x < n$. Since for the determination of $D_{W^{p_{n-1}}}$ the values of the function in an arbitrary finite interval are irrelevant if only the function is p_{n-1}-integrable in this interval, we get from (2)

$$D_{W^{p_{n-1}}}[f^*(x), f^*_n(x)] \leqq \varepsilon_n + \varepsilon_{n+1} + \cdots \tag{3}$$

which tends to 0 for $n \to \infty$. From (3) it results in particular that

$$D_W[f^*(x), f^*_n(x)] \to 0 \quad \text{for} \quad n \to \infty;$$

hence $f^*(x)$ belongs to $\mathfrak{A}$, a G-point considered as a set of functions being G-closed. Further we get from (3) that

$$D_{W^{p_{n-1}}}[f^*(x)] \leqq D_{W^{p_{n-1}}}[f^*_n(x)] + \varepsilon_n + \varepsilon_{n+1} + \cdots,$$

so that $f^*(x)$ is a $W^{p_{n-1}}$-function for every $n = 1, 2, \ldots$ and therefore a W^p-function for $p < P$.

CHAPTER VI.

The Mutual Relations of the B^p- and the B^p-a. p. Spaces.

§ 1.

Introduction.

In this Chapter we shall consider an arbitrary B-point as to the B^p-spaces and the B^p-a. p. spaces. We proceed in quite the same way as by the corresponding investigation in Chapter V of the behaviour of a W-point as to the W^p- and the W^p-a. p. spaces. On the one side we investigate what B^p-points belong to our B-point, and on the other side we consider the single functions in the B-point. Both the B^p-points and the functions are characterised by means of the B^p- and the B^p-a. p. spaces. In many respects also the results of our investigations will prove to be analogous to those of Chapter V. The results of Chapter V, § 2 on W-zero functions may even be transferred verbally to the B-zero functions, for by a retrospective glance we see immediately that we may replace »W« by »B« everywhere in the text without changing the examples. Also the general investigation in Chapter V, § 3 of the W^p-points in a given W-point can be transferred word for word; here too we may replace »W« by »B« everywhere. Whether the investigation of the single functions in Chapter V, § 3 can also be transferred, obviously depends on the question whether (analogously to the W-case) in a B-point there is always a through function as to the B^p-spaces and the B^p-a. p. spaces, i. e. a function which is a B^p-function for those p for which the B-point contains B^p-functions, and a B^p-a. p. function for those p for which the B-point contains B^p-a. p. functions. As we shall see, such a general theorem is really valid. Evidently, to establish this theorem it is, just as in the W-case, sufficient to prove that every B-point with the lifetime P which (if $P < \infty$) is dead at the time P contains a through function as to the B^p-spaces. By means of this theorem the investigation of the single functions in a given W-point can be transferred word for word to the given B-point. The proof of the theorem on the existence of a through function, however, is not analogous to that in the W-case, and it will be postponed to § 6.

But there is an interesting difference between the W-a. p. points and the B-a. p. points. In the W-case we gave an example of a W-a. p. point which is

alive at the time P as to the W^p-spaces but is dead as to the W^p-a. p. spaces (Main example 3, Chapter V, § 5); in the B-case, however, such an example does not exist, the theorem being valid that a B-a. p. point which is alive at the time P as to the B^p-spaces is also alive at the time P as to the B^p-a. p. spaces. Hence, if a B-a. p. point with the lifetime P possesses P-descendants at all, one (and of course only one) of them will always be a B^P-a. p. point.

As to the B-zero functions, we simply refer to the treatment in Chapter V, § 2 of the W-zero functions where, as mentioned above, the letter »W« may right away be changed to »B«. From systematical reasons, however, we shall (in spite of the complete analogy with the W-case) in § 2 give a brief account of the behaviour of the B^p-points and the functions in a given B-point as to the B^p- and the B^p-a. p. spaces. In § 3 we give the proof of the theorem on B-a. p. points indicated above. Next, in § 4 and § 5 we state all the possibilities for a B-point which is not a B-a. p. point, respectively is a B-a. p. point. Finally in § 6 we prove the theorem on the through function.

§ 2.

General Remarks on B-Points.

Already in § 1 we spoke about a lifetime P of a B-point as to the B^p-spaces and eventually the B^p-a. p. spaces, and used the fact *that a B-a. p. point has the same lifetime as to these spaces.* As in the W-case, we say that we know the behaviour of a B-point at the time p_1 as to the B^p-spaces and (eventually) the B^p-a. p. spaces, if we know whether the point is alive or dead at the time p_1 as to the B^p-spaces and the B^p-a. p. spaces. Further we speak of the p-descendants of our B-point (or of the »components« of our B-rocket) and for every p, $1<p<P$, of a p-generator (p-nucleus). *The p-generator is the only one of the p-descendants which has descendants itself,* all the other p-descendants (the still-born brothers, or p-sparks) dying at the time p in the moment they are born. *If the B-point is B-a. p., the p-generator is B^p-a. p.* As to the general situation we may refer to the Fig. 5 (with »W« replaced by »B«).

We now pass to the single functions in a B-point. We speak about a function being alive or dead as to the B^p-spaces and the B^p-a. p. spaces at a definite date, and we speak of its lifetime P_1. *If the B-point is B-a. p., a function in this point has the same lifetime as to the B^p- and the B^p-a. p. spaces.*

If the B-point (with the lifetime P) is not B-a. p., there are the following possibilities for a function $f(x)$ with the lifetime P_1 contained in the B-point. *The lifetime P_1 may be an arbitrary number*, $1 \leqq P_1 \leqq P$, and for each fixed P_1 there are, if $P_1 < \infty$, the *two possibilities:*

1. $f(x)$ is dead as to the B^p-spaces at the time P_1,
2. $f(x)$ is alive as to the B^p-spaces at the time P_1,

with exception, however, of the case $P_1 = 1$ where of course only 2. can occur, and the case $P_1 = P$ where 2. can only occur if the B-point is alive as to the B^p-spaces at the time P.

If the B-point is B-a. p., *the lifetime P_1 of a function in the point may be an arbitrary number in the interval* $1 \leqq P_1 \leqq P$, and for each fixed P_1 there are, if $P_1 < \infty$, the *three possibilities:*

1. $f(x)$ is dead as to the B^p-spaces at the time P_1,
2. $f(x)$ is alive as to the B^p-spaces, but dead as to the B^p-a. p. spaces at the time P_1,
3. $f(x)$ is alive as to the B^p-a. p. spaces at the time P_1,

with exception, however, of the case $P_1 = 1$ where of course only 3. can occur, and the case $P_1 = P$ where 3. can only occur if the B-a. p. point is alive as to the B^p-a. p. spaces at the time P, and 2. can only occur if the B-a. p. point is alive as to the B^p-spaces at the time P.

§ 3.

A Theorem on the Behaviour of B-a. p. Points in their Moments of Death.

In this paragraph we prove the following theorem concerning the B-a. p. points which has no analogue in the W-case.

Theorem. *A B-a. p. point which contains a B^p-function $f(x)$ contains also a B^p-a. p. function $g(x)$.*

The general proof of this theorem uses the notion of asymptotic distribution function of a real B-a. p. function. In the special case $P = 2$, however, the theorem can be proved in another and more simple way, namely by help of Besicovitch's Theorem on Fourier series of B^2-a. p. functions. We shall begin by giving this proof which is only applicable in the case $P = 2$.

The special case $P=2$. Let our B-a. p. and B^2-function $f(x)$ have the Fourier series $\Sigma A_n e^{i\Lambda_n x}$. We first show that

$$\Sigma |A_n|^2 \leqq (D_{B^2}[f(x)])^2. \tag{1}$$

Let

$$\sigma_q(x) = \sum_{n=1}^{N(q)} k_n^{(q)} A_n e^{i\Lambda_n x}$$

be a Bochner-Fejér sequence of $f(x)$. Then

$$\sum_{n=1}^{N(q)} \{k_n^{(q)}\}^2 |A_n|^2 = M\{|\sigma_q(x)|^2\} = (D_{B^2}[\sigma_q(x)])^2$$

Hence on account of the inequality (Chapter I)

$$D_{B^2}[\sigma_q(x)] \leqq D_{B^2}[f(x)]$$

we get

$$\sum_{n=1}^{N(q)} \{k_n^{(q)}\}^2 |A_n|^2 \leqq (D_{B^2}[f(x)])^2.$$

As $0 \leqq k_n^{(q)} \leqq 1$, and $k_n^{(q)} \to 1$ for fixed n and $q \to \infty$, we immediately get for $q \to \infty$ the desired inequality (1). In particular $\Sigma |A_n|^2$ is convergent and thus, in consequence of Besicovitch's Theorem, $\Sigma A_n e^{i\Lambda_n x}$ is the Fourier series of a B^2-a. p. function $g(x)$. As the two functions $g(x)$ and $f(x)$ (considered as B-a. p. functions) have the same Fourier series, the function $g(x)$ lies in our B-a. p. point around $f(x)$.

As mentioned above the proof of the theorem in the general case uses the notion of asymptotic distribution function of a real B-a. p. function. The asymptotic distribution functions for the different types of almost periodic functions are dealt with by Jessen and Wintner in their paper: Distribution Functions and the Riemann Zeta Function, Trans. of the Amer. Math. Soc., vol. 38. We shall only apply a single theorem of this paper, and as we shall not assume the knowledge of the paper we shall not merely state the theorem but also give a direct proof of it (communicated to us by Jessen).

To begin with we remind of two well known and elementary facts concerning real monotonic functions (in the wide sense) defined on the whole axis.

1°. A monotonic function has at most an enumerable number of discontinuity points.

2°. Let $\psi(\alpha)$ and $\psi_1(\alpha)$ be increasing functions with the following two properties:

$$\psi_1(\alpha) \leqq \psi(\alpha) \text{ for all } \alpha, \quad \text{and} \quad \psi_1(\beta) \geqq \psi(\alpha) \text{ for } \beta > \alpha.$$

Then $\psi(\alpha)$ and $\psi_1(\alpha)$ have the same discontinuity points, and $\psi_1(\alpha) = \psi(\alpha)$ in all the continuity points. (For, if α is a continuity point of $\psi_1(\alpha)$, it results from $\psi_1(\beta) \geqq \psi(\alpha)$ for $\beta > \alpha$ that $\psi_1(\alpha) = \psi_1(\alpha+) \geqq \psi(\alpha)$ which together with $\psi_1(\alpha) \leqq \psi(\alpha)$ gives $\psi_1(\alpha) = \psi(\alpha)$).

We say that a real function $f(x)$ defined on the whole x-axis has an asymptotic distribution function, if there exists an increasing function $\psi(\alpha)$ (in the wide sense) defined on the whole α-axis so that:

1) In a continuity point α of $\psi(\alpha)$ the two »relative measures«

$$m_{\text{rel}}\{[f(x) \leqq \alpha]\} = \lim_{T\to\infty} \frac{1}{2T} m\{[f(x) \leqq \alpha] \times [-T \leqq x \leqq T]\}$$

and

$$m_{\text{rel}}\{[f(x) < \alpha]\} = \lim_{T\to\infty} \frac{1}{2T} m\{[f(x) < \alpha] \times [-T \leqq x \leqq T]\}$$

both exist and are equal to $\psi(\alpha)$ (then obviously $0 \leqq \psi(\alpha) \leqq 1$).

2) $$\psi(\alpha) \to 1 \text{ for } \alpha \to \infty, \text{ and } \psi(\alpha) \to 0 \text{ for } \alpha \to -\infty.$$

By the distribution function of $f(x)$ we then understand the function $\psi(\alpha)$ in its continuity points.

We can now state the theorem of Jessen and Wintner:

Auxiliary theorem. *Every real B-a. p. function $f(x)$ possesses an asymptotic distribution function.*

Proof. Let

$$\psi(\alpha) = m_{\text{rel}}\{[f(x) \leqq \alpha]\} = \overline{\lim_{T\to\infty}} \frac{1}{2T} m\{[f(x) \leqq \alpha] \times [-T \leqq x \leqq T]\}.$$

Obviously the function $\psi(\alpha)$ is an increasing function of α (in the wide sense) defined on the whole α-axis. We shall show that the two relative measures $m_{\text{rel}}\{[f(x) \leqq \alpha]\}$ and $m_{\text{rel}}\{[f(x) < \alpha]\}$ exist in every continuity point of $\psi(\alpha)$ and are both equal to $\psi(\alpha)$, and that $\psi(\alpha) \to 1$ for $\alpha \to \infty$ and $\psi(\alpha) \to 0$ for $\alpha \to -\infty$. Then, according to our definition, the function $\psi(\alpha)$ considered in its continuity points is an asymptotic distribution function of $f(x)$.

Together with $\psi(\alpha)$ we consider the other increasing function

$$\psi_1(\alpha) = \underline{m}_{\mathrm{rel}}\{[f(x) < \alpha]\} = \lim_{\overline{T \to \infty}} \frac{1}{2T} m\{[f(x) < \alpha] \times [-T \leqq x \leqq T]\}.$$

First we shall show by help of 2° that $\psi(\alpha)$ and $\psi_1(\alpha)$ have the same discontinuity points and are equal in their continuity points. Obviously $\psi_1(\alpha) \leqq \psi(\alpha)$; hence it is sufficient to show that $\psi_1(\beta) \geqq \psi(\alpha)$ for $\beta > \alpha$. In order to do that we introduce the auxiliary function (see Fig. 9):

$$\Phi(z) = \begin{cases} 1 & \text{for } z \leqq \alpha \\ \dfrac{\beta - z}{\beta - \alpha} & \text{for } \alpha \leqq z \leqq \beta \\ 0 & \text{for } z \geqq \beta. \end{cases}$$

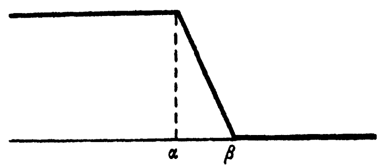


Fig. 9.

This continuous function $\Phi(z)$ (which for β »near to« α differs unessentially from the function which is 1 for $z \leqq \alpha$ and 0 for $z > \alpha$) has a bounded difference quotient. Hence $\Phi(f(x))$ is a B-a. p. function (Chapter I). In particular, what is of decisive importance in the proof, $\Phi(f(x))$ has a mean value $M\{\Phi(f(x))\}$. As $\Phi(f(x)) \leqq 1$ for $f(x) < \beta$ and $\Phi(f(x)) = 0$ for $f(x) \geqq \beta$, we have

$$\psi_1(\beta) = \underline{m}_{\mathrm{rel}}\{[f(x) < \beta]\} \geqq M\{\Phi(f(x))\},$$

and as $\Phi(f(x)) = 1$ for $f(x) \leqq \alpha$ and $\Phi(f(x)) \geqq 0$ for $f(x) > \alpha$, we have

$$\psi(\alpha) = \overline{m}_{\mathrm{rel}}\{[f(x) \leqq \alpha]\} \leqq M\{\Phi(f(x))\}.$$

From the two latter inequalities the desired inequality $\psi_1(\beta) \geqq \psi(\alpha)$ results.

Further, in an arbitrary one of the (common) continuity points for $\psi(\alpha)$ and $\psi_1(\alpha)$ we have

$$\psi_1(\alpha) = \underline{m}_{\mathrm{rel}}\{[f(x) < \alpha]\} \leqq \left\{ \begin{matrix} \underline{m}_{\mathrm{rel}}\{[f(x) \leqq \alpha]\} \\ \overline{m}_{\mathrm{rel}}\{[f(x) < \alpha]\} \end{matrix} \right\} \leqq \overline{m}_{\mathrm{rel}}\{[f(x) \leqq \alpha]\} = \psi(\alpha),$$

and as the first and the last term in this chain of inequalities are equal, all the terms must be equal. Consequently $m_{\mathrm{rel}}\{[f(x) \leqq \alpha]\}$ and $m_{\mathrm{rel}}\{[f(x) < \alpha]\}$ both exist and are equal to $\psi(\alpha)$.

It remains to prove that

$$\psi(\alpha) \to 1 \text{ for } \alpha \to \infty \quad \text{and} \quad \psi(\alpha) \to 0 \text{ for } \alpha \to -\infty.$$

We begin by proving the first of these limit relations. As $\psi(\alpha)$ is increasing and $\leqq 1$, the limit $\lim\limits_{\alpha \to \infty} \psi(\alpha)$ exists and is $\leqq 1$. Proceeding indirectly we assume that $\lim\limits_{\alpha \to \infty} \psi(\alpha) = g < 1$ and hence $\psi(\alpha) \leqq g < 1$ for every α. In the following we may let α avoid the discontinuity points of $\psi(\alpha)$. Obviously

$$m_{\text{rel}}\{[f(x) \geqq \alpha]\} = 1 - \psi(\alpha).$$

From our assumption it would follow that for every α

$$m_{\text{rel}}\{[f(x) \geqq \alpha]\} \geqq 1 - g > 0,$$

and hence for arbitrary large α (indeed for every $\alpha > 0$)

$$D_B[f(x)] = \overline{\lim_{T \to \infty}} \frac{1}{2T} \int_{-T}^{T} |f(x)|\, dx \geqq \underline{\lim_{T \to \infty}} \frac{1}{2T} \int\limits_{[f(x) \geqq \alpha] \times [-T \leqq x \leqq T]} f(x)\, dx \geqq$$

$$\lim_{T \to \infty} \alpha \cdot \frac{1}{2T} m\{[f(x) \geqq \alpha] \times [-T \leqq x \leqq T]\} = \alpha\, m_{\text{rel}}\{[f(x) \geqq \alpha]\} \geqq \alpha(1 - g)$$

in contradiction to $D_B[f(x)]$ being finite. The other limit relation $\psi(\alpha) \to 0$ for $\alpha \to -\infty$ follows immediately from the first limit relation $\psi(\alpha) \to 1$ for $\alpha \to \infty$ by applying the latter to the function $-f(x)$ and using that

$$m_{\text{rel}}\{[f(x) < \alpha]\} = 1 - m_{\text{rel}}\{[f(x) \geqq \alpha]\} = 1 - m_{\text{rel}}\{[-f(x) \leqq -\alpha]\}.$$

Having proved the auxiliary theorem, we now pass to the proof of our theorem in the general case i. e. for an arbitrary $P > 1$. This proof may be formulated in the shortest way by help of STIELTJES' integrals, but not having to use STIELTJES' integrals elsewhere in our paper we prefer to accomplish the proof in a more elementary manner.

Let $f(x)$ be the B-a. p. and B^p-function of the theorem. Then $|f(x)|$ is a real B-a. p. function and hence possesses an asymptotic distribution function $\psi(\alpha)$. For the sake of convenience we will assume that no point of the, at most enumerable, set of discontinuity points of $\psi(\alpha)$ is a positive integer; otherwise we might consider the function $kf(x)$, instead of $f(x)$, where k is a suitably chosen positive constant. (If $\psi(\alpha)$ has the discontinuity points d_n, the function $|kf(x)|$ has the distribution function $\psi\left(\frac{\alpha}{k}\right)$ with the discontinuity points $k d_n$, and disposing of k in a suitable way we can of course provide for none of these latter numbers being a positive integer).

For $n = 1, 2, \ldots$ we put

$$\mu_n = m_{\mathrm{rel}}\{[n \leqq |f(x)| < n + 1]\};$$

then

$$\mu_n = \psi(n + 1) - \psi(n).$$

We begin with two remarks which easily result from the fact that $|f(x)|$ has the distribution function $\psi(\alpha)$.

1°. It is evident that

$$m_{\mathrm{rel}}\{[n \leqq |f(x)| \leqq \infty]\} = \mu_n + \mu_{n+1} + \cdots;$$

for on the one hand

$$m_{\mathrm{rel}}\{[n \leqq |f(x)| \leqq \infty]\} = 1 - \psi(n)$$

and on the other hand

$$\mu_n + \mu_{n+1} + \cdots = (\psi(n+1) - \psi(n)) + (\psi(n+2) - \psi(n+1)) + \cdots = \lim_{\nu \to \infty} \psi(\nu) - \psi(n) = 1 - \psi(n).$$

2°. Further, *the series*

$$\mu_1 \cdot 1^p + \mu_2 \cdot 2^p + \cdots + \mu_n \cdot n^p + \cdots$$

is convergent with a sum $\leqq (D_{B^p}[f(x)])^p$, in other words, the inequality

$$\mu_1 \cdot 1^p + \mu_2 \cdot 2^p + \cdots + \mu_n \cdot n^p \leqq (D_{B^p}[f(x)])^p$$

holds for an arbitrary fixed n. In order to prove this latter inequality we estimate $(D_{B^p}[f(x)])^p$ from below in the following way: Taking only those x for which $1 \leqq |f(x)| < n + 1$ into consideration, we get

$$(D_{B^p}[f(x)])^p = \overline{\lim_{T\to\infty}} \frac{1}{2T}\int_{-T}^{T} |f(x)|^p\,dx \geqq \overline{\lim_{T\to\infty}} \frac{1}{2T} \int_{[-T \leqq x \leqq T] \times [1 \leqq |f(x)| < n+1]} |f(x)|^p\,dx.$$

Therefore we consider, for a fixed T, the integral

$$\frac{1}{2T} \int_{[-T \leqq x \leqq T] \times [1 \leqq |f(x)| < n+1]} |f(x)|^p\,dx.$$

We divide the range of integration $[-T \leqq x \leqq T] \times [1 \leqq |f(x)| < n + 1]$ into the n subsets $[-T \leqq x \leqq T] \times [\nu \leqq |f(x)| < \nu + 1]$ $(\nu = 1, 2, \ldots n)$, and correspondingly the integral into the n integrals

$$\frac{1}{2T} \int_{[-T \leqq x \leqq T] \times [\nu \leqq |f(x)| < \nu+1]} |f(x)|^p\,dx \qquad (\nu = 1, 2, \ldots n).$$

For each of these integral we have

$$\frac{1}{2T}\int\limits_{[-T\leq x\leq T]\times[\nu\leq|f(x)|<\nu+1]}|f(x)|^p\,dx \quad \geqq \nu^p\cdot m\{[-T\leqq x\leqq T]\times[\nu\leqq|f(x)|<\nu+1]\},$$

where the left-hand side tends to $\nu^p\cdot\mu_\nu$ for $T\to\infty$. Thus we get

$$\overline{\lim_{T\to\infty}}\,\frac{1}{2T}\int\limits_{[-T\leq x\leq T]\times[1\leq|f(x)|<n+1]}|f(x)|^p\,dx \quad \geqq \mu_1\cdot 1^p+\mu_2\cdot 2^p+\cdots+\mu_n\cdot n^p$$

and hence the wanted inequality

$$\mu_1\cdot 1^p+\mu_2\cdot 2^p+\cdots+\mu_n\cdot n^p\leqq(D_{B^p}[f(x)])^p.$$

Now we pass to the proper proof. The salient point is to demonstrate that the sequence $(f(x))_n$ is a B^p-fundamental sequence. To this purpose we have to estimate

$$\left(D_{B^p}[(f(x))_n,\ (f(x))_m]\right)^p=\overline{\lim_{T\to\infty}}\,\frac{1}{2T}\int_{-T}^{T}|(f(x))_m-(f(x))_n|^p\,dx$$

for $n<m$. For those x for which $|f(x)|<n$ we have $(f(x))_m-(f(x))_n=0$, for those x for which $\nu\leqq|f(x)|<\nu+1$ $(\nu=n,n+1,\ldots,m-1)$ we have $|(f(x))_m-(f(x))_n|<\nu+1-n$, and for those x for which $|f(x)|\geqq m$ we have $|(f(x))_m-(f(x))_n|=m-n$. Thus we get

$$\left(D_{B^p}[(f(x))_n,\ (f(x))_m]\right)^p=\overline{\lim_{T\to\infty}}\,\frac{1}{2T}\int_{-T}^{T}|(f(x))_m-(f(x))_n|^p\,dx\leqq$$

$$\sum_{\nu=n}^{m-1}\mu_\nu(\nu+1-n)^p+(m-n)^p\,m_{\mathrm{rel}}\{[m\leqq|f(x)|\leqq\infty]\}=$$

$$\sum_{\nu=n}^{m-1}\mu_\nu(\nu+1-n)^p+(m-n)^p\sum_{\nu=m}^{\infty}\mu_\nu.$$

Enlarging $\nu+1-n$ to ν in the first sum, and $(m-n)^p\mu_\nu$ to $\nu^p\mu_\nu$ in the last term, we get

$$\left(D_{B^p}[(f(x))_n,\ (f(x))_m]\right)^p\leqq\sum_{\nu=n}^{\infty}\mu_\nu\cdot\nu^p$$

where the right-hand side is independent of m and tends to 0 for $n\to\infty$ since, according to 2°, the series $\Sigma\mu_\nu\cdot\nu^p$ is convergent. Consequently $(f(x))_n$ is a B^p-fundamental sequence.

As $f(x)$ is B-a. p., the function $(f(x))_n$ is also B-a. p. (Chapter I) and, being bounded, it is therefore B^p-a. p. for all p, in particular it is B^P-a. p. Hence the sequence $(f(x))_n$ is a B^P-fundamental sequence of B^P-a. p. functions. The B^P-a. p. space being complete, the sequence $(f(x))_n$ thus B^P-converges to a B^P-a. p. function $g(x)$. This function $g(x)$ must lie in our B-a. p. point around $f(x)$ as the sequence $(f(x))_n$ B-converges to $g(x)$ and B-converges to $f(x)$, the latter because $f(x)$ is B-a. p. (Chapter I).

We observe that the »reason« why no corresponding theorem holds for the W-a. p. points is the incompleteness of the W^p-a. p. spaces; for as regards the distribution functions a wholly analogous notion exists for W-a. p. functions, only a relative measure in the W-sense being used instead of a relative measure in the B-sense. In the S-case we have completeness of the S^p-a. p. spaces but the notion asymptotic distribution function has no meaning in the S-case (and as we have seen in Chapter IV a function $f(x)$ may very well be an S-a. p. and S^p-function without being S^p-a. p.).

§ 4.

B-Points which are not B-a. p. Points.

In this paragraph we shall consider the B-points which are not B-a. p. points, and we shall investigate what possibilities may occur for such points concerning as well the lifetime P as the behaviour in the moment of death as to the B^p-spaces. We shall show that (as in the W-case) *all possibilities which are imaginable beforehand may occur*, viz.

1. $P = \infty$.
2. P arbitrarily finite, $1 \leqq P < \infty$.
   - 2 a. The point is dead as to the B^p-spaces at the time P $(P > 1)$.
   - 2 b. The point is alive as to the B^p-spaces at the time P $(P \geqq 1)$.

The examples which we shall give are quite similar to those used in the corresponding investigation in Chapter V, § 4 on W-points which are not W-a. p.

Example to 1.

Let

$$f(x) = \begin{cases} 1 & \text{for } x \geqq 0 \\ -1 & \text{for } x < 0. \end{cases}$$

The function $f(x)$ being bounded is a B^p-function for all p. Further $f(x)$ is no B-a. p. function as

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(x)\,dx = 1 \quad \text{while} \quad \lim_{T \to \infty} \frac{1}{T} \int_{-T}^0 f(x)\,dx = -1 \ (\neq 1).$$

Thus the B-point around $f(x)$ is not B-a. p. and has the lifetime $P = \infty$.

Example to 2 a.

In order to get a B-point (not B-a. p.) with an arbitrary finite lifetime $P (> 1)$ which is dead at the time P we add to the B-point of the first example a B-point around a periodic function $h(x)$ which is p-integrable for $p < P$ but not P-integrable. The B-point thus constructed is not B-a. p. as the B-point of the first example is not, while the B-point around $h(x)$ is. Further the point contains the function $f(x) + h(x)$ which is a B^p-function for $p < P$, but it does not contain any B^P-function, as the functions of the point can be obtained by adding to $f(x)$ all the functions in the B-point around $h(x)$, and $f(x)$ is a B^P-function whereas, in consequence of the theorem on the periodic points, the B-point around $h(x)$ does not contain any B^P-function.

Example to 2 b.

In order to get a B-point (not B-a. p.) with an arbitrary finite lifetime $P (\geqq 1)$ which is alive at the time P we add in an analogous way to the B-point from the first example a B-point around a periodic function $h(x)$ which is P-integrable but not p-integrable for $p > P$.

§ 5.

B-a. p. Points.

In this paragraph we consider an arbitrary B-a. p. point whose lifetime as to the B^p- and the B^p-a. p. spaces is denoted by P. In consequence of the theorem in § 3 it holds (in contrast to the W-case) that every B-a. p. point »behaves in the same way« as to the B^p- and the B^p-a. p. spaces in the following

sense: *If a B-a. p. point contains a B^p-function, it contains also a B^p-a. p. function.* We shall prove that there are *the following possibilities* for a B-a. p. point as regards its lifetime P and behaviour in the moment of death.

1. $P = \infty$.
2. P arbitrary finite, $1 \leqq P < \infty$.
   2 a. The point is dead as to the B^p- and the B^p-a. p. spaces at the time P $(P > 1)$.
   2 b. The point is alive as to the B^p- and the B^p-a. p. spaces at the time P $(P \geqq 1)$.

Example to 1.

The B-point around a bounded periodic function.

Example to 2 a.

The B-point around a periodic function which is p-integrable for $p < P$ but not P-integrable.

Example to 2 b.

The B-point around a periodic function which is P-integrable but not p-integrable for $p > P$.

§ 6.

Through Functions.

Finally in this paragraph we prove the following theorem which has already been used in § 2.

Theorem. *Let $\mathfrak{A}$ be a B-point with the lifetime P, $1 \leqq P \leqq \infty$, which (if $P < \infty$) is dead at the time P. Then there exists in $\mathfrak{A}$ a through function $f^*(x)$ as to the B^p-spaces, i. e. a function in $\mathfrak{A}$ which is a B^p-function for every $p < P$.*

In the proof of this theorem we use a remark made in the proof of the corresponding theorem on W-points, viz. that a 1-integrable function can always be modified by a W-zero function, and hence still more by a B-zero function, so that it becomes p-integrable for all p, and so that its modulus is not enlarged for any x. Further we shall use the operation of forming the minimum of two functions, in the sense indicated in the introduction.

Let $1 \leqq p_1 < p_2 < \cdots \to P$. We choose in $\mathfrak{A}$ a B^{p_1}-function $f_1(x)$, a B^{p_2}-function $f_2(x), \ldots$ and in consequence of the remark above we may assume these functions to be p-integrable for all p. We replace $f_1(x), f_2(x), \ldots$ by other functions $f_1^*(x), f_2^*(x), \ldots$ in $\mathfrak{A}$ where $f_n^*(x)$ like $f_n(x)$ is a B^{p_n}-function and p-integrable for all p and so that moreover the chain of inequalities

$$|f_1^*(x)| \geqq |f_2^*(x)| \geqq \cdots$$

holds for every x. As such functions $f_1^*(x), f_2^*(x), \ldots$ we may use

$$f_1^*(x) = f_1(x), \quad f_2^*(x) = \min\,[f_1^*(x), f_2(x)], \quad f_3^*(x) = \min\,[f_2^*(x), f_3(x)], \ldots.$$

In fact, firstly $|f_n^*(x)| \leqq |f_n(x)|$ for every x which involves that $f_n^*(x)$ like $f_n(x)$ is a B^{p_n}-function and p-integrable for all p, secondly $|f_1^*(x)| \geqq |f_2^*(x)| \geqq \cdots$ for every x, and thirdly $f_1^*(x), f_2^*(x), \ldots$ are all contained in $\mathfrak{A}$, as a G-point considered as a set of functions is closed with respect to the minimum-operation.

The functions $f_1^*(x), f_2^*(x), \ldots$ lying in $\mathfrak{A}$ form in particular a B-fundamental sequence. Now we make use of the special method of constructing a B-limit function of a B-fundamental sequence indicated in Chapter II in the proof of the completeness of the B^p-spaces. Constructing by this method (see Fig. 2) a B-limit function of our B-fundamental sequence $f_1^*(x), f_2^*(x), \ldots$ we get a function $f^*(x)$ which is a through function for our point $\mathfrak{A}$. On the one hand, this function $f^*(x)$ lies in $\mathfrak{A}$, as a G-point considered as a set of G-functions is G-closed. On the other hand, as $|f_n^*(x)| \geqq |f_{n+1}^*(x)| \geqq \cdots$ we have $|f^*(x)| \leqq |f_n^*(x)|$ for $x > T_{n-1}$ (and analogously for negative x with a large modulus) which, together with the fact that $f_1^*(x), f_2^*(x), \ldots$ are p-integrable for all p, shows that

$$D_{B^{p_n}}[f^*(x)] \leqq D_{B^{p_n}}[f_n^*(x)];$$

hence $f^*(x)$ is a B^{p_n}-function for every n and consequently a B^p-function for every $p < P$.

APPENDIX.

By

ERLING FØLNER.

In the proper paper the reciprocal interaction between the G^p- and the G^p-a. p. spaces was treated in every one of the three cases $G = S$, $G = W$ and $G = B$. As mentioned in the preface the reciprocal interaction between all the spaces will be investigated in a later paper. For this investigation a new series of main examples will be needed. In every one of these main examples the problem is to construct a B-a. p. point (represented by a B-a. p. function $F(x)$) with certain particular properties, and each example deals with an »extreme case«. The main examples serve as bricks in the construction of all the types of B-a. p. points, as the »medium cases« can be obtained by addition of different extreme cases. Naturally these main examples are more varied and complicated than our former main examples 1, 2 and 3, but on the other hand they are more or less analogous to them. Therefore we have preferred to indicate them — with exception of a single especially complicated one — in an appendix to the present paper. The examples in this appendix are numbered by Roman numerals I, II, ... with subsequent letters a, b, ... Every main example numbered by one of the Roman numerals I, II or III is nearly associated with the main example with the corresponding Arabian numeral in the paper itself. In connexion with main example II some lemmas concerning integral-estimations are proved which also will be used in the later paper. We shall not here try to give a comprehensive view of the examples, as such a view can first be properly gained in the course of the later paper where the examples are put in their natural places as counter examples to the general theorems.

Contents.

Main Example I.

In the main example I we constructed a sequence $F_1(x)$, $F_2(x)$, ... of bounded periodic functions with periods $h_1 = m_1$, $h_2 = m_1 m_2$, ... which is a W^p-fundamental sequence for every p, but not W-convergent. We shall now prove that there exists a function $F(x)$ such that the quoted sequence $F_n(x)$ is B^p-convergent to $F(x)$ for every p ($\geqq 1$) and such that the B-point around $F(x)$ does not contain any W-function. We remark that these latter properties involve in particular that the sequence $F_n(x)$ cannot W-converge to any W-function, as an eventual W-limit function of $F_n(x)$, being a B-limit function of $F_n(x)$, would lie in the B-point around $F(x)$. We emphasise, and this is the real content of the example, that we hereby get *a function $F(x)$ which is B^p-a. p. for all p, whereas the B-point around $F(x)$ does not contain W-functions.*

For $-\frac{1}{2} \leqq x \leqq \frac{1}{2}$ our sequence $F_n(x) = n$ tends to ∞, while for $-\infty < x < -\frac{1}{2}$ and $\frac{1}{2} < x < \infty$ the limit $\lim\limits_{n \to \infty} F_n(x)$ exists and is finite. In fact for $-h_{n+1} + \frac{1}{2} < x < -\frac{1}{2}$ and $\frac{1}{2} < x < h_{n+1} - \frac{1}{2}$ we have $F_n(x) = F_{n+1}(x)$ and hence for the same x (as $h_1 < h_2 < \cdots$)

$$F_n(x) = F_{n+1}(x) = F_{n+2}(x) = \cdots$$

and for n sufficiently large every $x < -\frac{1}{2}$ and $x > \frac{1}{2}$ is caught in the quoted intervals. For $-h_{n+1} + \frac{1}{2} < x < -\frac{1}{2}$ and $\frac{1}{2} < x < h_{n+1} - \frac{1}{2}$ we get

$$\lim_{\nu \to \infty} F_\nu(x) = F_n(x).$$

We shall see that as our $F(x)$ we can use the function

$$F(x)=\begin{cases}\text{say } 0 \text{ for } -\frac{1}{2}\leqq x\leqq \frac{1}{2}\\ \lim\limits_{n\to\infty} F_n(x) \text{ for } -\infty < x < -\frac{1}{2} \text{ and } \frac{1}{2}<x<\infty.\end{cases}$$

Thus $F(x)$ consists of:

Towers of the breadth 1 and the height 1 placed on all the numbers $\equiv 0 \pmod{h_1}$ but $\not\equiv 0 \pmod{h_2}$,

towers of the breadth 1 and the height 2 placed on all the numbers $\equiv 0 \pmod{h_2}$ but $\not\equiv 0 \pmod{h_3}$,

towers of the breadth 1 and the height 3 placed on all the numbers $\equiv 0 \pmod{h_3}$ but $\not\equiv 0 \pmod{h_4}$,

$\vdots$

We shall first show that $F_n(x)\xrightarrow{B^p} F(x)$ for every $p\geqq 1$, so that $F(x)$ is B^p-a. p. for all p.

We have

$$(D_{B^p}[F(x), F_n(x)])^p = \overline{\lim_{T\to\infty}}\frac{1}{2T}\int_{-T}^{T}|F(x)-F_n(x)|^p\,dx.$$

Thus we shall estimate

$$\frac{1}{2T}\int_{-T}^{T}|F(x)-F_n(x)|^p\,dx$$

for fixed n and large T, say $T\geqq h_n$. For a given $T\geqq h_n$ we determine first $q\geqq 0$ such that $h_{n+q}\leqq T<h_{n+q+1}$ and then ν among the numbers $1, 2, \ldots, m_{n+q+1}-1$ such that $\nu h_{n+q}\leqq T<(\nu+1)h_{n+q}$.

To begin with we distinguish between the case $\nu\leqq m_{n+q+1}-2$ and the case $\nu=m_{n+q+1}-1$.

In the first case we get

$$\frac{1}{2T}\int_{-T}^{T}|F(x)-F_n(x)|^p\,dx\leqq\frac{1}{2\nu h_{n+q}}\int_{-(\nu+1)h_{n+q}}^{(\nu+1)h_{n+q}}|F(x)-F_n(x)|^p\,dx\leqq$$

$$\frac{1}{2\nu h_{n+q}}\left[\int_{-\frac{1}{2}}^{\frac{1}{2}}|0-n|^p\,dx+\int_{-(\nu+1)h_{n+q}}^{(\nu+1)h_{n+q}}(F_{n+q}(x)-F_n(x))^p\,dx\right]$$

(as $F(x) = F_{n+q}(x)$ for $-(\nu+1)h_{n+q} \leqq x < -\frac{1}{2}$ and for $\frac{1}{2} < x \leqq (\nu+1)h_{n+q}$, $\nu = 1, 2, \ldots, m_{n+q+1} - 2$). Since $(F_{n+q}(x) - F_n(x))^p$ is periodic with the period h_{n+q}, the last term of the inequality is equal to

$$\frac{n^p}{2\nu h_{n+q}} + \frac{\nu+1}{\nu} M\{(F_{n+q}(x) - F_n(x))^p\}.$$

In the second case ($\nu = m_{n+q+1} - 1$) we get, as $F(x) = F_{n+q+1}(x)$ for $-h_{n+q+1} \leqq x < -\frac{1}{2}$ and $\frac{1}{2} < x \leqq h_{n+q+1}$,

$$\frac{1}{2T}\int_{-T}^{T} |F(x) - F_n(x)|^p dx \leqq \frac{1}{2\nu h_{n+q}} \int_{-h_{n+q+1}}^{h_{n+q+1}} |F(x) - F_n(x)|^p dx \leqq$$

$$\frac{1}{2\nu h_{n+q}}\left[\int_{-\frac{1}{2}}^{\frac{1}{2}} |0 - n|^p dx + \int_{-h_{n+q+1}}^{h_{n+q+1}} (F_{n+q+1}(x) - F_n(x))^p dx\right].$$

As $(F_{n+q+1}(x) - F_n(x))^p$ is periodic with the period h_{n+q+1}, the last term of the inequality is equal to

$$\frac{n^p}{2\nu h_{n+q}} + \frac{\nu+1}{\nu} M\{(F_{n+q+1}(x) - F_n(x))^p\}.$$

Using the estimation of $M\{(F_{n+q}(x) - F_n(x))^p\}$ from main example 1 (see pages 59—60) we get in both cases, as $\nu \geqq 1$ and $\frac{\nu+1}{\nu} \leqq 2$,

$$\frac{1}{2T}\int_{-T}^{T} |F(x) - F_n(x)|^p dx \leqq \frac{n^p}{2h_{n+q}} + 2R_n^p$$

where R_n is the remainder after the n-th term in the geometrical series $\sum_{1}^{\infty} \frac{1}{2^{\frac{n}{p}}}$. We let now T and consequently $q \to \infty$. Then we get

$$\overline{\lim_{T\to\infty}} \frac{1}{2T}\int_{-T}^{T} |F(x) - F_n(x)|^p dx \leqq 2R_n^p,$$

i. e.

$$D_{B^p}[F(x), F_n(x)] \leqq \sqrt[p]{2}\, R_n.$$

The last inequality shows that $F_n(x) \xrightarrow{B^p} F(x)$ for every $p \geqq 1$.

Next we shall show that the B-point around $F(x)$ does not contain any W-function. The proof runs in a similar way as the proof (given in main example 1) of the more special fact that the sequence $F_n(x)$ is not W-convergent. Proceeding indirectly we assume that there exists a B-zero function $J(x)$ such that the function

$$F(x) + J(x) = G(x)$$

is a W-function (i. e. an S-function). Let

$$D_S[G(x)] = K < \infty .$$

We choose a fixed integer $N > K$. In $F(x)$ on all the numbers $m \equiv 0 \pmod{h_N}$ except the number 0 there are standing towers of the breadth 1 and a height $\geqq N$. As $\int\limits_x^{x+1} |G(t)|\,dt \leqq K$ for all x, the inequality

$$\int\limits_{m-\frac{1}{2}}^{m+\frac{1}{2}} |F(x) - G(x)|\,dx \geqq \int\limits_{m-\frac{1}{2}}^{m+\frac{1}{2}} F(x)\,dx - \int\limits_{m-\frac{1}{2}}^{m+\frac{1}{2}} |G(x)|\,dx \geqq N - K$$

is valid for every one of the quoted m. Hence

$$D_B[J(x)] = D_B[G(x) - F(x)] = \overline{\lim_{T\to\infty}}\, \frac{1}{2T} \int\limits_{-T}^{T} |F(x) - G(x)|\,dx \geqq \frac{N-K}{h_N} > 0,$$

and consequently $D_B[J(x)] > 0$, in contradiction to $J(x)$ being a B-zero function.

Finally we observe that as in main example 2 we might have chosen all the numbers $m_1, m_2, \ldots$ equal to 2.

Main Examples II a, II b and II c.

In main example 2 we constructed a function $F(x)$ which is an S^p-a. p. function for $p < P$, an S^P-function, but not an S^P-a. p. function. The following three main examples II a, II b and II c are generalisations of this main example.

In main example II a the B-point around the function $F(x)$ of main example 2 is considered, the numbers $m_1, m_2, \ldots$ occurring in it being only assumed to increase suitably strongly to ∞. In this way we get, as we shall

see, *a function* $F(x)$ *which is* S^p*-a. p. for* $p < P$, W^P*-a. p.*, B^p*-a. p. for all* p, *and such that the B-point around* $F(x)$ *does not contain any* S^P*-a. p. function.* Here P is an arbitrarily given number, $1 < P < \infty$.

In the main examples II b and II c other types of towers are used than in main example 2, but apart from this the construction is quite the same.

In main example II b we construct *a function* $F(x)$ *which is* S^p*-a. p. for* $p<P$, B^p*-a. p. for all* p *and such that the B-point around* $F(x)$ *does not contain any* S^P*-function.* Here P is an arbitrarily given number, $1 < P < \infty$.

Finally in main example II c we construct *a function* $F(x)$ *which is* S^P*-a. p.*, B^p*-a. p. for all* p *and such that the B-point around* $F(x)$ *does not contain any* S^p*-function for any* $p > P$. Here P is an arbitrarily given number, $1 \leqq P < \infty$.

Main Example II a.

As mentioned, in main example II a we consider the B-a. p. point around the function $F(x)$ of main example 2. In this latter example we saw that $F(x)$ is an S^p-a. p. function for $p<P$ and an S^P-function, but not an S^P-a. p. function. From this it can easily be concluded that the B-a. p. point around $F(x)$ does not contain any S^P-a. p. function. Indirectly we assume that the point contains an S^P-a. p. function $G(x)$. Then $G(x)$ has the same Fourier series as $F(x)$, and both $F(x)$ and $G(x)$ are S-a. p. functions; in consequence of the uniqueness theorem they can therefore only differ by an S-zero function, which is also an S^P-zero function, and $G(x)$ being S^P-a. p., $F(x)$ would also be S^P-a. p., which is not the case.

Next we show that $F(x)$ is a W^P-a. p. function. As mentioned in main example 2, the function $F(x)$ differs from $F_n(x)$ only at the numbers $m \equiv 0$ (mod h_{n+1}). Therefore for all $m \equiv 0$ (mod h_1) but $\not\equiv 0$ (mod h_{n+1}) we have

$$\sqrt[P]{\int_{m-\frac{h_1}{2}}^{m+\frac{h_1}{2}} |F(x) - F_n(x)|^P \, dx} = 0;$$

and further, as all our towers have the P-integral 1, we get in consequence of MINKOWSKI's inequality for all $m \equiv 0$ (mod h_{n+1})

$$\sqrt[P]{\int_{m-\frac{h_1}{2}}^{m+\frac{h_1}{2}} |F(x)-F_n(x)|^P\,dx} \leqq$$

$$\sqrt[P]{\int_{m-\frac{h^1}{2}}^{m+\frac{h_1}{2}} |F(x)|^P\,dx} + \sqrt[P]{\int_{m-\frac{h_1}{2}}^{m+\frac{h_1}{2}} |F_n(x)|^P\,dx} \leqq 2.$$

Hence

$$D_{W^P}[F(x), F_n(x)] \leqq \frac{2}{\sqrt[P]{h_{n+1}}},$$

where the right-hand side tends to 0 for $n \to \infty$.

Finally we shall show that by letting the numbers $m_1, m_2, \ldots$ increase sufficiently strongly to infinity we can obtain that $F(x)$ becomes B^p-a. p. for all p. However, as it will be convenient to prove this property of $F(x)$, which is common for the main examples II a, II b and II c, simultaneously for all three main examples, we postpone the proof.

In the main examples II b and II c we shall use the following

Lemma 1 a. *Let $f(x)$ be a function, defined in a finite interval $a \leqq x \leqq a+L$, which consists of a number of congruent towers placed in some way or other in the interval. Then every function $t(x)$, satisfying the inequality*

$$\sqrt[P]{\frac{1}{L}\int_a^{a+L} |f(x)+t(x)|^P\,dx} \leqq k \sqrt[P]{\frac{1}{L}\int_a^{a+L} (f(x))^P\,dx}$$

where $0 < k < 1$ and $1 \leqq P < \infty$, will satisfy the inequality

$$\frac{1}{L}\int_a^{a+L} |t(x)|^\alpha\,dx \geqq (1-k)^P \frac{1}{L}\int_a^{a+L} (f(x))^\alpha\,dx$$

for an arbitrary α, $1 \leqq \alpha \leqq P$.

In the main examples II b and II c, however, the lemma will only be applied in the case where $\alpha = 1$ and $f(x)$ consists of only one tower.

Proof. Let the towers of $f(x)$ have the breadth b and the height h and let their number be ν. We may assume that $|t(x)| \leqq h$; otherwise we consider $(t(x))_h$ which in consequence of the inequality

$$|f(x) + (t(x))_h| = |(f(x))_h + (t(x))_h| \leqq |f(x) + t(x)|$$

satisfies the same assumption as $t(x)$; on account of $|t(x)| \geqq |(t(x))_h|$ the conclusion for $t(x)$ results from the conclusion for $(t(x))_h$. Then (i. e. for $|t(x)| \leqq h$) we have for every α, $1 \leqq \alpha \leqq P$,

$$\frac{1}{L}\int_a^{a+L} |t(x)|^\alpha dx \geqq \frac{1}{h^{P-\alpha}}\frac{1}{L}\int_a^{a+L} |t(x)|^P dx = \frac{1}{h^{P-\alpha}}\left\{\sqrt[P]{\frac{1}{L}\int_a^{a+L} |t(x)|^P dx}\right\}^P \geqq$$

$$\frac{1}{h^{P-\alpha}}\left\{\sqrt[P]{\frac{1}{L}\int_a^{a+L} |f(x)|^P dx} - \sqrt[P]{\frac{1}{L}\int_a^{a+L} |f(x) + t(x)|^P dx}\right\}^P \geqq$$

$$\frac{1}{h^{P-\alpha}}(1-k)^P \frac{1}{L}\int_a^{a+L} |f(x)|^P dx = \frac{1}{h^{P-\alpha}}(1-k)^P \frac{\nu}{L} b h^P = (1-k)^P \frac{\nu}{L} b h^\alpha =$$

$$(1-k)^P \frac{1}{L}\int_a^{a+L} (f(x))^\alpha dx.$$

Main Example II b.

We construct a sequence $F_1(x), F_2(x), \ldots$ and a function $F(x)$ in exactly the same way as in main example II a. Only by »a tower of type n« we shall now understand a tower with the p_n-integral ε_n and the P-integral n where the sequence $p_1, p_2, \ldots$ is chosen such that $1 \leqq p_1 < p_2 < \cdots \to P$. As before $1 > \varepsilon_1 > \varepsilon_2 > \cdots \to 0$. The n-th function $F_n(x)$ is a bounded periodic function with the period h_n.

First, we shall show that the sequence $F_n(x)$ is S^p-convergent to $F(x)$ for every $p < P$, so that $F(x)$ is an S^p-limit periodic function for $p < P$. For every $m \equiv 0 \pmod{h_1}$ and $\not\equiv 0 \pmod{h_{n+1}}$ the quantity

$$\sqrt[p_n]{\int_{m-\frac{h_1}{2}}^{m+\frac{h_1}{2}} |F(x) - F_n(x)|^{p_n} dx}$$

is equal to 0 (cp. page 115) and for $m \equiv 0 \pmod{h_{n+1}}$ we have

$$\sqrt[p_n]{\int_{m-\frac{h_1}{2}}^{m+\frac{h_1}{2}} |F(x)-F_n(x)|^{p_n}\,dx} \leqq \sqrt[p_n]{\int_{m-\frac{h_1}{2}}^{m+\frac{h_1}{2}} |F(x)|^{p_n}\,dx} + \sqrt[p_n]{\int_{m-\frac{h_1}{2}}^{m+\frac{h_1}{2}} |F_n(x)|^{p_n}\,dx},$$

which for $m=0$ is equal to $\sqrt[p_n]{\varepsilon_n}$, while for $m \neq 0$, denoting by $n+q(m)$, $q(m) \geqq 1$, the type of the tower placed on the number m in $F(x)$, it is

$$\leqq \sqrt[p_{n+q(m)}]{\int_{m-\frac{h_1}{2}}^{m+\frac{h_1}{2}} |F(x)|^{p_{n+q(m)}}\,dx} + \sqrt[p_n]{\int_{m-\frac{h_1}{2}}^{m+\frac{h_1}{2}} |F_n(x)|^{p_n}\,dx} = \sqrt[p_{n+q(m)}]{\varepsilon_{n+q(m)}} + \sqrt[p_n]{\varepsilon_n}.$$

As $1 > \varepsilon_1 > \varepsilon_2 > \cdots$, we have consequently for every $m \equiv 0 \pmod{h_1}$

$$\sqrt[p_n]{\int_{m-\frac{h_1}{2}}^{m+\frac{h_1}{2}} |F(x)-F_n(x)|^{p_n}\,dx} \leqq 2\sqrt[P]{\varepsilon_n},$$

and hence for every x

$$\sqrt[p_n]{\int_{x-\frac{h_1}{2}}^{x+\frac{h_1}{2}} |F(t)-F_n(t)|^{p_n}\,dt} \leqq \sqrt[p_n]{2}\cdot 2\sqrt[P]{\varepsilon_n}.$$

Thus

$$D_{S_{h_1}^{p_n}}[F(x), F_n(x)] \leqq \sqrt[p_n]{\frac{2}{h_1}}\cdot 2\sqrt[P]{\varepsilon_n} \leqq 2\sqrt[P]{\varepsilon_n},$$

which tends to 0 for $n \to \infty$. From this it results that

$$D_{S_{h_1}^{p}}[F(x), F_n(x)] \to 0 \quad \text{for} \quad p < P,$$

as for sufficiently large n we have $p_n > p$ and therefore

$$D_{S_{h_1}^{p}}[F(x), F_n(x)] \leqq D_{S_{h_1}^{p_n}}[F(x), F_n(x)] \to 0.$$

Next we show that the B-point around $F(x)$ does not contain any S^p-function. Proceeding indirectly we assume that the B-point around $F(x)$ contains an S^p-function $G(x)$. Then $D_{S^p}[G(x)] = K < \infty$. Let N be a fixed number so that $\sqrt[p]{N} \geqq 2K$. In $F(x)$ on all numbers $m \equiv 0 \pmod{h_N}$ but $\not\equiv 0 \pmod{h_{N+1}}$ towers of type N are standing and these towers have the P-integral N. Thus we have for the m in question

$$\sqrt[p]{\int_{m-\frac{1}{2}}^{m+\frac{1}{2}} |F(x)|^p\,dx} = \sqrt[p]{N} \geqq 2K.$$

At the same time the inequality

$$\sqrt[p]{\int_{m-\frac{1}{2}}^{m+\frac{1}{2}} |G(x)|^p\,dx} \leqq K$$

is valid so that for the quoted m we get the inequality

$$\sqrt[p]{\int_{m-\frac{1}{2}}^{m+\frac{1}{2}} |G(x)|^p\,dx} \leqq \frac{1}{2}\sqrt[p]{\int_{m-\frac{1}{2}}^{m+\frac{1}{2}} |F(x)|^p\,dx}.$$

Thus we have, on account of the lemma 1 a above, for these m

$$\int_{m-\frac{1}{2}}^{m+\frac{1}{2}} |G(x) - F(x)|\,dx \geqq \left(\frac{1}{2}\right)^p \int_{m-\frac{1}{2}}^{m+\frac{1}{2}} |F(x)|\,dx = \left(\frac{1}{2}\right)^p k' > 0,$$

where k' denotes the 1-integral of a tower of type N. Hence

$$D_B[G(x), F(x)] = \overline{\lim_{T\to\infty}}\, \frac{1}{2T}\int_{-T}^{T} |G(x) - F(x)|\,dx \geqq \frac{\left(\frac{1}{2}\right)^p k'(m_{N+1} - 1)}{h_{N+1}} > 0,$$

which contradicts the fact that $F(x)$ and $G(x)$ lie in the same B-point.

Main Example II c.

In just the same way as in the main examples II a and II b a sequence $F_1(x), F_2(x), \ldots$ and a function $F(x)$ are constructed. Only by a »tower of type n« we shall now understand a tower with the P-integral ε_n and the p_n-integral n where the sequence of numbers $p_1, p_2, \ldots$ is chosen such that $p_1 > p_2 > \cdots \to P$. As before $1 > \varepsilon_1 > \varepsilon_2 > \cdots \to 0$. The n-th function $F_n(x)$ is a bounded periodic function with the period h_n.

First, we shall show that the sequence $F_n(x)$ is S^P-convergent to $F(x)$, so that $F(x)$ is an S^P-limit periodic function. The quantity

$$\sqrt[P]{\int_{m-\frac{h_1}{2}}^{m+\frac{h_1}{2}} |F(x) - F_n(x)|^P\,dx}$$

is equal to 0 for every $m \equiv 0 \pmod{h_1}$ but $\not\equiv 0 \pmod{h_{n+1}}$, whereas for $m \equiv 0 \pmod{h_{n+1}}$ we have, denoting for $m \neq 0$ by $n + q(m)$, $q(m) \geqq 1$, the type of the tower which stands in $F(x)$ on the number m,

$$\sqrt[P]{\int_{m-\frac{h_1}{2}}^{m+\frac{h_1}{2}} |F(x)-F_n(x)|^P\,dx} \leqq \sqrt[P]{\int_{m-\frac{h_1}{2}}^{m+\frac{h_1}{2}} |F(x)|^P\,dx} + \sqrt[P]{\int_{m-\frac{h_1}{2}}^{m+\frac{h_1}{2}} |F_n(x)|^P\,dx} =$$

$$\begin{cases} \sqrt[P]{\varepsilon_n} & \text{for } m = 0 \\ \sqrt[P]{\varepsilon_{n+q(m)}} + \sqrt[P]{\varepsilon_n} & \text{for } m \neq 0 \end{cases}$$

which is

$$\leqq 2\sqrt[P]{\varepsilon_n}.$$

Hence for every $m \equiv 0 \pmod{h_1}$

$$\sqrt[P]{\int_{m-\frac{h_1}{2}}^{m+\frac{h_1}{2}} |F(x) - F_n(x)|^P\,dx} \leqq 2\sqrt[P]{\varepsilon_n},$$

and hence for every x

$$\sqrt[P]{\int_{x-\frac{h_1}{2}}^{x+\frac{h_1}{2}} |F(t)-F_n(t)|^P\,dt} \leqq \sqrt[P]{2}\cdot 2\sqrt[P]{\varepsilon_n}.$$

Consequently

$$D_{S^P_{h_1}}[F(x), F_n(x)] \leqq \sqrt[P]{\frac{2}{h_1}}\cdot 2\sqrt[P]{\varepsilon_n},$$

which tends to o for $n \to \infty$.

Next we shall prove that the B-point around $F(x)$ does not contain S^p-functions for any $p > P$. Proceeding indirectly we assume that the B-point around $F(x)$ contains an S^p-function $G(x)$ for a $p > P$. Then

$$D_{S^p}[G(x)] = K < \infty.$$

Let N be a fixed number so that $\sqrt[p]{N} \geqq 2K$ and $p_N \leqq p$. In $F(x)$ on all the numbers $m \equiv 0 \pmod{h_N}$ but $\not\equiv 0 \pmod{h_{N+1}}$ there are standing towers of type N and these towers have the p_N-integral N and therefore a p-integral which is $\geqq N$. Hence for the m in question

$$\sqrt[p]{\int_{m-\frac{1}{2}}^{m+\frac{1}{2}} (F(x))^p\,dx} \geqq \sqrt[p]{N} \geqq 2K.$$

At the same time the inequality

$$\sqrt[p]{\int_{m-\frac{1}{2}}^{m+\frac{1}{2}} |G(x)|^p\,dx} \leqq K$$

holds, so that for the quoted m

$$\sqrt[p]{\int_{m-\frac{1}{2}}^{m+\frac{1}{2}} |G(x)|^p\,dx} \leqq \frac{1}{2}\sqrt[p]{\int_{m-\frac{1}{2}}^{m+\frac{1}{2}} (F(x))^p\,dx}.$$

Consequently, by the lemma, we have for these m

$$\int_{m-\frac{1}{2}}^{m+\frac{1}{2}} |G(x) - F(x)|\,dx \geqq \left(\frac{1}{2}\right)^p \int_{m-\frac{1}{2}}^{m+\frac{1}{2}} F(x)\,dx = \left(\frac{1}{2}\right)^p k' > 0,$$

where k' denotes the 1-integral of a tower of type N. Hence

$$D_B[G(x), F(x)] = \overline{\lim_{T\to\infty}} \frac{1}{2T}\int_{-T}^{T} |G(x) - F(x)|\,dx \geqq \frac{\left(\frac{1}{2}\right)^p k'(m_{N+1} - 1)}{h_{N+1}} > 0$$

which contradicts the fact that $F(x)$ and $G(x)$ lie in the same B-point.

»Spreading» of the Towers in the Main Examples II a, II b and II c.

Finally we show, simultaneously for all three main examples, that by letting $m_1, m_2, \ldots$ increase sufficiently strongly we can obtain that the sequence $F_1(x), F_2(x), \ldots$ is B^p-convergent to $F(x)$ for every p, so that $F(x)$ is B^p-limit periodic for every p.

We put (cp. main example 1)

$$f_1(x) = F_1(x),\quad f_2(x) = F_2(x) - F_1(x),\quad f_3(x) = F_3(x) - F_2(x), \ldots.$$

Let $\delta_1, \delta_2, \ldots$ be a sequence of positive numbers so that $\sum_1^\infty \delta_n$ is convergent, and let $P_1, P_2, \ldots$ be a sequence of numbers, $1 \leqq P_1 < P_2 < \cdots$, tending to ∞. Successively we may choose $m_1, m_2, \ldots$ so large that

$$D_{B^{P_n}}[f_n(x)] < \delta_n \quad \text{for} \quad n = 1, 2, \ldots.$$

In fact $f_n(x) = F_n(x) - F_{n-1}(x)$ differs from 0 only at the numbers $\equiv 0 \pmod{h_n}$, and on these numbers in $F_{n-1}(x)$ there stand towers of type $n-1$, whereas in $F_n(x)$ towers of type n are standing, so that

$$D_{B^{P_n}}[f_n(x)] = D_{B^{P_n}}[F_n(x) - F_{n-1}(x)] \leqq \frac{\sqrt[P_n]{I_n'} + \sqrt[P_n]{I_n}}{h_n},$$

where I_n' denotes the P_n-integral of a tower of type $n-1$ and I_n denotes the P_n-integral of a tower of type n; assuming the numbers $m_1, m_2, \ldots, m_{n-1}$ already

fixed, the number m_n and therewith $h_n = m_1 m_2 \ldots m_n$ may evidently be chosen so large that the right-hand side of the inequality and therefore $D_{B^{P_n}}[f_n(x)]$ becomes $< \delta_n$.

After this choice of $m_1, m_2, \ldots$ we can prove that

$$D_{B^{P_n}}[F(x), F_n(x)] \to 0 \quad \text{for} \quad n \to \infty.$$

From this we get immediately the desired relation $D_{B^p}[F(x), F_n(x)] \to 0$ for every fixed p, as for n sufficiently large $P_n > p$ and therefore $D_{B^p}[F(x), F_n(x)] \leqq D_{B^{P_n}}[F(x), F_n(x)] \to 0$.

In order to prove that

$$D_{B^{P_n}}[F(x), F_n(x)] = \overline{\lim_{T \to \infty}} \sqrt[P_n]{\frac{1}{2T}\int_{-T}^{T} |F(x) - F_n(x)|^{P_n}\, dx} \to 0 \quad \text{for} \quad n \to \infty$$

we estimate (cp. main example 1)

$$\frac{1}{2T}\int_{-T}^{T} |F(x) - F_n(x)|^{P_n}\, dx$$

for fixed n and large T, say $T \geqq h_n$. First we determine $q \geqq 0$ so that $h_{n+q} \leqq T < h_{n+q+1}$, and then ν among the numbers $1, 2, \ldots, m_{n+q+1} - 1$ so that $\nu h_{n+q} \leqq T < (\nu + 1) h_{n+q}$.

To begin with we distinguish between the two cases $\nu \leqq m_{n+q+1} - 2$ and $\nu = m_{n+q+1} - 1$.

In the first case we have

$$\frac{1}{2T}\int_{-T}^{T} |F(x) - F_n(x)|^{P_n}\, dx \leqq \frac{1}{2\nu h_{n+q}} \int_{-(\nu+1) h_{n+q}}^{(\nu+1) h_{n+q}} |F(x) - F_n(x)|^{P_n}\, dx \leqq$$

$$\frac{1}{2\nu h_{n+q}}\left[I_n + \int_{-(\nu+1) h_{n+q}}^{(\nu+1) h_{n+q}} |F_{n+q}(x) - F_n(x)|^{P_n}\, dx\right],$$

where, as before, I_n denotes the P_n-integral of a tower of type n; for on 0 there is standing no tower in $F(x)$, whereas a tower of type n is standing in $F_n(x)$, and $F(x) = F_{n+q}(x)$ for $-(\nu + 1) h_{n+q} < x < -\frac{h_1}{2}$ and for $\frac{h_1}{2} < x < (\nu + 1) h_{n+q}$,

$\nu = 1, 2, \ldots, m_{n+q+1}-2$. As $|F_{n+q}(x)-F_n(x)|^{p_n}$ is periodic with the period h_{n+q}, the right-hand side is equal to

$$\frac{I_n}{2\nu h_{n+q}} + \frac{\nu+1}{\nu} M\{|F_{n+q}(x) - F_n(x)|^{p_n}\}.$$

In the second case ($\nu = m_{n+q+1} - 1$) we get

$$\frac{1}{2T}\int_{-T}^{T} |F(x) - F_n(x)|^{p_n}\, dx \leqq \frac{1}{2\nu h_{n+q}} \int_{-h_{n+q+1}}^{h_{n+q+1}} |F(x) - F_n(x)|^{p_n}\, dx \leqq$$

$$\frac{1}{2\nu h_{n+q}}\left[I_n + \int_{-h_{n+q+1}}^{h_{n+q+1}} |F_{n+q+1}(x) - F_n(x)|^{p_n}\, dx\right],$$

as $F(x) = F_{n+q+1}(x)$ for $-h_{n+q+1} < x < -\frac{h_1}{2}$ and for $\frac{h_1}{2} < x < h_{n+q+1}$. Since $|F_{n+q+1}(x)-F_n(x)|^{p_n}$ is periodic with the period h_{n+q+1}, the last term is equal to

$$\frac{I_n}{2\nu h_{n+q}} + \frac{\nu+1}{\nu} M\{|F_{n+q+1}(x) - F_n(x)|^{p_n}\}.$$

An estimation of $M\{|F_{n+q}(x) - F_n(x)|^{p_n}\}$ is got in the following way:

$$\sqrt[p_n]{M\{|F_{n+q}(x) - F_n(x)|^{p_n}\}} = \sqrt[p_n]{M\{|f_{n+1}(x) + f_{n+2}(x) + \cdots + f_{n+q}(x)|^{p_n}\}} \leqq$$

$$\sqrt[p_n]{M\{|f_{n+1}(x)|^{p_n}\}} + \sqrt[p_n]{M\{|f_{n+2}(x)|^{p_n}\}} + \cdots + \sqrt[p_n]{M\{|f_{n+q}(x)|^{p_n}\}} \leqq$$

$$\sqrt[p_{n+1}]{M\{|f_{n+1}(x)|^{p_{n+1}}\}} + \sqrt[p_{n+2}]{M\{|f_{n+2}(x)|^{p_{n+2}}\}} + \cdots + \sqrt[p_{n+q}]{M\{|f_{n+q}(x)|^{p_{n+q}}\}} \leqq$$

$$\delta_{n+1} + \delta_{n+2} + \cdots + \delta_{n+q} \leqq \delta_{n+1} + \delta_{n+2} + \cdots.$$

Thus we have

$$M\{|F_{n+q}(x) - F_n(x)|^{p_n}\} \leqq (\delta_{n+1} + \delta_{n+2} + \cdots)^{p_n}.$$

Using this estimation in each of the two above cases, we get

$$\frac{1}{2T}\int_{-T}^{T} |F(x) - F_n(x)|^{p_n}\, dx \leqq \frac{I_n}{2h_{n+q}} + 2(\delta_{n+1} + \delta_{n+2} + \cdots)^{p_n}.$$

For $T \to \infty$ and therefore $q \to \infty$, we have $\frac{I_n}{2 h_{n+q}} \to 0$; hence

$$\overline{\lim_{T \to \infty}} \frac{1}{2T} \int_{-T}^{T} |F(x) - F_n(x)|^{P_n} dx \leqq 2 (\delta_{n+1} + \delta_{n+2} + \cdots)^{P_n}$$

or

$$D_B^{P_n}[F(x), F_n(x)] \leqq \sqrt[P_n]{2} (\delta_{n+1} + \delta_{n+2} + \cdots),$$

which shows that

$$D_B^{P_n}[F(x), F_n(x)] \to 0 \quad \text{for} \quad n \to \infty.$$

In connexion with the lemma 1 a we insert here two lemmas of similar character which will be applied in the later paper.

Lemma 1 b. *Let $f(x)$ be a function, defined in a finite interval $a \leqq x \leqq a + L$, which consists of a number of congruent towers placed in some way or other in the interval. Then every function $t(x)$ satisfying the inequality*

$$\sqrt[P]{\frac{1}{L} \int_a^{a+L} |f(x) + t(x)|^P dx} \leqq k \sqrt[P]{\frac{1}{L} \int_a^{a+L} (f(x))^P dx},$$

where $0 < k < 1$ and $1 \leqq P < \infty$, will satisfy the inequality

$$\frac{1}{L} \int_a^{a+L} |t(x)|^\alpha dx \geqq (1 - k)^\alpha \frac{1}{L} \int_a^{a+L} (f(x))^\alpha dx$$

for an arbitrary α, $P \leqq \alpha < \infty$.

Proof. We may assume that $t(x) = 0$ where $f(x) = 0$. Otherwise we may consider the function $t^*(x)$ which is equal to 0 where $f(x) = 0$ and equal to $t(x)$ where $f(x) \neq 0$; like $t(x)$ this function $t^*(x)$ satisfies the assumption of the lemma 1 b, since $|f(x) + t^*(x)| \leqq |f(x) + t(x)|$, and the conclusion for $t(x)$ results from the conclusion for $t^*(x)$, as $|t(x)| \geqq |t^*(x)|$. Let the towers of $f(x)$ have the breadth b and the height h, let the number of the towers be ν, and denote by $e(x)$ the function which has towers in the same places and with the same

breadth as the towers of $f(x)$ but with the height 1. In consequence of HÖLDER's inequality we have for every $\alpha \geqq P$

$$\frac{1}{L}\int_a^{a+L} |t(x)|^P\,dx = \frac{1}{L}\int_a^{a+L} |t(x)|^P e(x)\,dx \leqq$$

$$\left(\frac{1}{L}\int_a^{a+L} |t(x)|^\alpha\,dx\right)^{\frac{P}{\alpha}} \left(\frac{1}{L}\int_a^{a+L} (e(x))^{\frac{1}{1-\frac{P}{\alpha}}}\,dx\right)^{1-\frac{P}{\alpha}} = \left(\frac{\nu}{L}b\right)^{1-\frac{P}{\alpha}} \left(\frac{1}{L}\int_a^{a+L} |t(x)|^\alpha\,dx\right)^{\frac{P}{\alpha}}.$$

Hence

$$\frac{1}{L}\int_a^{a+L} |t(x)|^\alpha\,dx \geqq \frac{1}{\left(\frac{\nu}{L}b\right)^{\frac{\alpha}{P}-1}} \left(\frac{1}{L}\int_a^{a+L} |t(x)|^P\,dx\right)^{\frac{\alpha}{P}} =$$

$$\frac{1}{\left(\frac{\nu}{L}b\right)^{\frac{\alpha}{P}-1}} \left[\sqrt[P]{\frac{1}{L}\int_a^{a+L} |t(x)|^P\,dx}\right]^\alpha \geqq$$

$$\frac{1}{\left(\frac{\nu}{L}b\right)^{\frac{\alpha}{P}-1}} \left[\sqrt[P]{\frac{1}{L}\int_a^{a+L} (f(x))^P\,dx} - \sqrt[P]{\frac{1}{L}\int_a^{a+L} |f(x)+t(x)|^P\,dx}\right]^\alpha \geqq$$

$$\frac{1}{\left(\frac{\nu}{L}b\right)^{\frac{\alpha}{P}-1}} \left[(1-k)\sqrt[P]{\frac{1}{L}\int_a^{a+L} (f(x))^P\,dx}\right]^\alpha = \frac{1}{\left(\frac{\nu}{L}b\right)^{\frac{\alpha}{P}-1}} (1-k)^\alpha \left(\frac{\nu}{L}b\,h^P\right)^{\frac{\alpha}{P}} =$$

$$(1-k)^\alpha \left(\frac{\nu}{L}b\right) h^\alpha = (1-k)^\alpha \frac{1}{L}\int_a^{a+L} (f(x))^\alpha\,dx.$$

Lemma 2. *Let the function $f(x)$ be defined in a finite interval $a \leqq x \leqq a+L$ and let P be a number, $1 \leqq P < \infty$. Let further $A \geqq 0$ be given so that*

$$A < \sqrt[P]{\frac{1}{L}\int_a^{a+L} |f(x)|^P\,dx}.$$

Then there exists a constant $c > 0$ such that every function $t(x)$ which satisfies the inequality

$$\sqrt[p]{\frac{1}{L}\int_a^{a+L} |f(x) + t(x)|^p\, dx} \leqq A$$

also satisfies the inequality

$$\frac{1}{L}\int_a^{a+L} |t(x)|\, dx \geqq c \qquad .$$

Proof. We determine N so large that

$$\sqrt[p]{\frac{1}{L}\int_a^{a+L} |(f(x))_N|^p\, dx} = A_1 > A.$$

Then

$$\frac{1}{L}\int_a^{a+L} |t(x)|\, dx \geqq \frac{1}{L}\int_a^{a+L} |(t(x))_N|\, dx \geqq$$

$$\frac{1}{N^{p-1}}\,\frac{1}{L}\int_a^{a+L} |(t(x))_N|^p\, dx = \frac{1}{N^{p-1}}\left\{\sqrt[p]{\frac{1}{L}\int_a^{a+L} |(t(x))_N|^p\, dx}\right\}^p \geqq$$

$$\frac{1}{N^{p-1}}\left\{\sqrt[p]{\frac{1}{L}\int_a^{a+L} |(f(x))_N|^p\, dx} - \sqrt[p]{\frac{1}{L}\int_a^{a+L} |(f(x))_N + (t(x))_N|^p\, dx}\right\}^p \geqq$$

$$\frac{1}{N^{p-1}}\left\{\sqrt[p]{\frac{1}{L}\int_a^{a+L} |(f(x))_N|^p\, dx} - \sqrt[p]{\frac{1}{L}\int_a^{a+L} |f(x) + t(x)|^p\, dx}\right\}^p \geqq$$

$$\frac{1}{N^{p-1}}\{A_1 - A\}^p = c > 0.$$

It may be observed that lemma 2 is of a somewhat other character than the two lemmas 1 a and 1 b. In fact the lower bound c indicated for $\frac{1}{L}\int_a^{a+L} |t(x)|\, dx$ in lemma 2 depends on the function $f(x)$; it is easily seen that there exists no form of lemma 2 corresponding to the lemmas 1 a and 1 b (where the indicated lower bounds are independent of $f(x)$).

Main Examples III a and III b.

Main Example III a.

In main example 3 we constructed *a function* $F(x)$ *which is an* S^p*-a. p. function for* $p < P$, *an* S^P*-function and such that the* B*-point around* $F(x)$ *does not contain* W^P*-a. p. functions.* Now we shall show *that* $F(x)$ *becomes* B^p*-a. p. for all* p, if the numbers $m_1, m_2, \ldots$ increase sufficiently rapidly to ∞. We remark, that it was in order to be able to obtain this property that already in main example 3 we chose to fill out just the central subintervals.

Let $1 \leqq P_1 < P_2 < \cdots \to \infty$ and let $\sum_1^\infty \delta_n$ be a convergent series of positive numbers. We put (cp. main examples I and II)

$$f_1(x) = F_1(x),\ f_2(x) = F_2(x) - F_1(x),\ f_3(x) = F_3(x) - F_2(x), \ldots.$$

The function $f_n(x)$ is periodic with the period h_n and consists in a period interval $\nu h_n \leqq x < (\nu + 1) h_n$ of the towers of type n which by the transition from $F_{n-1}(x)$ to $F_n(x)$ we filled into the central one of the subintervals $\mu h_{n-1} \leqq x < (\mu + 1) h_{n-1}$. We have calculated the number of these towers exactly, but here we need only observe that it is (of course) at most equal to the total number of subintervals $\eta \leqq x < \eta + 1$ in the mentioned central interval, viz. $\leqq h_{n-1}$. The P_n-integral of a tower of type n being denoted by I_n we have the estimation

$$\sqrt[P_n]{M\{(f_n(x))^{P_n}\}} \leqq \sqrt[P_n]{\frac{h_{n-1} I_n}{h_n}} = \sqrt[P_n]{\frac{I_n}{m_n}}.$$

Now we choose m_n so large that

$$\sqrt[P_n]{\frac{I_n}{m_n}} < \delta_n, \qquad n = 1, 2, \ldots.$$

In particular we have

$$\sqrt[P_n]{M\{(f_n(x))^{P_n}\}} < \delta_n, \qquad n = 1, 2, \ldots.$$

We shall prove that

$$D_{B^{P_n}}[F(x), F_n(x)] \to 0$$

for such a choice of $m_1, m_2, \ldots$, which involves (on account of $P_n \to \infty$, cp. main example II) that $D_{B^p}[F(x), F_n(x)] \to 0$ for every fixed p, so that $F(x)$ is B^p-a. p. for all p.

We have

$$D_{B^{P_n}}[F(x), F_n(x)] = \overline{\lim_{T\to\infty}} \sqrt[P_n]{\frac{1}{2T}\int_{-T}^{T} (F(x) - F_n(x))^{P_n}\, dx}.$$

Thus we shall estimate

$$\frac{1}{2T}\int_{-T}^{T} (F(x) - F_n(x))^{P_n}\, dx$$

for fixed n and large T, say $T \geqq h_n$ (cp. the main examples I and II). We choose first $q \geqq 0$ such that $h_{n+q} \leqq T < h_{n+q+1}$ and then ν among the numbers $1, 2, \ldots, m_{n+q+1} - 1$ such that $\nu h_{n+q} \leqq T < (\nu + 1) h_{n+q}$. As $F(x) = F_{n+q+1}(x)$ for $-h_{n+q+1} \leqq x < h_{n+q+1}$, we have

$$\sqrt[P_n]{\frac{1}{2T}\int_{-T}^{T} (F(x) - F_n(x))^{P_n}\, dx} = \sqrt[P_n]{\frac{1}{2T}\int_{-T}^{T} (F_{n+q+1}(x) - F_n(x))^{P_n}\, dx} =$$

$$\sqrt[P_n]{\frac{1}{2T}\int_{-T}^{T} (F_{n+q}(x) - F_n(x) + f_{n+q+1}(x))^{P_n}\, dx} \leqq$$

$$\sqrt[P_n]{\frac{1}{2T}\int_{-T}^{T} (F_{n+q}(x) - F_n(x))^{P_n}\, dx} + \sqrt[P_n]{\frac{1}{2T}\int_{-T}^{T} (f_{n+q+1}(x))^{P_n}\, dx};$$

and as $f_{n+q+1}(x) = 0$ for

$$-\frac{m_{n+q+1} - 1}{2} h_{n+q} \leqq x < \frac{m_{n+q+1} - 1}{2} h_{n+q},$$

we plainly have

$$\sqrt[P_n]{\frac{1}{2T}\int_{-T}^{T} (f_{n+q+1}(x))^{P_n}\, dx} \leqq \sqrt[P_n]{\frac{1}{2\,\frac{m_{n+q+1} - 1}{2}\, h_{n+q}}\int_{-h_{n+q+1}}^{h_{n+q+1}} (f_{n+q+1}(x))^{P_n}\, dx},$$

and the above quantity is thus

$$\leqq \sqrt[p_n]{\frac{1}{2\nu h_{n+q}}\int_{-(\nu+1)h_{n+q}}^{(\nu+1)h_{n+q}}(F_{n+q}(x)-F_n(x))^{p_n}\,dx} + \sqrt[p_n]{\frac{1}{2\,\frac{m_{n+q+1}-1}{2}\,h_{n+q}}\int_{-h_{n+q+1}}^{h_{n+q+1}}(f_{n+q+1}(x))^{p_n}\,dx}.$$

As $(F_{n+q}(x)-F_n(x))^{p_n}$ is periodic with the period h_{n+q} and $(f_{n+q+1}(x))^{p_n}$ is periodic with the period h_{n+q+1}, the latter quantity is

$$= \sqrt[p_n]{\frac{\nu+1}{\nu}M\{(F_{n+q}(x)-F_n(x))^{p_n}\}} + \sqrt[p_n]{\frac{m_{n+q+1}}{\frac{m_{n+q+1}-1}{2}}M\{(f_{n+q+1}(x))^{p_n}\}} \leqq$$

$$\sqrt[p_n]{2\,M\{(F_{n+q}(x)-F_n(x))^{p_n}\}} + \sqrt[p_n]{4\,M\{(f_{n+q+1}(x))^{p_n}\}} \leqq$$

$$\sqrt[p_n]{4}\left(\sqrt[p_n]{M\{(F_{n+q}(x)-F_n(x))^{p_n}\}} + \sqrt[p_n]{M\{(f_{n+q+1}(x))^{p_n}\}}\right).$$

By an estimation of

$$\sqrt[p_n]{M\{(F_{n+q}(x)-F_n(x))^{p_n}\}}$$

in a way quite analogous to that on page 124 we see that the right-hand side is

$$\leqq \sqrt[p_n]{4}\left(\sqrt[p_{n+1}]{M\{(f_{n+1}(x))^{p_{n+1}}\}} + \sqrt[p_{n+2}]{M\{(f_{n+2}(x))^{p_{n+2}}\}} + \cdots + \sqrt[p_{n+q+1}]{M\{(f_{n+q+1}(x))^{p_{n+q+1}}\}}\right) \leqq$$

$$\sqrt[p_n]{4}\,(\delta_{n+1}+\delta_{n+2}+ \quad +\delta_{n+q+1}) \leqq \sqrt[p_n]{4}\,(\delta_{n+1}+\delta_{n+2}+\cdots).$$

Hence

$$\sqrt[p_n]{\frac{1}{2T}\int_{-T}^{T}(F(x)-F_n(x))^{p_n}\,dx} \leqq \sqrt[p_n]{4}\,(\delta_{n+1}+\delta_{n+2}+\cdots) \quad \text{for} \quad T \geqq h_n.$$

It results for $T\to\infty$ that $D_{B^{p_n}}[F(x), F_n(x)] \leqq \sqrt[p_n]{4}\,(\delta_{n+1}+\delta_{n+2}+\cdots)$ so that $D_{B^{p_n}}[F(x), F_n(x)] \to 0$ for $n\to\infty$.

Main Example III b.

This main example is formed in a manner quite analogous to main example III a, but is somewhat simpler in so far as all the towers are taken to be congruent, viz. with the breadth 1 and the height 1. The function $F(x)$ thus constructed will have the same properties corresponding to $P = 1$ as the function $F(x)$ of main example III a (where P was > 1) — with exception of course of the function being S^p-a. p. for $p < P$ — and the proofs can directly be transferred. It may be remarked, however, that, on account of $F(x)$ here being bounded, in order to prove that $F(x)$ is B^p-a. p. for all p, we need only to show that $F(x)$ is B-a. p.; hence in the proof we may put $P_1 = P_2 = \cdots = 1$. In this way we get *a function $F(x)$ which is an S^p-function (even bounded) and a B^p-a. p. function for all p and such that the B-point around $F(x)$ does not contain any W-a. p. function.*

Main Example IV.

We construct *a function $F(x)$ which is W^p-a. p. for all p and such that the B-point around $F(x)$ does not contain any S-a. p. function.*

Throughout this main example by a »tower« we shall always understand a tower with the height 1 and the breadth 1. Let $m_1, m_2, \ldots$ be arbitrary integers $\geqq 2$. As usual, we put $h_1 = m_1$, $h_2 = m_1 m_2$, $h_3 = m_1 m_2 m_3, \ldots$, and construct (cp. main examples I and II) a sequence of functions $F_1(x), F_2(x), \ldots$ in the following way:

$F_1(x)$:	On all numbers	$\equiv 0 \pmod{h_1}$		a tower is placed.
$F_2(x)$:	On all numbers	$\equiv 0 \pmod{h_1}$	but $\not\equiv 0 \pmod{h_2}$	a tower is placed.
	» » »	$\equiv 0 \pmod{h_2}$		no » » » .
$\vdots$				
$F_{2n}(x)$:	On all numbers	$\equiv 0 \pmod{h_1}$	but $\not\equiv 0 \pmod{h_2}$	a tower is placed.
	» » »	$\equiv 0 \pmod{h_2}$	» $\not\equiv 0 \pmod{h_3}$	no » » » .
	$\vdots$			
	» » »	$\equiv 0 \pmod{h_{2n-1}}$	» $\not\equiv 0 \pmod{h_{2n}}$	a » » » .
	» » »	$\equiv 0 \pmod{h_{2n}}$		no » » » .
$F_{2n+1}(x)$:	On all numbers	$\equiv 0 \pmod{h_1}$	but $\not\equiv 0 \pmod{h_2}$	a tower is placed.
	» » »	$\equiv 0 \pmod{h_2}$	» $\not\equiv 0 \pmod{h_3}$	no » » » .
	$\vdots$			
	» » »	$\equiv 0 \pmod{h_{2n}}$	» $\not\equiv 0 \pmod{h_{2n+1}}$	no » » » .
	» » »	$\equiv 0 \pmod{h_{2n+1}}$		a » » » .
$\vdots$				

(see Fig. 10 which represents $F_3(x)$ for $m_1 = m_2 = m_3 = 2$).

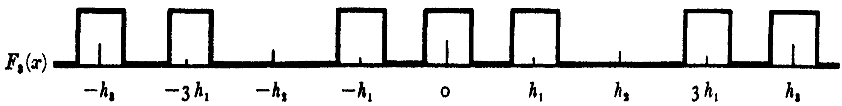


Fig. 10.

Obviously $F_n(x)$ is a bounded periodic function with the period h_n.

We begin by proving that $F_n(x)$ is W-convergent to the following function:

$$
\begin{array}{lllll}
F(x)\text{: On all numbers} & \equiv 0 \pmod{h_1} & \text{but } \not\equiv 0 \pmod{h_2} & \text{a tower is placed.} \\
\quad \text{»} \quad \text{»} \quad \text{»} & \equiv 0 \pmod{h_2} & \text{»} \quad \not\equiv 0 \pmod{h_3} & \text{no} \quad \text{»} \quad \text{»} \quad \text{»} \quad . \\
\quad\quad \vdots & & & \\
\quad \text{»} \quad \text{»} \quad \text{»} & \equiv 0 \pmod{h_{2n-1}} & \text{»} \quad \not\equiv 0 \pmod{h_{2n}} & \text{a} \quad \text{»} \quad \text{»} \quad \text{»} \quad . \\
\quad \text{»} \quad \text{»} \quad \text{»} & \equiv 0 \pmod{h_{2n}} & \text{»} \quad \not\equiv 0 \pmod{h_{2n+1}} & \text{no} \quad \text{»} \quad \text{»} \quad \text{»} \quad . \\
\quad\quad \vdots & & &
\end{array}
$$

If we leave the interval $-\frac{1}{2} \leqq x \leqq \frac{1}{2}$ out of account, obviously $F(x)$ can also be defined as $\lim\limits_{n \to \infty} F_n(x)$ (cp. the main examples I and II). $F(x)$ is a bounded function and differs from $F_n(x)$ at most on the numbers $m \equiv 0 \pmod{h_{n+1}}$, and we have for each such m

$$\int_{m-\frac{1}{2}}^{m+\frac{1}{2}} |F(x) - F_n(x)|\, dx \leqq 1,$$

viz. either 0 or 1. Hence

$$D_W[F(x), F_n(x)] \leqq \frac{1}{h_{n+1}}$$

which tends to 0 for $n \to \infty$, so that $F_n(x) \xrightarrow{W} F(x)$ for $n \to \infty$. Thus the function $F(x)$ is a W-limit periodic function, and $F(x)$ being bounded is therefore W^p-a. p. for all p.

Next we show that the B-point around $F(x)$ does not contain any S-a. p. function. Indirectly, we assume that $G(x)$ is such a function. Then we have

$$G(x) = F(x) + J(x)$$

where $J(x)$ is a B-zero function. Further, $F_n(x)$ being a sequence of periodic functions with the periods h_n which B-converge to $G(x)$, the period h_n is, in

consequence of Theorem 1a of Chapter I, for n sufficiently large, an S-translation number of $G(x)$ belonging to an arbitrary given $\varepsilon > 0$. We choose $\varepsilon = \frac{1}{2}$ and determine a fixed N so large that

$$(1) \qquad D_S[G(x+h_N), G(x)] \leqq \tfrac{1}{2}.$$

Let now m' denote numbers $\equiv 0 \pmod{h_N}$ but $\not\equiv 0 \pmod{h_{N+1}}$, and let m'' denote numbers $\equiv 0 \pmod{h_{N+1}}$ but $\not\equiv 0 \pmod{h_{N+2}}$. Either, in $F(x)$, there are towers on all numbers m' and none on the numbers m'', or conversely. By a translation h_N all m''-points are translated into certain of the m'-points, as

$$m'' + h_N \equiv 0 + h_N \not\equiv 0 \pmod{h_{N+1}}$$

and

$$m'' + h_N \equiv 0 + 0 = 0 \pmod{h_N}.$$

Hence

$$(2) \qquad \int\limits_{m''-\frac{1}{2}}^{m''+\frac{1}{2}} |F(x+h_N) - F(x)|\, dx = 1$$

for all the numbers m''. In consequence of (1) we have in particular

$$(3) \qquad \int\limits_{m''-\frac{1}{2}}^{m''+\frac{1}{2}} |G(x+h_N) - G(x)|\, dx \leqq \tfrac{1}{2}.$$

By (2) and (3) we get for the function $J(x) = G(x) - F(x)$

$$\int\limits_{m''-\frac{1}{2}}^{m''+\frac{1}{2}} |J(x+h_N) - J(x)|\, dx \geqq \tfrac{1}{2}.$$

Hence

$$D_B[J(x+h_N), J(x)] = \overline{\lim_{T\to\infty}}\, \frac{1}{2T} \int\limits_{-T}^{T} |J(x+h_N) - J(x)|\, dx \geqq \tfrac{1}{2}\, \frac{m_{N+2}-1}{h_{N+2}} > 0.$$

Consequently $D_B[J(x+h_N), J(x)] > 0$ which contradicts the fact that $J(x)$ is a B-zero function.

We observe that in this main example we were not forced to impose additional conditions on the sequence $m_1, m_2, \ldots$.

Main Examples V a, V b and VI.

In main example V a *a function* $F(x)$ *is constructed which is an* S^p*-a. p. function for* $p < P$, *not a* B^P*-function and such that the B-point around* $F(x)$ *contains a function* $G(x)$ *which is* B^p*-a. p. for all* p. Here P is an arbitrarily given number, $1 < P < \infty$.

In main example VI *a function* $F(x)$ *is constructed which is an* S^p*-a. p. function for* $p<P$, *an* S^P*-function, but no* B^P*-a. p. function and such that the B-point around* $F(x)$ *contains a function* $G(x)$ *which is* B^p*-a. p. for all* p. Here P is an arbitrarily given number, $1 < P < \infty$.

In main example V b *a function* $F(x)$ *is constructed which is an* S^P*-a. p. function, but not a* B^p*-function for any* $p > P$ *and such that the B-point around* $F(x)$ *contains a function* $G(x)$ *which is* B^p*-a. p. for all* p. Here P is an arbitrarily given number, $1 \leqq P < \infty$.

Main Examples V a and V b.

The two main examples V a and V b are constructed in an analogous way. In both cases we start from a positive function $t(x)$ defined for $-\frac{1}{2} \leqq x < \frac{1}{2}$ which is bounded in every interval $-\frac{1}{2} \leqq x \leqq \alpha < \frac{1}{2}$; in main example V a this function $t(x)$ is p-integrable for $p < P$ but not for $p = P$, while in main example V b the function $t(x)$ is P-integrable, but not p-integrable for $p > P$.

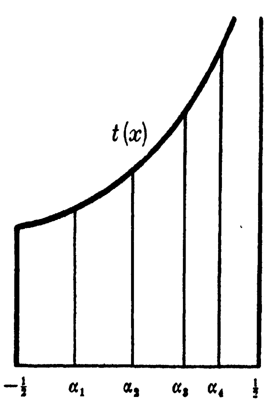


Fig. 11.

Let $-\frac{1}{2} < \alpha_1 < \alpha_2 < \cdots \to \frac{1}{2}$ (see Fig. 11). For $-\frac{1}{2} \leqq x < \frac{1}{2}$ we define

$$t_1(x) = \begin{cases} t(x) & \text{for } -\frac{1}{2} \leqq x < \alpha_1 \\ 0 & \text{elsewhere,} \end{cases}$$

$$t_2(x) = \begin{cases} t(x) & \text{for } \alpha_1 \leqq x < \alpha_2 \\ 0 & \text{elsewhere,} \end{cases}$$

$$t_3(x) = \begin{cases} t(x) & \text{for } \alpha_2 \leqq x < \alpha_3 \\ 0 & \text{elsewhere.} \end{cases}$$

$$\vdots$$

Let $m_1, m_2, \ldots$ be a sequence of integers $\geqq 2$ and let $h_1 = m_1$, $h_2 = m_1 m_2$, $h_3 = m_1 m_2 m_3, \ldots$. By $f_n(x)$ we denote the function arising from the function $t_n(x)$ by repeating it periodically with the period h_n, $n = 1, 2, \ldots$. We put (cp. main example I)

$$F_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x)$$

(see Fig. 12 where $m_1 = m_2 = m_3 = 2$ and $n = 3$).

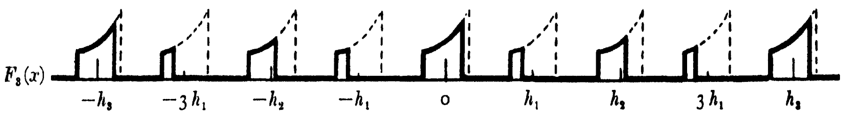


Fig. 12.

Further we put

$$F(x) = f_1(x) + f_2(x) + \cdots;$$

this last series is convergent for every x, since at most one of the terms is different from 0 for a given x. The function $F_n(x)$ is bounded and periodic with the period h_n.

It is easily seen that

$$F_n(x) \xrightarrow{S^p} F(x)$$

for $p<P$, respectively for $p=P$, so that $F(x)$ is S^p-a. p. for $p<P$, respectively for $p=P$; in fact for an arbitrary $\varepsilon>0$ we have

$$D_{S^p}[f_n(x) + f_{n+1}(x) + \cdots] < \varepsilon$$

for $p<P$, respectively for $p=P$, when $n >$ some $N = N(\varepsilon, p)$, as for $n \to \infty$

$$\int_{a_n}^{\frac{1}{2}} (t(x))^p \, dx \to 0$$

for $p<P$, respectively $p=P$.

The function $F(x)$ is no B^p-function for $p=P$, respectively for $p>P$, since $F(x) = t(x)$ for $-\frac{1}{2} \leqq x < \frac{1}{2}$ and $t(x)$ is not p-integrable for $p=P$, respectively for $p>P$.

Finally we shall show that the B-point (even the W-point) around $F(x)$ contains a function $G(x)$ which is B^p-a. p. for all p, if we only let the numbers $m_1, m_2, \ldots$ increase sufficiently rapidly to ∞.

Let $1 \leqq P_1 < P_2 < \cdots \to \infty$ and let $\sum_1^\infty \delta_n$ be a convergent series of positive numbers. We choose the number m_1 so large that $\sqrt[P_1]{M\{(f_1(x))^{P_1}\}} < \delta_1$, the number m_2 so large that $\sqrt[P_2]{M\{(f_2(x))^{P_2}\}} < \delta_2, \ldots .$

Subtracting from $F(x)$ the W-zero function

$$j(x) = \begin{cases} t(x) & \text{for } -\frac{1}{2} \leqq x < \frac{1}{2} \\ 0 & \text{elsewhere} \end{cases}$$

we get a function $G(x) = F(x) - j(x)$ which will prove to be B^p-a. p. for all p.

Putting

$$j_n(x) = \begin{cases} t(x) & \text{for } -\frac{1}{2} \leqq x < a_n \\ 0 & \text{elsewhere,} \end{cases}$$

and $G_n(x) = F_n(x) - j_n(x)$, we have

$$D_{B^{P_n}}[G(x), F_n(x)] = D_{B^{P_n}}[G(x), G_n(x)],$$

since $j_n(x)$ is a W^p-zero function for all p. Further

$$D_{B^{P_n}}[G(x), G_n(x)] = \overline{\lim_{T \to \infty}} \sqrt[P_n]{\frac{1}{2T} \int_{-T}^{T} (G(x) - G_n(x))^{P_n}\, dx},$$

and we shall therefore estimate

$$\frac{1}{2T} \int_{-T}^{T} (G(x) - G_n(x))^{P_n}\, dx$$

for fixed n and large T, say $T \geqq h_n$, proceeding in a similar way as in the main examples I, II and III. We determine first $q \geqq 0$ so that $h_{n+q} \leqq T < h_{n+q+1}$ and then ν among the numbers $1, 2, \ldots, m_{n+q+1} - 1$ so that $\nu h_{n+q} \leqq T < (\nu + 1) h_{n+q}$.

To begin with we distinguish between the two cases $\nu \leqq m_{n+q+1} - 2$ and $\nu = m_{n+q+1} - 1$.

In the first case we get

$$\frac{1}{2T} \int_{-T}^{T} (G(x) - G_n(x))^{P_n}\, dx \leqq \frac{1}{2\nu h_{n+q}} \int_{-(\nu+1)h_{n+q}}^{(\nu+1)h_{n+q}} (G(x) - G_n(x))^{P_n}\, dx =$$

$$\frac{1}{2\nu h_{n+q}} \int_{-(\nu+1)h_{n+q}}^{(\nu+1)h_{n+q}} (G_{n+q}(x) - G_n(x))^{P_n}\, dx,$$

as $G(x) = G_{n+q}(x)$ for

$$-(\nu+1)h_{n+q} \leqq x < (\nu+1)h_{n+q}, \quad \nu = 1, 2, \ldots, m_{n+q+1} - 2.$$

Here the right-hand side is

$$\leqq \frac{1}{2\nu h_{n+q}} \int\limits_{-(\nu+1)h_{n+q}}^{(\nu+1)h_{n+q}} (F_{n+q}(x) - F_n(x))^{p_n}\, dx,$$

as

$$0 \leqq G_{n+q}(x) - G_n(x) = F_{n+q}(x) - F_n(x) - (j_{n+q}(x) - j_n(x)) \leqq F_{n+q}(x) - F_n(x),$$

and this quantity is

$$= \frac{\nu+1}{\nu} M\{(F_{n+q}(x) - F_n(x))^{p_n}\},$$

as $(F_{n+q}(x) - F_n(x))^{p_n}$ is periodic with the period h_{n+q}.

In the other case we get

$$\frac{1}{2T}\int\limits_{-T}^{T} (G(x) - G_n(x))^{p_n}\, dx \leqq \frac{1}{2\nu h_{n+q}} \int\limits_{-h_{n+q+1}}^{h_{n+q+1}} (G(x) - G_n(x))^{p_n}\, dx =$$

$$\frac{1}{2\nu h_{n+q}} \int\limits_{-h_{n+q+1}}^{h_{n+q+1}} (G_{n+q+1}(x) - G_n(x))^{p_n}\, dx,$$

as $G(x) = G_{n+q+1}(x)$ for $-h_{n+q+1} \leqq x < h_{n+q+1}$. Here the right-hand side is

$$\leqq \frac{1}{2\nu h_{n+q}} \int\limits_{-h_{n+q+1}}^{h_{n+q+1}} (F_{n+q+1}(x) - F_n(x))^{p_n}\, dx,$$

as $0 \leqq G_{n+q+1}(x) - G_n(x) \leqq F_{n+q+1}(x) - F_n(x)$, and this is further

$$= \frac{\nu+1}{\nu} M\{(F_{n+q+1}(x) - F_n(x))^{p_n}\},$$

as $(F_{n+q+1}(x) - F_n(x))^{p_n}$ is periodic with the period h_{n+q+1}.

Estimating $M\{(F_{n+q}(x) - F_n(x))^{p_n}\}$ in the same manner as on page 124 we get

$$\sqrt[p_n]{M\{(F_{n+q}(x) - F_n(x))^{p_n}\}} \leqq \sqrt[p_{n+1}]{M\{(f_{n+1}(x))^{p_{n+1}}\}} + \sqrt[p_{n+2}]{M\{(f_{n+2}(x))^{p_{n+2}}\}} + \cdots$$

$$+ \sqrt[p_{n+q}]{M\{(f_{n+q}(x))^{p_{n+q}}\}} \leqq \delta_{n+1} + \delta_{n+2} + \cdots + \delta_{n+q} \leqq \delta_{n+1} + \delta_{n+2} + \cdots.$$

Thus we get in both cases, as $\frac{\nu + 1}{\nu} \leqq 2$,

$$\sqrt[p_n]{\frac{1}{2T}\int_{-T}^{T}\left(G(x) - G_n(x)\right)^{p_n} dx} \leqq \sqrt[p_n]{2}\,(\delta_{n+1} + \delta_{n+2} + \cdots)$$

for $T \geqq h_n$. Letting $T \to \infty$ we get

$$D_{B^{p_n}}[G(x),\, G_n(x)] \leqq \sqrt[p_n]{2}\,(\delta_{n+1} + \delta_{n+2} + \cdots).$$

Since $D_{B^{p_n}}[G(x), F_n(x)] = D_{B^{p_n}}[G(x), G_n(x)]$, it results that $D_{B^{p_n}}[G(x), F_n(x)] \to 0$ for $n \to \infty$. Hence

$$F_n(x) \xrightarrow{B^p} G(x) \text{ for all } p,$$

and consequently $G(x)$ is a B^p-a. p. function for all p.

Main Example VI.

This main example is constructed in a similar way as main example III a. As in that example we construct a sequence $F_1(x), F_2(x), \ldots$ of bounded periodic functions with the periods $h_1 = m_1$, $h_2 = m_1 m_2, \ldots$ and consider $F(x) = \lim_{n\to\infty} F_n(x)$. In main example III a we obtained $F_{n+1}(x)$ from $F_n(x)$ by filling out the central one of the subintervals $\eta h_n \leqq x < (\eta + 1) h_n$ of every interval $\nu h_{n+1} \leqq x < (\nu + 1) h_{n+1}$ by towers of type $n+1$, i. e. by towers with the 1-integral ε_{n+1} and the P-integral 1. Now, however, instead of filling out the central one of these subintervals we fill out the first, i. e. that farthest to the left (see Fig. 13 where $m_1 = m_2 = 3$ and $n = 2$).

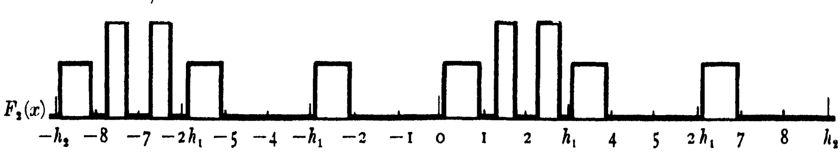


Fig. 13.

As in main example III a we denote the added function $F_{n+1}(x) - F_n(x)$ by $f_{n+1}(x)$ $(f_1(x) = F_1(x))$ and assume that

$$\left(1 - \frac{1}{m_1}\right)\left(1 - \frac{1}{m_2}\right)\cdots$$

is convergent. Since this time we fill out the first interval instead of the central one, it does not hold that $F(x) = F_n(x)$ for $-h_n \leqq x < h_n$, but only that $F(x) = F_n(x)$ for $-h_n \leqq x < h_{n-1}$; but obviously it is still valid that $F_n(x) \xrightarrow{S} F(x)$ so that $F(x)$ is S-a. p., and that $F(x)$ is an S^p-function.

While $F(x)$ of main example III a is B^p-a. p. for all p for a suitable choice of $m_1, m_2, \ldots$ we shall now show that in the present case $F(x)$ is not B^p-a. p. We prove this by showing that

$$D^*_{B^p}\left[F(x), (F(x))_N\right] \geqq \frac{1}{2}\sqrt[p]{\left(1-\frac{1}{m_1}\right)\left(1-\frac{1}{m_2}\right)\cdots} > 0$$

for every $N > 0$; this involves that $F(x)$ is not B^p-a. p., as otherwise (see Chapter I) $D^*_{B^p}[F(x), (F(x))_N] \to 0$ for $N \to \infty$.

Let, then, N be an arbitrary number > 0. The height k_n of a tower of type n being equal to $\left(\frac{1}{\varepsilon_n}\right)^{\frac{1}{p-1}}$ (see page 43) tends to ∞ for $n \to \infty$. We choose N_1 so large that $k_n \geqq 2N$ for $n \geqq N_1$. If $t(x)$ denotes a tower of type n for $n \geqq N_1$ standing on the interval $\eta \leqq x < \eta + 1$ we have

$$\sqrt[p]{\int_{\eta}^{\eta+1}\left(t(x) - (t(x))_N\right)^p dx} \geqq \tfrac{1}{2}.$$

We consider $F(x)$ in the interval $0 \leqq x < h_{n-1}$. In this interval $F(x) = F_n(x)$, and $F_n(x)$ contains $h_{n-1}\left(1-\frac{1}{m_1}\right)\left(1-\frac{1}{m_2}\right)\cdots\left(1-\frac{1}{m_{n-1}}\right)$ towers of type n, namely all the towers of type n which were filled into the first of the subintervals $\nu h_{n-1} \leqq x < (\nu+1) h_{n-1}$ of the interval $0 \leqq x < h_n$ when passing from $F_{n-1}(x)$ to $F_n(x)$ (cp. page 86). Therefore for all $n \geqq N_1$ we have

$$\sqrt[p]{\frac{1}{h_{n-1}}\int_0^{h_{n-1}}\left(F(x) - (F(x))_N\right)^p dx} \geqq$$

$$\sqrt[p]{\frac{1}{h_{n-1}} h_{n-1}\left(1-\frac{1}{m_1}\right)\left(1-\frac{1}{m_2}\right)\cdots\left(1-\frac{1}{m_{n-1}}\right)\left(\frac{1}{2}\right)^p} =$$

$$\frac{1}{2}\sqrt[p]{\left(1-\frac{1}{m_1}\right)\left(1-\frac{1}{m_2}\right)\cdots\left(1-\frac{1}{m_{n-1}}\right)} \geqq \frac{1}{2}\sqrt[p]{\left(1-\frac{1}{m_1}\right)\left(1-\frac{1}{m_2}\right)\cdots};$$

and thus

$$\overline{\lim_{T\to\infty}} \sqrt[P]{\frac{1}{T}\int_0^T (F(x)-(F(x))_N)^P\,dx} \geqq \frac{1}{2}\sqrt[P]{\left(1-\frac{1}{m_1}\right)\left(1-\frac{1}{m_2}\right)\cdots}.$$

Hence

$$D^*_{B^P}[F(x),\,(F(x))_N] \geqq \frac{1}{2}\sqrt[P]{\left(1-\frac{1}{m_1}\right)\left(1-\frac{1}{m_2}\right)\cdots}.$$

Finally we shall prove that, by letting $m_1, m_2, \ldots$ increase sufficiently rapidly to ∞, we can obtain, that the B-point (even the W-point) around $F(x)$ contains a function $G(x)$ which is B^p-a. p. for all p.

Let $1 \leqq P_1 < P_2 < \cdots \to \infty$ and let $\sum_1^\infty \delta_n$ be a convergent series of positive numbers. We denote the P_n-integral of a tower of type n by I_n and choose m_n, $n = 1, 2, \ldots$, so large that

$$\sqrt[P_n]{\frac{I_n}{m_n}} < \delta_n$$

and thus (see page 128)

$$\sqrt[P_n]{M\{(f_n(x))^{P_n}\}} < \delta_n.$$

Let

$$f_n^*(x) = \begin{cases} f_n(x) & \text{for } -h_n \leqq x < h_n \\ 0 & \text{elsewhere} \end{cases}, \quad n = 1, 2, \ldots.$$

Corresponding to $F(x) = f_1(x) + f_2(x) + \cdots$ we form the function

$$j(x) = f_1^*(x) + f_2^*(x) + \cdots$$

(the series is convergent, since for a given x at most one of the terms is $\neq 0$).

We shall prove that $j(x)$ is a W-zero function and that the difference $G(x) = F(x) - j(x)$ is B^p-a. p. for all p.

It is easily seen that $j(x)$ is a W-zero function; for outside the interval $-h_n \leqq x < h_n$ the towers of $j(x)$ are all of types $\geqq n+1$ and such towers have 1-integrals $\leqq \varepsilon_{n+1}$.

Next we prove that $G(x) = F(x) - j(x)$ is B^p-a. p. for all p by showing that $F_n(x) \overset{B^p}{\to} G(x)$ for all p. To this purpose, corresponding to

$$F_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x) \quad \text{we put} \quad j_n(x) = f_1^*(x) + f_2^*(x) + \cdots + f_n^*(x)$$

and consider the function $G_n(x) = F_n(x) - j_n(x)$, $n = 1, 2, \ldots$. Obviously $j_n(x)$ is a W^p-zero function for all p and $G(x) = \lim\limits_{n\to\infty} G_n(x)$. Further

$$G_{n+1}(x) = G_n(x) + f_{n+1}(x) - f_{n+1}^*(x),$$

since

$$G_{n+1}(x) = F_{n+1}(x) - j_{n+1}(x) = F_n(x) + f_{n+1}(x) - j_n(x) - f_{n+1}^*(x) = G_n(x) + f_{n+1}(x) - f_{n+1}^*(x).$$

Hence we have, on account of the definition of $f_{n+1}^*(x)$,

$$G_{n+1}(x) = G_n(x) \quad \text{for} \quad -h_{n+1} \leqq x < h_{n+1};$$

successively applying this equation, we get

$$G(x) = G_n(x) \quad \text{for} \quad -h_{n+1} \leqq x < h_{n+1}.$$

As $j_n(x)$ is a W^p-zero function (and hence a fortiori a B^p-zero function) for all p, we have

$$D_{B^{p_n}}[G(x), F_n(x)] = D_{B^{p_n}}[(G(x), G_n(x)] = \overline{\lim_{T\to\infty}} \sqrt[p_n]{\frac{1}{2T}\int_{-T}^{T} (G(x) - G_n(x))^{p_n}\, dx}.$$

Thus we shall estimate

$$\frac{1}{2T}\int_{-T}^{T} (G(x) - G_n(x))^{p_n}\, dx$$

for fixed n and large T, say $T \geqq h_n$ (cp. the main examples I, II, III, V). First $q \geqq 0$ is determined so that $h_{n+q} \leqq T < h_{n+q+1}$, and next ν among the numbers $1, 2, \ldots, m_{n+q+1} - 1$ so that $\nu h_{n+q} \leqq T < (\nu+1) h_{n+q}$. Then we have

$$\frac{1}{2T}\int_{-T}^{T} (G(x) - G_n(x))^{p_n}\, dx \leqq \frac{1}{2\nu h_{n+q}} \int_{-(\nu+1)h_{n+q}}^{(\nu+1)h_{n+q}} (G(x) - G_n(x))^{p_n}\, dx =$$

$$\frac{1}{2\nu h_{n+q}} \int_{-(\nu+1)h_{n+q}}^{(\nu+1)h_{n+q}} (G_{n+q}(x) - G_n(x))^{p_n}\, dx,$$

since

$$G(x) = G_{n+q}(x) \quad \text{for} \quad -h_{n+q+1} \leqq x < h_{n+q+1}.$$

As $0 \leqq G_{n+q}(x) - G_n(x) \leqq F_{n+q}(x) - F_n(x)$, the right-hand side is

$$\leqq \frac{1}{2\nu h_{n+q}} \int\limits_{-(\nu+1)h_{n+q}}^{(\nu+1)h_{n+q}} (F_{n+q}(x) - F_n(x))^{p_n}\, dx,$$

and this is

$$= \frac{\nu+1}{\nu} M\{(F_{n+q}(x) - F_n(x))^{p_n}\},$$

as $(F_{n+q}(x) - F_n(x))^{p_n}$ is periodic with the period h_{n+q}. Further, in consequence of the estimation on page 124 the last quantity is

$$\leqq 2\left(\sqrt[p_{n+1}]{M\{(f_{n+1}(x))^{p_{n+1}}\}} + \sqrt[p_{n+2}]{M\{(f_{n+2}(x))^{p_{n+2}}\}} + \cdots + \sqrt[p_{n+q}]{M\{(f_{n+q}(x))^{p_{n+q}}\}}\right)^{p_n} \leqq$$

$$2(\delta_{n+1} + \delta_{n+2} + \cdots + \delta_{n+q})^{p_n} \leqq 2(\delta_{n+1} + \delta_{n+2} + \cdots)^{p_n}.$$

Hence for $T \geqq h_n$ we have

$$\sqrt[p_n]{\frac{1}{2T}\int\limits_{-T}^{T} (G(x) - G_n(x))^{p_n}\, dx} \leqq \sqrt[p_n]{2}\,(\delta_{n+1} + \delta_{n+2} + \cdots);$$

letting $T \to \infty$, we get

$$D_{B^{p_n}}[G(x), G_n(x)] \leqq \sqrt[p_n]{2}\,(\delta_{n+1} + \delta_{n+2} + \cdots).$$

Since

$$D_{B^{p_n}}[G(x), F_n(x)] = D_{B^{p_n}}[G(x), G_n(x)],$$

we conclude that $D_{B^{p_n}}[G(x), F_n(x)] \to 0$ for $n \to \infty$ and consequently that $F_n(x) \xrightarrow{B^p} G(x)$ for all p.

Main Examples VII a, VII b and VII c.

The number α, $1 < \alpha < \infty$, being arbitrarily given, in all the three main examples we construct *a function* $F(x)$ *which is an* S^p*-a. p. function for* $p<\alpha$, *an* S^α*-function and such that the B-point around* $F(x)$ *contains a function* $G(x)$ *which is* B^p*-a. p. for all* p.

In main example VII a the number P being arbitrarily given such that $\alpha < P < \infty$, and in the main examples VII b and VII c the number P being arbitrarily given such that $\alpha \leqq P < \infty$, in the different examples the function $F(x)$ has further the following properties.

In main example VII a: $F(x)$ *is* B^p*-a. p. for* $p<P$, *and the* W^α*-point around* $F(x)$ *contains no* B^P*-functions.*

In main example VII b: $F(x)$ *is a* B^P*-function, and the* W^α*-point around* $F(x)$ *contains no* B^P*-a. p. functions.* In the case $P=\alpha$ it results already from the above that $F(x)$ is a B^α-function; in fact it is even an S^α-function.

In main example VII c: $F(x)$ *is a* B^P*-a. p. function, and the* W^α*-point around* $F(x)$ *contains no* B^p*-functions for* $p > P$.

We remark that in the later paper, where the examples of the appendix will be used, we shall see that, on account of general theorems, the B-points around the functions $F(x)$ of the three main examples cannot contain W^α-a. p. functions.

The main examples VII a, VII b and VII c are constructed in an analogous way, and they are of a similar type as the main examples III a and VI. Just as in these examples, we construct a sequence $F_1(x)$, $F_2(x)$, ... of bounded periodic functions with the periods $h_1 = m_1$, $h_2 = m_1 m_2$, ... where

$$\left(1-\frac{1}{m_1}\right)\left(1-\frac{1}{m_2}\right)\cdots$$

is convergent and consider $F(x) = \lim\limits_{n\to\infty} F_n(x)$. In main example III a, respectively VI, we passed from $F_n(x)$ to $F_{n+1}(x)$ by filling out the central, respectively the first, of the subintervals $\mu h_n \leqq x < (\mu+1) h_n$ of every interval $\nu h_{n+1} \leqq x < (\nu+1) h_{n+1}$ by towers of type $n+1$, i. e. towers with the 1-integral ε_{n+1} and the P-integral 1. In the present construction, however, α takes the place of P, so that a tower of type n means a tower with the 1-integral ε_n and the α-integral 1. Further,

by the transition from $F_n(x)$ to $F_{n+1}(x)$ we do not fill out just the central or the first of the subintervals $\mu h_n \leqq x < (\mu + 1) h_n$ by towers of type $n + 1$, but another of the subintervals, later precisely indicated. The subintervals to be filled out shall of course as usual lie periodically with the period h_{n+1}. As we shall see, by a suitable choice of these intervals, we can obtain that $F(x)$ gets the desired »B-properties«. Let the subinterval $\nu h_n \leqq x < (\nu + 1) h_n$ of the interval $0 \leqq x < h_{n+1}$ which is filled out at the transition from $F_n(x)$ to $F_{n+1}(x)$ be denoted by $\nu_{n+1} h_n \leqq x < (\nu_{n+1} + 1) h_n$. For the sake of convenience we shall choose

$$\nu_{n+1} < \frac{m_{n+1}}{2}$$

so that the interval $\nu_{n+1} h_n \leqq x < (\nu_{n+1} + 1) h_n$ is that (or eventually one of the two) of the subintervals filled out at the mentioned transition which lies nearest to 0.

It is plain that $F_n(x) \xrightarrow{S} F(x)$ for $n \to \infty$, as the 1-integral ε_n of a tower of type n tends to 0 for $n \to \infty$, and thus the function $F(x)$ is an S-a. p. function. Further all towers of $F(x)$ having the α-integral 1, the function $F(x)$ is an S^α-function.

We introduce similar notions as in main example VI. We put $f_1(x) = F_1(x)$, $f_2(x) = F_2(x) - F_1(x)$, $f_3(x) = F_3(x) - F_2(x), \ldots$, so that $F_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x)$ and $F(x) = f_1(x) + f_2(x) + \cdots$. Further we put

$$f_n^*(x) = \begin{cases} f_n(x) & \text{for } -h_n \leqq x < h_n \\ 0 & \text{elsewhere,} \end{cases}$$

$$j_n(x) = f_1^*(x) + f_2^*(x) + \quad + f_n^*(x), \quad j(x) = f_1^*(x) + f_2^*(x) + \cdots,$$

$$G_n(x) = F_n(x) - j_n(x), \quad G(x) = F(x) - j(x).$$

For $h_{n+q} \leqq T < h_{n+q+1}$ we get the estimation (cp. the analogous estimation on pages 129—130)

$$\sqrt[p]{\frac{1}{2T}\int_{-T}^{T} (F(x) - F_n(x))^p \, dx} \leqq \sqrt[p]{2} \left(\sqrt[p]{M\{(f_{n+1}(x))^p\}} + \cdots + \sqrt[p]{M\{(f_{n+q}(x))^p\}} \right) + \sqrt[p]{\frac{1}{2\,\nu_{n+q+1} h_{n+q}} \int_{-h_{n+q+1}}^{h_{n+q+1}} (f_{n+q+1}^*(x))^p \, dx}$$

$\left(\text{if } \nu_{n+q+1} = 0 \text{ we put } \frac{1}{0} = \infty\right)$, or, introducing the notion

$$A_p(n) = \sqrt[p]{\frac{1}{2\,\nu_n h_{n-1}} \int_{-h_n}^{h_n} (f_n^*(x))^p\, dx},$$

the estimation

$$(1) \quad \sqrt[p]{\frac{1}{2\,T} \int_{-T}^{T} (F(x) - F_n(x))^p\, dx} \leqq$$
$$\sqrt[p]{2}\left(\sqrt[p]{M\{(f_{n+1}(x))^p\}} + \cdots + \sqrt[p]{M\{(f_{n+q}(x))^p\}}\right) + A_p(n+q+1).$$

Further we have (cp. page 137)

$$(2) \quad \sqrt[p]{\frac{1}{2\,T} \int_{-T}^{T} (G(x) - G_n(x))^p\, dx} \leqq \sqrt[p]{2}\left(\sqrt[p]{M\{(f_{n+1}(x))^p\}} + \cdots + \sqrt[p]{M\{(f_{n+q}(x))^p\}}\right).$$

Let $1 \leqq P_1 < P_2 < \cdots \to \infty$ and let $\sum_{1}^{\infty} \delta_n$ be a convergent series of positive numbers. We let $m_1, m_2, \ldots$ increase so strongly that (cp. page 128)

$$(3) \quad \sqrt[P_n]{M\{(f_n(x))^{P_n}\}} < \delta_n.$$

For $h_{n+q} \leqq T < h_{n+q+1}$ we get from (2) and (3)

$$\sqrt[P_n]{\frac{1}{2\,T} \int_{-T}^{T} (G(x) - G_n(x))^{P_n}\, dx} \leqq \sqrt[P_n]{2}\left(\sqrt[P_n]{M\{(f_{n+1}(x))^{P_n}\}} + \cdots + \sqrt[P_n]{M\{(f_{n+q}(x))^{P_n}\}}\right) \leqq$$
$$\sqrt[P_n]{2}\left(\sqrt[P_{n+1}]{M\{(f_{n+1}(x))^{P_{n+1}}\}} + \cdots + \sqrt[P_{n+q}]{M\{(f_{n+q}(x))^{P_{n+q}}\}}\right) \leqq$$
$$\sqrt[P_n]{2}\,(\delta_{n+1} + \cdots + \delta_{n+q}) \leqq \sqrt[P_n]{2}\,(\delta_{n+1} + \delta_{n+2} + \cdots).$$

Hence

$$D_B^{P_n}[G(x), G_n(x)] \leqq \sqrt[P_n]{2}\,(\delta_{n+1} + \delta_{n+2} + \cdots),$$

and, $j_n(x)$ being a W^p-zero function for all p, we have

$$D_{B^{p_n}}[G(x), F_n(x)] = D_{B^{p_n}}[G(x), G_n(x)] \leqq \sqrt[p_n]{2}\,(\delta_{n+1} + \delta_{n+2} + \cdots)$$

so that $D_{B^{p_n}}[G(x), F_n(x)] \to 0$ for $n \to \infty$. Thus $G(x)$ is a B^p-a. p. function for all p. Since $j(x)$ is a W-zero function (cp. main example VI), $G(x)$ lies in the W-point around $F(x)$ and in particular in the B-point around $F(x)$.

Having discussed the common properties of the main examples VII a, VII b and VII c we now pass to consider these examples separately, as regards their mutual differences.

Main Example VII a.

We wish to choose the numbers $\nu_1, \nu_2, \ldots$ defined above so that $F(x)$ becomes B^p-a. p. for $p < P$ and so that the W^α-point around $F(x)$ does not contain any B^P-function.

We shall first show that we can choose $\nu_1, \nu_2, \ldots$ so that

$$A_p(n) \to 0 \quad \text{for} \quad p < P \quad \text{and} \quad n \to \infty$$

and

$$A_P(n) \to \infty \quad \text{for} \quad n \to \infty;$$

later we shall show that $F(x)$ then gets the desired properties.

A necessary condition for $A_P(n) \to \infty$ is that $\frac{\nu_n}{m_n} \to 0$. For, as

$$A_P(n) = \sqrt[P]{\frac{1}{2\,\nu_n\,h_{n-1}} \int_{-h_n}^{h_n} (f_n^*(x))^P\,dx} =$$

$$\sqrt[P]{\frac{m_n}{\nu_n} \cdot \frac{1}{2\,h_n} \int_{-h_n}^{h_n} (f_n^*(x))^P\,dx} = \sqrt[P]{\frac{m_n}{\nu_n}} \cdot \sqrt[P]{M\{(f_n(x))^P\}},$$

the relation $A_P(n) \to \infty$ obviously involves the relation $\frac{m_n}{\nu_n} \to \infty$, since, on account of $\sqrt[p_n]{M\{(f_n(x))^{p_n}\}} \to 0$, we have $\sqrt[P]{M\{(f_n(x))^P\}} \to 0$.

As the p-integral of a tower of type n is equal to $\varepsilon_n^{\frac{\alpha-p}{\alpha-1}}$ we have

$$A_p(n)=\sqrt[p]{\frac{1}{\nu_n}\,d_{n-1}\,\varepsilon_n^{\frac{\alpha-p}{\alpha-1}}}=\sqrt[p]{\frac{1}{\nu_n}\,d_{n-1}\left(\frac{1}{\varepsilon_n}\right)^{\frac{p-\alpha}{\alpha-1}}}$$

where d_{n-1} has the same meaning as in main example 3, i. e. indicates the relative density of the empty intervals $\eta \leqq x < \eta + 1$ in the function $F_{n-1}(x)$. Thus, denoting $d_{n-1}\left(\frac{1}{\varepsilon_n}\right)^{\frac{p-\alpha}{\alpha-1}}$ by $B_p(n)$, we have

$$A_p(n)=\sqrt[p]{\frac{1}{\nu_n}B_p(n)}.$$

Obviously we can choose a sequence of numbers $(\alpha <)p_1 < p_2 < \cdots \rightarrow P$ which converges so slowly to P that

$$\frac{B_P(n)}{B_{p_n}(n)}=\frac{d_{n-1}\left(\frac{1}{\varepsilon_n}\right)^{\frac{P-\alpha}{\alpha-1}}}{d_{n-1}\left(\frac{1}{\varepsilon_n}\right)^{\frac{p_n-\alpha}{\alpha-1}}}=\left(\frac{1}{\varepsilon_n}\right)^{\frac{P-p_n}{\alpha-1}}\rightarrow\infty \quad \text{for} \quad n\rightarrow\infty.$$

We shall show that as our ν_n we may, from a certain step N (to be indicated below), use

$$\nu_n=\left[\frac{1}{\log\frac{B_P(n)}{B_{p_n}(n)}}B_P(n)\right]$$

(where $[x]$ denotes the greatest integer $\leqq x$). In fact we shall show that, choosing ν_n in this manner, $A_p(n)\rightarrow 0$ for $p<P$, $A_P(n)\rightarrow\infty$ and (for N sufficiently large) $\nu_n < \frac{m_n}{2}$.

We start by observing that $\nu_n\rightarrow\infty$ (and thus especially $\nu_n \geqq 1$ for n sufficiently large). This results from

$$\nu_n=\left[\frac{1}{\log\frac{B_P(n)}{B_{p_n}(n)}}B_P(n)\right]=\left[\frac{1}{\frac{P-p_n}{\alpha-1}\log\frac{1}{\varepsilon_n}}d_{n-1}\left(\frac{1}{\varepsilon_n}\right)^{\frac{P-\alpha}{\alpha-1}}\right]=\left[\frac{1}{\frac{P-p_n}{\alpha-1}}d_{n-1}\frac{\left(\frac{1}{\varepsilon_n}\right)^{\frac{P-\alpha}{\alpha-1}}}{\log\left(\frac{1}{\varepsilon_n}\right)}\right]$$

where

$$\frac{P-p_n}{\alpha-1} \to 0, \qquad d_{n-1} \to \left(1-\frac{1}{m_1}\right)\left(1-\frac{1}{m_2}\right)\cdots > 0$$

and

$$\frac{\left(\frac{1}{\varepsilon_n}\right)^{\frac{P-\alpha}{\alpha-1}}}{\log \frac{1}{\varepsilon_n}} \to \infty.$$

In particular

$$\nu_n \sim \frac{1}{\log \frac{B_P(n)}{B_{p_n}(n)}} B_P(n) \quad \text{for} \quad n \to \infty.$$

Then we have

$$A_{p_n}(n) = \sqrt[p_n]{\frac{1}{\nu_n} B_{p_n}(n)} \sim \sqrt[p_n]{\log \frac{B_P(n)}{B_{p_n}(n)} \cdot \frac{B_{p_n}(n)}{B_P(n)}} \to 0,$$

since

$$\frac{B_P(n)}{B_{p_n}(n)} \to \infty,$$

so that $A_{p_n}(n) \to 0$. Now, for a fixed $p < P$ and n being chosen so large that $p_n > p$ (and $\nu_n \geqq 1$), we have

$$A_p(n) \leqq A_{p_n}(n);$$

in fact, as

$$A_p(n) = \sqrt[p]{\frac{1}{2\,\nu_n\, h_{n-1}} \int_{-h_n}^{h_n} (f_n^*(x))^p\, dx},$$

the inequality $A_p(n) \leqq A_{p_n}(n)$ follows from Hölder's inequality, since $f_n^*(x)$ is different from 0 at most on intervals with total length $\leqq 2h_{n-1}$ and a fortiori with total length $\leqq 2\nu_n h_{n-1}$. Thus, as $A_{p_n}(n) \to 0$, we have

$$A_p(n) \to 0 \quad \text{for} \quad p < P.$$

Further we have

$$A_P(n) = \sqrt[P]{\frac{1}{\nu_n} B_P(n)} \sim \sqrt[P]{\log \frac{B_P(n)}{B_{p_n}(n)}} \to \infty$$

so that

$$A_P(n) \to \infty.$$

As mentioned above this involves $\frac{\nu_n}{m_n} \to 0$, and therefore we can determine our N so large that the last claim $\nu_n < \frac{m_n}{2}$ for $n \geqq N$ is satisfied. For $n < N$ we choose the ν_n arbitrarily so that merely $\nu_n < \frac{m_n}{2}$.

We shall now show that by this choice of the numbers ν_n the function $F(x)$ gets the desired properties.

First we shall see that $F(x)$ is B^p-a. p. for $p < P$. This results from (1) and (3), since for $h_{n+q} \leqq T < h_{n+q+1}$ and n so large that $P_{n+1} > p$ we have

$$\sqrt[p]{\frac{1}{2T}\int_{-T}^{T}(F(x)-F_n(x))^p\,dx} \leqq$$

$$\sqrt[p]{2}\left(\sqrt[p]{M\{(f_{n+1}(x))^p\}} + \cdots + \sqrt[p]{M\{(f_{n+q}(x))^p\}}\right) + A_p(n+q+1) \leqq$$

$$\sqrt[p]{2}\left(\sqrt[P_{n+1}]{M\{(f_{n+1}(x))^{P_{n+1}}\}} + \cdots + \sqrt[P_{n+q}]{M\{(f_{n+q}(x))^{P_{n+q}}\}}\right) + A_p(n+q+1) \leqq$$

$$\sqrt[p]{2}\,(\delta_{n+1} + \delta_{n+2} + \cdots + \delta_{n+q}) + A_p(n+q+1),$$

and hence, letting $q \to \infty$,

$$D_{B^p}[F(x), F_n(x)] \leqq \sqrt[p]{2}\,(\delta_{n+1} + \delta_{n+2} + \cdots).$$

Thus $D_{B^p}[F(x), F_n(x)] \to 0$ for $n \to \infty$, and consequently $F(x)$ is a B^p-a. p. function for $p < P$.

Next we show that the W^α-point around $F(x)$ does not contain any B^P-function. Proceeding indirectly, we suppose that there is a W^α-zero function $J(x)$ so that $D^*_{B^P}[F(x)+J(x)] < \infty$. Denoting by $J^*(x)$ the function which is equal to $J(x)$ where $j(x) \neq 0$ and equal to 0 elsewhere, we have

$$D^*_{B^P}[j(x) + J^*(x)] \leqq D^*_{B^P}[F(x) + J(x)] < \infty,$$

since $F(x) = j(x)$ where $j(x) \neq 0$. Let $D^*_{B^P}[j(x) + J^*(x)] = K$. For $n \geqq$ some sufficiently large N_1 we have

$$\sqrt[P]{\frac{1}{(\nu_n+1)h_{n-1}}\int_0^{(\nu_n+1)h_{n-1}} |j(x) + J^*(x)|^P\,dx} \leqq 2K;$$

in particular we have, denoting by $J_n^*(x)$ the function which is equal to $J^*(x)$ where $f_n^*(x) \neq 0$ and equal to 0 elsewhere,

$$\sqrt[p]{\frac{1}{(\nu_n+1)h_{n-1}} \int_{\nu_n h_{n-1}}^{(\nu_n+1)h_{n-1}} |f_n^*(x) + J_n^*(x)|^p\, dx} \leqq 2K.$$

For $n \to \infty$ we have also

$$\sqrt[p]{\frac{1}{(\nu_n+1)h_{n-1}} \int_{\nu_n h_{n-1}}^{(\nu_n+1)h_{n-1}} (f_n^*(x))^p\, dx} =$$

$$\sqrt[p]{\frac{1}{2(\nu_n+1)h_{n-1}} \int_{-h_n}^{h_n} (f_n^*(x))^p\, dx} \sim \sqrt[p]{\frac{1}{2\nu_n h_{n-1}} \int_{-h_n}^{h_n} (f_n^*(x))^p\, dx} = A_p(n) \to \infty$$

so that for $n \geqq$ some sufficiently large N_2

$$\sqrt[p]{\frac{1}{(\nu_n+1)h_{n-1}} \int_{\nu_n h_{n-1}}^{(\nu_n+1)h_{n-1}} (f_n^*(x))^p\, dx} \geqq 4K.$$

Thus for $n \geqq \max(N_1, N_2)$

$$\sqrt[p]{\frac{1}{h_{n-1}} \int_{\nu_n h_{n-1}}^{(\nu_n+1)h_{n-1}} |f_n^*(x) + J_n^*(x)|^p\, dx} \leqq \frac{1}{2} \sqrt[p]{\frac{1}{h_{n-1}} \int_{\nu_n h_{n-1}}^{(\nu_n+1)h_{n-1}} (f_n^*(x))^p\, dx}.$$

Hence, by help of the lemma on page 116, we get

$$\frac{1}{h_{n-1}} \int_{\nu_n h_{n-1}}^{(\nu_n+1)h_{n-1}} |J_n^*(x)|^\alpha\, dx \geqq \left(\frac{1}{2}\right)^p \frac{1}{h_{n-1}} \int_{\nu_n h_{n-1}}^{(\nu_n+1)h_{n-1}} (f_n^*(x))^\alpha dx = \left(\frac{1}{2}\right)^p d_{n-1} \geqq \left(\frac{1}{2}\right)^p \left(1 - \frac{1}{m_1}\right)\left(1 - \frac{1}{m_2}\right) \cdots$$

and therefore further

$$\frac{1}{h_{n-1}} \int_{\nu_n h_{n-1}}^{(\nu_n+1)h_{n-1}} |J(x)|^\alpha\, dx \geqq \left(\frac{1}{2}\right)^p \left(1 - \frac{1}{m_1}\right)\left(1 - \frac{1}{m_2}\right) \cdots.$$

This inequality, holding for every sufficiently large n, contradicts the fact that $J(x)$ is a W^α-zero function.

Main Example VII b.

Here we wish to choose the numbers $\nu_1, \nu_2, \ldots$ so that $F(x)$ becomes a B^P-function and so that the W^α-point around $F(x)$ does not contain any B^P-a. p. function.

We begin by proving that $\nu_1, \nu_2, \ldots$ can be chosen so that

$$A_P(n) \to k \quad \text{for} \quad n \to \infty$$

where k is a constant > 0. In a similar way as in main example VII a it is seen that a necessary condition for $A_P(n) \to k$ is that $\frac{\nu_n}{m_n} \to 0$. For all n from a certain step N (which will be indicated below) we put

$$\nu_n = \left[\frac{B_P(n)}{d_{n-1}}\right] = \left[\left(\frac{1}{\varepsilon_n}\right)^{\frac{P-\alpha}{\alpha-1}}\right].$$

We observe immediately that $\nu_n \geqq 1$. If $P = \alpha$, we have $B_P(n) = d_{n-1}$, $\nu_n = 1$, and hence

$$A_P(n) = \sqrt[P]{\frac{1}{\nu_n} B_P(n)} = \sqrt[P]{d_{n-1}} \to \sqrt[P]{\left(1 - \frac{1}{m_1}\right)\left(1 - \frac{1}{m_2}\right)\cdots}.$$

If $P > \alpha$, we have

$$\frac{B_P(n)}{d_{n-1}} = \left(\frac{1}{\varepsilon_n}\right)^{\frac{P-\alpha}{\alpha-1}} \to \infty$$

so that

$$\nu_n \sim \frac{B_P(n)}{d_{n-1}}$$

and therefore

$$A_P(n) = \sqrt[P]{\frac{1}{\nu_n} B_P(n)} \sim \sqrt[P]{d_{n-1}} \to \sqrt[P]{\left(1 - \frac{1}{m_1}\right)\left(1 - \frac{1}{m_2}\right)\cdots}.$$

As mentioned we have then $\frac{\nu_n}{m_n} \to 0$ and therefore our above N can be determined so that the claim $\nu_n < \frac{m_n}{2}$ is satisfied for $n \geqq N$. For $n < N$ the numbers ν_n are chosen arbitrarily so that merely $\nu_n < \frac{m_n}{2}$ is satisfied.

We shall show that by this choice of the numbers ν_n the function $F(x)$ gets the desired properties.

Firstly $F(x)$ is a B^P-function. In fact for $h_{n+q} \leqq T < h_{n+q+1}$ and sufficiently large n we have

$$\sqrt[p]{\frac{1}{2T}\int_{-T}^{T}(F(x)-F_n(x))^p\,dx} \leqq$$

$$\sqrt[p]{2}\left(\sqrt[p]{M\{(f_{n+1}(x))^p\}}+\cdots+\sqrt[p]{M\{(f_{n+q}(x))^p\,dx\}}\right)+A_P(n+q+1) \leqq$$

$$\sqrt[p]{2}\left(\sqrt[p_{n+1}]{M\{(f_{n+1}(x))^{p_{n+1}}\}}+\cdots+\sqrt[p_{n+q}]{M\{(f_{n+q}(x))^{p_{n+q}}\}}\right)+A_P(n+q+1) \leqq$$

$$\sqrt[p]{2}\,(\delta_{n+1}+\delta_{n+2}+\cdots+\delta_{n+q})+2k < \sqrt[p]{2}\,(\delta_{n+1}+\delta_{n+2}+\cdots)+2k.$$

Letting $q \to \infty$, we get

$$D_{B^P}[F(x), F_n(x)] \leqq \sqrt[p]{2}\,(\delta_{n+1}+\delta_{n+2}+\cdots)+2k$$

and thus

$$D_{B^P}[F(x)] \leqq D_{B^P}[F_n(x)] + \sqrt[p]{2}\,(\delta_{n+1}+\delta_{n+2}+\cdots)+2k.$$

Secondly we shall show that the W^α-point around $F(x)$ does not contain any B^P-a. p. function. Proceeding indirectly we assume that there exists a W^α-zero function $J(x)$ so that $F(x)+J(x)$ is a B^P-a. p. function or, which is equivalent (as $F(x)$ is B-a. p.), that

$$(F(x)+J(x))_N \overset{B^P}{\to} F(x)+J(x).$$

Denoting by $J^*(x)$ the function which is equal to $J(x)$ in the points where $j(x) \neq 0$ and equal to 0 elsewhere, we have

$$(j(x)+J^*(x))_N \overset{B^P}{\to} j(x)+J^*(x),$$

since $F(x)=j(x)$ in the points where $j(x) \neq 0$. $J(x)$ being a W^α-zero function, $J^*(x)$ is also a W^α-zero function. Then $j(x)+J^*(x)$ is a W-zero function, in particular a B-zero function. Consequently $(j(x)+J^*(x))_N$ is a B^p-zero function for all p and especially a B^P-zero function. As the B^P-point around 0, considered as a set of functions, is B^P-closed, the function $j(x)+J^*(x)$ is also a B^P-zero function. Consequently we have

$$\sqrt[p]{\frac{1}{(\nu_n+1)\,h_{n-1}}\int_0^{(\nu_n+1)\,h_{n-1}}|j(x)+J^*(x)|^p\,dx} \leqq \frac{k}{8} \quad \text{for} \quad n \geqq \text{some } N_1,$$

in particular

$$\sqrt[p]{\frac{1}{(\nu_n+1)\,h_{n-1}}\int\limits_{\nu_n h_{n-1}}^{(\nu_n+1)\,h_{n-1}} |f_n^*(x)+J_n^*(x)|^p\,dx} \leqq \frac{k}{8} \quad \text{for} \quad n \geqq N_1,$$

where $J_n^*(x)$ denotes the function which is equal to $J(x)$ in the points where $f_n^*(x) \neq 0$ and 0 elsewhere. Further, for sufficiently large n,

$$\sqrt[p]{\frac{1}{(\nu_n+1)\,h_{n-1}}\int\limits_{\nu_n h_{n-1}}^{(\nu_n+1)\,h_{n-1}} (f_n^*(x))^p\,dx} = \sqrt[p]{\frac{1}{2\,(\nu_n+1)\,h_{n-1}}\int\limits_{-h_n}^{h_n} (f_n^*(x))^p\,dx} =$$

$$\sqrt[p]{\frac{\nu_n}{\nu_n+1}}\sqrt[p]{\frac{1}{2\,\nu_n\,h_{n-1}}\int\limits_{-h_n}^{h_n} (f_n^*(x))^p\,dx} = \sqrt[p]{\frac{\nu_n}{\nu_n+1}}\,A_p(n) \geqq \frac{k}{4}$$

and thus

$$\sqrt[p]{\frac{1}{(\nu_n+1)\,h_{n-1}}\int\limits_{\nu_n h_{n-1}}^{(\nu_n+1)\,h_{n-1}} (f_n^*(x))^p\,dx} \geqq \frac{k}{4} \quad \text{for} \quad n \geqq \text{some } N_2.$$

Hence for $n \geqq \max\,(N_1, N_2)$

$$\sqrt[p]{\frac{1}{h_{n-1}}\int\limits_{\nu_n h_{n-1}}^{(\nu_n+1)\,h_{n-1}} |f_n^*(x)+J_n^*(x)|^p\,dx} \leqq \frac{1}{2}\sqrt[p]{\frac{1}{h_{n-1}}\int\limits_{\nu_n h_{n-1}}^{(\nu_n+1)\,h_{n-1}} (f_n^*(x))^p\,dx}.$$

By help of the lemma of page 116 we get

$$\frac{1}{h_{n-1}}\int\limits_{\nu_n h_{n-1}}^{(\nu_n+1)\,h_{n-1}} |J_n^*(x)|^\alpha\,dx \geqq \left(\frac{1}{2}\right)^p \frac{1}{h_{n-1}}\int\limits_{\nu_n h_{n-1}}^{(\nu_n+1)\,h_{n-1}} (f_n^*(x))^\alpha\,dx \geqq \left(\frac{1}{2}\right)^p \left(1-\frac{1}{m_1}\right)\left(1-\frac{1}{m_2}\right)\cdots.$$

Hence, for sufficiently large n,

$$\frac{1}{h_{n-1}}\int\limits_{\nu_n h_{n-1}}^{(\nu_n+1)\,h_{n-1}} |J(x)|^\alpha\,dx \geqq \left(\frac{1}{2}\right)^p \left(1-\frac{1}{m_1}\right)\left(1-\frac{1}{m_2}\right)\cdots,$$

which contradicts the fact that $J(x)$ is a W^α-zero function.

Main Example VII c.

Finally, in this main example, we wish to choose the numbers $\nu_1, \nu_2, \ldots$ such that $F(x)$ becomes B^P-a. p. and so that the W^α-point around $F(x)$ does not contain B^p-functions for $p > P$.

We shall show that we can determine $\nu_1, \nu_2, \ldots$ such that

$$A_P(n) \to 0 \quad \text{for} \quad n \to \infty$$

and

$$A_p(n) \to \infty \quad \text{for} \quad p > P \quad \text{and} \quad n \to \infty.$$

A necessary condition for this last relation is that $\frac{\nu_n}{m_n} \to 0$. We have

$$A_p(n) = \sqrt[p]{\frac{1}{\nu_n} d_{n-1} \left(\frac{1}{\varepsilon_n}\right)^{\frac{p-\alpha}{\alpha-1}}} = \sqrt[p]{\frac{1}{\nu_n} B_p(n)}.$$

First we choose $p_1 > p_2 > \cdots \to P$ converging so slowly to P that

$$\frac{B_{p_n}(n)}{B_P(n)} = \left(\frac{1}{\varepsilon_n}\right)^{\frac{p_n - P}{\alpha-1}} \to \infty.$$

Then from a certain step N (which will be indicated below) we put

$$\nu_n = \left[\frac{1}{\log \frac{B_{p_n}(n)}{B_P(n)}} B_{p_n}(n)\right].$$

Then on the one hand, as $\nu_n \to \infty$ and therefore

$$\nu_n \sim \frac{1}{\log \frac{B_{p_n}(n)}{B_P(n)}} B_{p_n}(n),$$

we have

$$A_P(n) = \sqrt[P]{\frac{1}{\nu_n} B_P(n)} \sim \sqrt[P]{\log \frac{B_{p_n}(n)}{B_P(n)} \cdot \frac{B_P(n)}{B_{p_n}(n)}} \to 0,$$

while on the other hand

$$A_{p_n}(n) = \sqrt[p_n]{\frac{1}{\nu_n} B_{p_n}(n)} \sim \sqrt[p_n]{\log \frac{B_{p_n}(n)}{B_P(n)}} \to \infty,$$

which involves that $A_p(n) \to \infty$ for $p > P$. From this it follows as mentioned that $\frac{\nu_n}{m_n} \to 0$ for $n \to \infty$, and we can therefore determine our above N so that the claim $\nu_n < \frac{m_n}{2}$ for $n \geqq N$ is satisfied. For $n < N$ we choose ν_n arbitrarily so that merely $\nu_n < \frac{m_n}{2}$.

Now it results (in a way quite analogous to main example VII a) that $F(x)$ is B^P-a. p. and that the W^α-point around $F(x)$ does not contain B^p-functions for $p > P$.

Om S-næstenperiodiske Funktioner med lineært uafhængige Exponenter.

Af

Harald Bohr.

En for $-\infty < x < \infty$ kontinuert Funktion $f(x) = u(x) + iv(x)$ kaldes som bekendt n æ s t e n p e r i o d i s k (afkortet n.p.), dersom der til ethvert $\varepsilon > 0$ findes en «relativ tæt» Mængde af til ε hørende «Forskydningstal» $\tau = \tau_f(\varepsilon)$ for $f(x)$. Herved kaldes τ et til ε hørende Forskydningstal for $f(x)$, hvis $|f(x+\tau) - f(x)| \leqq \varepsilon$ for $-\infty < x < \infty$, og en Mængde af reelle Tal kaldes relativ tæt, hvis der findes en fast Længde l saaledes, at ethvert Interval $a < x < a + l$ af denne Længde indeholder mindst et Tal i Mængden.

En Hovedsætning indenfor de n.p. Funktioners Teori, den saakaldte A p p r o x i m a t i o n s s æ t n i n g, udsiger, at Mængden af n.p. Funktioner er identisk med Mængden af de Funktioner $f(x)$, der kan approximeres ligeligt for alle x ved Exponentialpolynomier, d.v.s. endelige Summer af Formen $s(x) = \sum_{\nu=1}^{n} a_\nu e^{i\lambda_\nu x}$, hvor Exponenterne λ_ν er reelle Tal, medens Koefficienterne a_ν er komplekse Tal.

Idet vi som «Normen» $\|\varphi\|$ for en for $-\infty < x < \infty$ defineret Funktion $\varphi(x)$ betegner Tallet

$$\|\varphi(x)\| = \|\varphi\| = \underset{-\infty<x<\infty}{\text{Øvr. Gr.}} |\varphi(x)|,$$

kan Betingelsen for, at τ er et til ε hørende Forskydningstal for $f(x)$, udtrykkes ved, at Normen $\|f(x+\tau) - f(x)\|$ skal være $\leqq \varepsilon$, og Approximationssætningen udsiger, at de n.p. Funktioner er identiske med de Funktioner $f(x)$, for hvilke der til ethvert $\varepsilon > 0$

findes et Exponentialpolynomium $s(x)$ saaledes, at Normen $\|f(x) - s(x)\|$ er $\leqq \varepsilon$.

Til enhver n.p. Funktion $f(x)$ er knyttet en uendelig Række $\sum_1^\infty A_n e^{i\Lambda_n x}$ med reelle Exponenter Λ_n og komplekse Koefficienter A_n som Funktionens Fourierrække. Denne Række bestemmes udfra Funktionen $f(x)$ paa følgende Maade: For ethvert reelt λ betragtes Middelværdien

$$a(\lambda) = M\{f(x)\, e^{-i\lambda x}\} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(x)\, e^{-i\lambda x}\, dx.$$

Det viser sig, at denne Funktion $a(\lambda)$ er 0 for alle λ paa nær et (højst) tælleligt Antal Værdier; disse Værdier er da Funktionens «Fourierexponenter» $\Lambda_1, \Lambda_2, \ldots$, og de tilsvarende Værdier $a(\Lambda_n) = A_n$ Funktionens «Fourierkonstanter». Fourierrækken for en n.p. Funktion vil i Almindelighed ikke være konvergent (dette behøver jo ikke engang at gælde i det specielle Tilfælde, hvor $f(x)$ er en kontinuert periodisk Funktion). Der findes imidlertid et vigtigt Specialtilfælde, hvor Fourierrækken altid er konvergent, endda ligelig konvergent for alle x, nemlig det, hvor Fourierexponenterne $\Lambda_1, \Lambda_2, \ldots$ er lineært uafhængige, d.v.s. hvor der ikke findes nogen Relation af Formen

$$h_1\Lambda_1 + h_2\Lambda_2 + \ldots + h_m\Lambda_m = 0$$

med hele Koefficienter $h_1, \ldots, h_m$ (som ikke alle er 0); i Tilfælde af lineært uafhængige Fourierexponenter Λ_n gælder det nemlig, at Rækken $\sum |A_n|$ altid er konvergent.

Foruden de sædvanlige (kontinuerte) n.p. Funktioner har man ogsaa studeret forskellige Klasser af Funktioner, der er n.p. i en eller anden generaliseret Forstand. I denne Afhandling skal kun omtales de af Stepanoff indførte generaliserede n.p. Funktioner, de saakaldte S-n.p. Funktioner. Idet vi for en vilkaarlig (maalelig) Funktion $\varphi(x)$, givet paa hele Aksen $-\infty < x < \infty$, som Funktionens «S-Norm» betegner Tallet

$$\|\varphi(x)\|_S = \|\varphi\|_S = \underset{-\infty<x<\infty}{\text{Øvr. Gr.}} \int_x^{x+1} |\varphi(t)|\, dt \quad (\leqq \infty),$$

kan en S-n.p. Funktion defineres som en Funktion $f(x)$, der til ethvert $\varepsilon > 0$ besidder en relativ tæt Mængde af «S-Forskyd-

ningstal» $\tau = \tau_f(\varepsilon)$, d.v.s. af Tal τ, for hvilke $\|f(x+\tau) - f(x)\|_S \leqq \varepsilon$; og ganske svarende til Hovedsætningen (Approximationssætningen) for de sædvanlige n.p. Funktioner gælder det her, at en nødvendig og tilstrækkelig Betingelse for, at en Funktion $f(x)$ er en S-n.p. Funktion, er, at den «i S-Forstand» kan approximeres ved Exponentialpolynomier, d.v.s. at der til ethvert $\varepsilon > 0$ findes et Exponentialpolynomium $s(x)$ saaledes, at $\|f(x) - s(x)\|_S \leqq \varepsilon$. Ogsaa en S-n.p. Funktion $f(x)$ besidder en Fourierrække $\sum A_n e^{i\Lambda_n x}$, der bestemmes paa nøjagtig samme Maade som for en sædvanlig n.p. Funktion.

For S-Normen $\|\varphi\|_S$ gælder (ligesom for den ovenfor betragtede simplere Norm $\|\varphi\|$) den saakaldte Trekantsulighed $\|\varphi + \psi\|_S \leqq \|\varphi\|_S + \|\psi\|_S$. Idet et Exponentialpolynomium $s(x)$ jo er begrænset og derfor har en endelig S-Norm, følger det umiddelbart af Approximationssætningen, ved Hjælp af Trekantsuligheden, at enhver S-n.p. Funktion har en endelig S-Norm. Vi nævner endvidere (se f. Eks. H. Bohr og E. Følner: On some types of functional spaces, Acta Mathematica 76), at S-Rummet, d.v.s. Mængden af alle Funktioner med endelig S-Norm, er «fuldstændigt»; herved menes, at en Funktionsfølge $f_1(x), f_2(x), \ldots$, der er en «S-Fundamentalfølge», d.v.s. $\|f_n - f_m\|_S \to 0$ for n og $m \to \infty$, altid tillige er «S-konvergent», d.v.s. der findes en Funktion $f(x)$, saaledes at $\|f - f_n\|_S \to 0$ for $n \to \infty$.

I det følgende skal behandles S-n.p. F u n k t i o n e r m e d l i n e æ r t u a f h æ n g i g e F o u r i e r e x p o n e n t e r. Visse Forhold kunde tyde paa, at det ogsaa for en S-n.p. Function (ligesom for en sædvanlig n.p. Funktion) maatte gælde, at dersom dens Fourierrække $\sum A_n e^{i\Lambda_n x}$ havde lineært uafhængige Exponenter Λ_n, da vilde $\sum |A_n|$ være konvergent (og Funktionen derfor i Virkeligheden være en sædvanlig n.p. Funktion). Som vi skal se, behøver dette imidlertid ikke at være Tilfældet. Derimod gælder det, at hvis Fourierexponenterne Λ_n, foruden at være lineært uafhængige, tillige er b e g r æ n s e d e, $|\Lambda_n| \leqq c$ for alle n, da vil $\sum |A_n|$ være konvergent. Vi vil begynde med at bevise dette, altsaa

Sætning 1. *Dersom Rækken* $\sum A_n e^{i.\Lambda_n x}$ *med lineært uafhængige og begrænsede Exponenter* Λ_n *er Fourierrække for en S-n.p. Funktion* $f(x)$, *da er* $\sum |A_n|$ *konvergent.*

Vi bemærker først, at det af Beviset for Approximationssætningen (ved Hjælp af de saakaldte BOCHNER-FEJÉR'ske Summer) fremgaar, at dersom en S-n.p. Funktion $f(x)$ har en Fourierrække $\sum A_n e^{i.\Lambda_n x}$ med lineært uafhængige Exponenter, da kan man som approximerende Exponentialpolynomier til $f(x)$ simpelthen benytte Fourierrækkens Afsnit

$$s_n(x) = \sum_{\nu=1}^{n} A_\nu e^{i.\Lambda_\nu x},$$

d.v.s. det vil gælde, at

$$\| f - s_n \|_S \to 0 \quad \text{for } n \to \infty.$$

Idet $\| s_n \|_S \leqq \| f \|_S + \| f - s_n \|_S$ følger heraf umiddelbart, at der findes en Konstant C, saaledes at $\| s_n \|_S \leqq C$ for alle n, d.v.s.

$$\int_x^{x+1} | s_n(t) | \, dt \leqq C \text{ for } n = 1, 2, 3, \ldots \text{ og } -\infty < x < \infty,$$

eller anderledes skrevet

$$(1) \qquad \int_0^1 | s_n(x+t) | \, dt \leqq C \text{ for alle } n \text{ og alle } x.$$

Lad nu $A_\nu = \varrho_\nu e^{i\theta_\nu}$ $(\nu = 1, 2, \ldots)$. Det drejer sig om at bevise, at $\sum \varrho_\nu$ er konvergent. Vi betragter det ν^{te} Led i Fourierrækken i Punktet $x + t$, altsaa

$$A_\nu e^{i.\Lambda_\nu (x+t)} = \varrho_\nu \, e^{i(\theta_\nu + \Lambda_\nu x + \Lambda_\nu t)}.$$

Idet Tallene $\Lambda_1, \Lambda_2, \ldots, \Lambda_n$ er lineært uafhængige, kan vi i Følge KRONECKER's klassiske Sætning om Diophantiske Approximationer bestemme et Tal $x_0 = x_0(n)$ saaledes, at

$$| \theta_\nu + \Lambda_\nu x_0 | < \frac{\pi}{6} \pmod{2\pi} \qquad (\nu = 1, 2, \ldots, n).$$

I Følge Antagelse er $|\Lambda_\nu| \leqq$ en Konstant c for alle ν, og følgelig er $|\Lambda_\nu t| < \frac{\pi}{6}$ for $|t| < \frac{\pi}{6c}$. For ethvert $\nu = 1, 2, \ldots, n$ og $|t| < \frac{\pi}{6c}$ gælder det derfor, at

$$|\varepsilon_\nu + A_\nu x_0 + A_\nu t| < \frac{\pi}{6} + \frac{\pi}{6} = \frac{\pi}{3} \pmod{2\pi},$$

og dermed (idet vi med $\Re(z)$ betegner den reelle Del af det komplekse Tal z), at

$$\Re\left\{A_\nu e^{iA_\nu(x_0+t)}\right\} = \Re\left\{\varrho_\nu e^{i(\Theta_\nu + A_\nu x_0 + A_\nu t)}\right\} > \varrho_\nu \cos\frac{\pi}{3} = \frac{1}{2}\varrho_\nu.$$

Følgelig er

$$(2) \qquad \Re\left\{s_n(x_0+t)\right\} = \sum_{\nu=1}^{n} \Re\left\{A_\nu e^{iA_\nu(x_0+t)}\right\} > \frac{1}{2}\sum_{\nu=1}^{n} \varrho_\nu \text{ for } |t| < \frac{\pi}{6c}.$$

Nu er imidlertid, i Følge (1), $\int_0^1 |\Re\{s_n(x_0+t)\}| \, dt \leqq C$, altsaa yderligere, idet vi med d betegner Min $\left(\frac{\pi}{6c}, 1\right)$,

$$(3) \qquad \int_0^d \Re\left\{s_n(x_0+t)\right\} dt \leqq C.$$

Af (2) og (3) følger

$$d \cdot \frac{1}{2}\sum_{\nu=1}^{n} \varrho_\nu < \int_0^d \Re\left\{s_n(x_0+t)\right\} dt \leqq C,$$

d.v.s. $\sum_{\nu=1}^{n} \varrho_\nu < \frac{2C}{d}$. Da denne Ulighed gælder for alle n, er hermed Konvergensen av $\sum_1^\infty \varrho_\nu$ godtgjort.

Vi skal dernæst bevise, at det **ikke** for **enhver** Række $\sum A_n e^{iA_n x}$ med lineært uafhængige Exponenter, der er Fourierrække for en S-n.p. Funktion, gælder, at $\sum |A_n|$ er konvergent. Vi vil imidlertid bevise væsentlig mere. Hertil betragtes Klassen af de saakaldte S_2-n.p. Funktioner, der er en Underklasse i Klassen af S-n.p. Funktioner. En S_2-n.p. Funktion defineres paa ganske tilsvarende Maade som en S-n.p. Funktion, og der gælder en fuldkommen tilsvarende Hovedsætning (Approximationssætning), idet blot overalt — saavel i Definitionen som i Hovedsætningen — S-normen $\|\varphi\|_S$ erstattes af S_2-Normen

$$\|\varphi(x)\|_{S_2} = \|\varphi\|_{S_2} = \sqrt{\underset{-\infty<x<\infty}{\text{Øvr. Gr.}} \int_0^1 |\varphi(t)|^2 \, dt}.$$

For S_2-Normen gælder (ligesom for S-Normen) Trekantsuligheden $\|\varphi + \psi\|_{S_2} \leqq \|\varphi\|_{S_2} + \|\psi\|_{S_2}$, og S_2-Rummet er (ligesom

S-Rummet) et fuldstændigt Rum. Ogsaa en S_2-n.p. Funktion $f(x)$ besidder en Fourierrække $\sum A_n e^{i\Lambda_n x}$. Om S_2-n.p. Funktioner gælder det, at en nødvendig (men ikke tilstrækkelig) Betingelse for, at $\sum A_n e^{i\Lambda_n x}$ er Fourrierrække for en S_2-n.p. Funktion er, at $\sum |A_n|^2$ er konvergent. Vi vil bevise, at der findes S_2-n.p. Funktioner (og dermed a fortiori S-n.p. Funktioner), hvis Fourierrække $\sum A_n e^{i\Lambda_n x}$ har lineært uafhængige Exponenter Λ_n, men hvor $\sum |A_n|$ dog er d i v e r g e n t, ja vi vil endog bevise, at der findes Fourierrækker for S_2-n.p. Funktioner med lineært uafhængige Exponenter Λ_n, hvori Koefficienterne A_n er fuldkommen vilkaarligt opgivne komplekse Tal, som blot (selvfølgelig) opfylder den førnævnte nødvendige Betingelse $\sum |A_n|^2$ konvergent.

Sætning 2. *Til en vilkaarlig opgiven Følge af komplekse Tal A_1, A_2 ..., for hvilken $\sum |A_n|^2$ er konvergent, kan bestemmes en Følge af lineært uafhængige reelle Tal $\Lambda_1, \Lambda_2, \ldots$ saaledes, at den uendelige Række $\sum A_n e^{i\Lambda_n x}$ er Fourierrække for en S_2-n.p. Funktion $f(x)$.*

Ved Beviset benyttes bl. a., at det som bekendt — og som man umiddelbart viser — gælder for et vilkaarligt Exponentialpolynomium $s(x) = \sum_{\nu=1}^{n} a_\nu e^{i\lambda_\nu x}$, at

$$\lim_{L\to\infty} \frac{1}{L}\int_x^{x+L} |s(t)|^2\, dt = \sum_{\nu=1}^{n} |a_\nu|^2 \quad \text{ligeligt i } -\infty < x < \infty.$$

Lad $\mu_1, \mu_2, \ldots$ være en vilkaarlig opgiven Følge af lineært uafhængige reelle Tal. Vi vil da bevise, at vi som Exponenter $\Lambda_1, \Lambda_2, \ldots$ i vor Sætning kan benytte de (lineært uafhængige) Tal $\Lambda_n = p_n \mu_n$, hvor de paahæftede Faktorer $p_1, p_2, \ldots$ er passende valgte positive hele Tal. I det følgende vil vi til Afkortning betegne Exponentialpolynomierne

$$\sum_{\nu=1}^{n} A_\nu e^{i\mu_\nu x} \quad \text{og} \quad \sum_{\nu=1}^{n} A_\nu e^{i\Lambda_\nu x} = \sum_{\nu=1}^{n} A_\nu e^{ip_\nu \mu_\nu x}$$

med henholdsvis $\sigma_n(x)$ og $s_n(x)$. Vi begynder med at bestemme positive hele Tal $m_1 < m_2 < m_3 \ldots$ saaledes at, naar vi sætter

$$b_1 = \sqrt{|A_1|^2 + \ldots + |A_{m_1}|^2}, \ldots, \; b_q = \sqrt{|A_{m_{q-1}+1}|^2 + \ldots + |A_{m_q}|^2}, \ldots,$$

da vil $\sum_1^{\infty} b_q$ være konvergent. At et saadant Valg af $m_1, m_2, \ldots$

er muligt, er klart; thi idet $|A_{m+1}|^2 + |A_{m+2}|^2 + \ldots \to 0$ for $m \to \infty$, kan vi jo successivt vælge $m_1, m_2, \ldots$ saa store, at f. Eks. $b_2 < \frac{1}{2^2}$, $b_3 < \frac{1}{2^3}, \ldots$. For ethvert fast $q > 1$ gælder det, ligelig i $-\infty < x < \infty$, at

$$\lim_{L \to \infty} \frac{1}{L} \int_x^{x+L} |\sigma_{m_q}(t) - \sigma_{m_{q-1}}(t)|^2 \, dt = \sum_{\nu = m_{q-1}+1}^{m_q} |A_\nu|^2 = b_q^2,$$

og vi kan derfor bestemme et positivt helt Tal N_q saaledes, at

$$\frac{1}{N_q} \int_x^{x+N_q} |\sigma_{m_q}(t) - \sigma_{m_{q-1}}(t)|^2 \, dt < 4 b_q^2 \quad \text{for alle } x.$$

I Stedet for at skulle tage Middelværdien over et Interval af Længden N_q (som maaske er meget stor for store q), vil vi imidlertid gerne kunne benytte den faste Intervallængde 1; dette kan vi opnaa ved at multiplicere de indgaaende Exponenter μ_ν $(m_{q-1} < \nu \leqq m_q)$ med Faktoren N_q, og vi vælger derfor de før omtalte Faktorer $p_\nu = N_q$ for $\nu = m_{q-1} + 1, \ldots, m_q$, og sætter altsaa $\Lambda_\nu = N_q \mu_\nu$ for disse ν (medens vi vælger de m_1 første Faktorer $p_1, \ldots, p_{m_1}$ vilkaarligt). Herved bliver

$$\sigma_{m_q}(N_q t) - \sigma_{m_{q-1}}(N_q t) = s_{m_q}(t) - s_{m_{q-1}}(t),$$

og vi faar for ethvert x

$$\int_x^{x+1} |s_{m_q}(t) - s_{m_{q-1}}(t)|^2 \, dt = \int_x^{x+1} |\sigma_{m_q}(N_q t) - \sigma_{m_{q-1}}(N_q t)|^2 \, dt$$

$$= \frac{1}{N_q} \int_{xN_q}^{xN_q+N_q} |\sigma_{m_q}(t) - \sigma_{m_{q-1}}(t)|^2 \, dt < 4 b_q^2.$$

Det gælder altsaa for ethvert $q > 1$, at

$$\| s_{m_q} - s_{m_{q-1}} \|_{S_2} \leqq \sqrt{4 b_q^2} = 2 b_q,$$

og dermed, i Følge Trekantsuligheden, at

$$\| s_{m_{q+l}} - s_{m_q} \|_{S_2} \leqq \| s_{m_{q+1}} - s_{m_q} \|_{S_2} + \| s_{m_{q+2}} - s_{m_{q+1}} \|_{S_2} + \ldots + \| s_{m_{q+l}} - s_{m_{q+l-1}} \|_{S_2}$$

$$\leqq 2 (b_{q+1} + b_{q+2} + \ldots + b_{q+l}) < \varepsilon \text{ for } q > Q_0, \; l > 0.$$

Følgen af Exponentialpolynomier $s_{m_1}(x)$, $s_{m_2}(x), \ldots$ er saaledes en S_2-Fundamentalfølge, og da S_2-Rummet, som ovenfor nævnt, er fuldstændigt, følger heraf, at denne Følge er S_2-konvergent, d.v.s. der findes en Funktion $f(x)$, saaledes at

$$\|f - s_{m_q}\|_{S_2} \to 0 \quad \text{for } q \to \infty .$$

Denne Funktion $f(x)$ vil da have de i Sætningen omhandlede Egenskaber. Thi for det første er $f(x)$ en S_2-n.p. Funktion, idet den kan S_2-approximeres ved Exponentialpolynomier, nemlig ved Exponentialpolynomierne $s_{m_q}(x)$. Og for det andet vil $f(x)$ til Fourierrække netop have Rækken $\sum_{n=1}^{\infty} A_n e^{i \Lambda_n x}$, fordi Fourierrækken for $f(x)$ (i Følge en kendt let beviselig Sætning) fremkommer ved formel Grænseovergang udfra de approximerende Exponentialpolynomier $s_{m_q}(x) = \sum_{n=1}^{m_q} A_n e^{i \Lambda_n x}$. Hermed er Sætning 2 bevist.

On some functional spaces.

By

Harald Bohr.

A (metric) space is a set of elements $f, g, \ldots$ which in the following sense is organized by help of a distance notion: To two arbitrary elements f and g in the set a real non-negative number $D[f, g]$ is associated which is called the distance from f to g, and which satisfies the following conditions

$$1)\ D[f,g] = D[g,f]\,, \quad 2)\ D[f,g] = 0 \text{ if and only if } f = g\,,$$

$$3)\ D[f,g] \leqq D[f,h] + D[h,g] \quad \text{(the triangel inequality).}$$

The notion of convergence immediately presents itself in such a space; we say of a sequence of elements f_n that $f_n \to f$ if $D[f, f_n] \to 0$ for $n \to \infty$. The space is called complete if every fundamental sequence f_n (i.e. every sequence satisfying $D[f_n, f_m] \to 0$ for n and $m \to \infty$) is also a convergent sequence.

In this lecture I shall treat some functional spaces, namely spaces whose elements $f = f(x)$, $g = g(x), \ldots$ are complex functions of a real variable. These spaces were met with by the generalization of the theory of almost periodic functions and have been studied in a joined paper by Erling Følner and the lecturer[1]), and later more deeply in Følner's dissertation[2]). The functions in question are defined for all x, i.e. for $-\infty < x < \infty$. As starting point, however, we shall first consider some classical functional spaces, the functions of which are defined in a finite interval, say the interval $0 \leqq x < 1$, or—more conveniently—defined in the whole interval $-\infty < x < \infty$ but periodic with the period 1.

---

[1]) Harald Bohr and Erling Følner, On some types of functional spaces, Acta Mathematica 76 p. 31–155.

[2]) Erling Følner, Bidrag til de generaliserede næstenperiodiske Funktioners Teori, København 1944.

Functions defined in $0 \leqq x < 1$ (and periodically continued).

1. As an especially simple example of a functional space we first mention the set of all functions bounded in $0 \leqq x < 1$ and organized by help of the distance notion

$$d[f, g] = \underset{0 \leqq x < 1}{\text{u. b.}} |f(x) - g(x)| .$$

Here, evidently, convergence of a sequence $f_n \to f$ is equivalent to the uniform convergence of the sequence of functions $f_n(x)$ towards $f(x)$, and the space is complete. An important subspace of this space is the set $\boldsymbol{r}$ of the (periodic) continuous functions. The classical approximation theorem of WEIERSTRASS may be expressed by saying that in this space $\boldsymbol{r}$ the periodic exponential polynomials with the period 1, i. e. the functions

$$s(x) = \sum_{-N}^{N} a_n e^{2\pi i n x}$$

are lying everywhere dense. We may also express this theorem by the equation $\boldsymbol{r} = H\{s(x)\}$ where $H\{s(x)\}$ denotes the closure of the set $\{s(x)\}$.

2. Another important functional space is the space $\boldsymbol{r}_1$ consisting of all in $0 \leqq x < 1$ measurable functions $f(x)$ for which $\int_0^1 |f(x)|\, dx < \infty$ where the distance is defined as a mean distance, namely by

$$d_1[f, g] = \int_0^1 |f(x) - g(x)|\, dx .$$

Also in this space the set of all our exponential polynomials $s(x) = \sum_{-N}^{N} a_n e^{2\pi i n x}$ is lying everywhere dense, i. e. $\boldsymbol{r}_1 = H\{s(x)\}$, and $\boldsymbol{r}_1$ is (as $\boldsymbol{r}$) a complete space.

As well-known, we can more generally for an arbitrary $p \geqq 1$ consider the space $\boldsymbol{r}_p$ of all in $0 \leqq x < 1$ measurable functions $f(x)$ for which $\int_0^1 |f(x)|^p dx < \infty$ where the distance notion is defined by

$$d_p[f, g] = \sqrt[p]{\int_0^1 |f(x) - g(x)|^p dx} .$$

For every $p \geqq 1$ the space $\boldsymbol{r}_p$ is a complete space, and our exponential polynomials $s(x)$ are lying every where dense in this space. For increasing p the space $\boldsymbol{r}_p$ will decrease, i. e. $\boldsymbol{r}_{p_2}$ is contained in $\boldsymbol{r}_{p_1}$

for $p_2 > p_1$. It will be convenient to consider the parameter p as a time-parameter which increases from 1 to ∞, and we shall say about a function $f(x)$ in $\boldsymbol{r}_1$ that it is still "alive" at the time p if $f(x)$ belongs to $\boldsymbol{r}_p$, while $f(x)$ is "dead" at the time p if $f(x)$ is not contained in $\boldsymbol{r}_p$. By the "lifetime" of a function $f(x)$ in $\boldsymbol{r}_1$ we then naturally understand the upper bound P $(1 \leqq P \leqq \infty)$ of all p's for which $f(x)$ is alive. In its "moment of death" (i. e. at the time P) the function $f(x)$ may either be alive or be dead.

Concerning these functional spaces $\boldsymbol{r}_p$ there is, however, a remark to be made—rather unimportant in itself but essential in view of the functional spaces $\boldsymbol{R}_p$ to be considered below—namely that our set of functions $\boldsymbol{r}_p$ in reality are not functional spaces according to the definition of such a space, since (for every $p \geqq 1$) there exist functions $f(x)$ and $g(x)$ which have the distance $d_p[f, g] = 0$ without being identical. This, however, is only the case when $f(x)$ is equal to $g(x)$ almost everywhere, i. e. outside a set of measure 0. Strictly speaking, a point in the space $\boldsymbol{r}_p$ is not a single function but a class of equivalent functions, i. e. functions with mutual distances 0. In view of the more general spaces to be considered below we introduce the notion of a zero-function whereby is meant a function which has the distance 0 from the function identical 0. Two functions belong to the same point if and only if their difference is a zero function. In case of the distance d_p the zero-functions are the functions which are 0 almost everywhere. It is of little importance, however, whether as points in $\boldsymbol{r}_p$ we think of the single functions or—as we ought to do—of the (small) classes of equivalent functions. This is connected with the fact that the zero-functions are the same for every p so that two functions equivalent in $\boldsymbol{r}_1$ are either both alive at the time p or both dead at the time p, and (if alive) equivalent in $\boldsymbol{r}_p$, too.

After these introductory remarks on functions defined in a finite interval I now pass to my proper subject:

Functions defined in the interval $-\infty < x < \infty$.

We shall treat different generalizations of the above mentioned functional spaces. The fact that now the whole axis is at our disposal, and not only a finite interval, will give rise to a larger richness of possibilities presenting some quite new features. The class of all periodic exponential polynomials with the period 1 is here replaced

by the class of all exponential polynomials, i. e. all finite sums of the form

$$S(x) = \sum_{1}^{N} A_n e^{i\Lambda_n x}$$

where the exponents Λ_n are arbitrary real numbers.

I. Firstly, in analogy to **1**, we consider the set of all in $-\infty < x < \infty$ bounded functions $f(x)$ organized to a space by the distance notion

$$D[f, g] = \underset{-\infty < x < \infty}{\text{u. b.}} |f(x) - g(x)| .$$

Here, as in **1**, convergence means uniform convergence, and the space is complete. By $\boldsymbol{R}$ we denote the subspace which consists of all in $-\infty < x < \infty$ continuous (bounded) functions. In contrast with **1**, the set $\{S(x)\}$ of all exponential polynomials is here not every where dense in $\boldsymbol{R}$, i. e. the closure $H\{S(x)\}$ is not the whole space $\boldsymbol{R}$ but only a certain subspace $\boldsymbol{R}^*$ in $\boldsymbol{R}$. This subspace $\boldsymbol{R}^*$ consists just of the almost periodic functions, the different properties of which shall not, however, be treated in this lecture.

II. Next, in analogy to **2**, we consider functional spaces where the distance between two functions $f = f(x)$ and $g = g(x)$ is defined as a mean distance, but now of course taken over the whole axis $-\infty < x < \infty$. This distance notion is for periodic functions to coincide with the one considered in **2**. Several possibilities of definition are here available, each with its special properties and its special interest. We shall mention three such distance notions which have all been introduced in connection with the study of generalized almost periodic functions. We denote them

$$D^S[f, g], \quad D^W[f, g] \quad \text{and} \quad D^B[f, g]$$

where the letters S, W and B refer to the names of STEPANOFF, WEYL and BESICOVITCH.

STEPANOFF's distance is defined by

$$D^{S, L}[f, g] = \underset{-\infty < x < \infty}{\text{u. b.}} \frac{1}{L} \int_{x}^{x+L} |f(x) - g(x)| \, dx$$

where L is a fixed length; its value is unessential, as a change of L certainly involves a change of the distance $D^{S, L}$ but not of the convergence notion $f_n \to f$ established by $D^{S, L}$.

As to BESICOVITCH's distance the mean value is at once extended over the whole axis $-\infty < x < \infty$, viz.

$$D^B[f, g] = \overline{\lim_{T\to\infty}} \frac{1}{2T}\int_{-T}^{T} |f(x)-g(x)|\,dx.$$

Finally, WEYL's distance is an "intermediate thing" between the two distances cited above. Like STEPANOFF, WEYL considers a fixed length L which he, however, lets increase to ∞, viz.

$$D^W[f, g] = \lim_{L\to\infty} D^{S,L}[f, g]\,.$$

As mentioned above, these three distances are all generalizations of the former one $d_1[f, g]$ for the finite interval $0 \leqq x < 1$ since they all (for the S-distance when $L = 1$) are reduced to that distance when $f(x)$ and $g(x)$ are periodic with the period 1.

The S-distance is the "finest" of the three distances, next comes the W-distance and finally the B-distance. This involves that among the three corresponding functional spaces $\boldsymbol{R}_1^S$, $\boldsymbol{R}_1^W$ and $\boldsymbol{R}_1^B$ the B-space is the most comprehensive, the W-space less comprehensive, and the S-space the narrowest one. As for the finite interval so also for the infinite interval we introduce distance notions corresponding to a parameter $p \geqq 1$, namely the distance notions

$$D_p^S[f, g],\quad D_p^W[f, g] \quad\text{and}\quad D_p^B[f, g]$$

where for instance $D_p^B[f, g]$ is defined by

$$D_p^B[f, g] = \overline{\lim_{T\to\infty}} \sqrt[p]{\frac{1}{2T}\int_{-T}^{T} |f(x)-g(x)|^p\,dx}\,.$$

As before we are interested in the changes which take place when the parameter p—which again is interpreted as a time-parameter—increases from 1 to ∞. All three functional spaces $\boldsymbol{R}_p^S$, $\boldsymbol{R}_p^W$ and $\boldsymbol{R}_p^B$ corresponding to the three distances will decrease when p increases, exactly as in the case of a finite interval.

S. In case of the S-distance, the situation is in many respects nearly as simple as for the finite interval. There is, however, as in **I**, reason to consider besides the space $\boldsymbol{R}_p^S$ itself also the so-called almost periodic subspace of this space which is defined as the closure $H\{S(x)\}$ of all exponential polynomials $S(x)$. We shall not treat this relatively simple S-case more detailed, but shall at once pass to the essentially more complicated and interesting spaces $\boldsymbol{R}_p^W$ and $\boldsymbol{R}_p^B$.

W and B. What in the following shall be said about the $\boldsymbol{R}_p^W$- and the $\boldsymbol{R}_p^B$-spaces is mainly the same in the W- and the B-case, and

hence we shall omit the letters W and B and simply write $D_p[f, g]$ and $\boldsymbol{R}_p$.

What distinguishes the W- and the B-case from the S-case is that in the W- and B-case the set of the zero-functions, i. e. functions $n(x)$ with $D_p[n(x), 0] = 0$, is much more comprehensive than in the S-case, and especially the fact that the class of zero-functions (the zero-point of the space) is not the same for different values of p but decreases for increasing p. In fact, this involves that two functions which belong to the same point in $\boldsymbol{R}_1$, i. e. has the D_1-distance 0, need not have the same lifetime, and further, if they are both alive at a certain time p they need not belong to the same point in $\boldsymbol{R}_p$ (since their D_p-distance may be > 0). Thus, besides the lifetime of a single function $f(x)$, also the notion of the lifetime of a point in $\boldsymbol{R}_1$ is needed. Naturally this lifetime is defined as the upper bound of the lifetimes of all the functions in the point in question. We are interested in and shall try to give an illustrative account of what could be called the "evolution" of an arbitrary point from $\boldsymbol{R}_1$ and, especially, of a point from the almost periodic subspace $\boldsymbol{R}_1^*$ which consists of all points which (or rather the functions of which) can be approximated by exponential polynomials $S(x)$.

We first consider an arbitrary point A in $\boldsymbol{R}_1$ with the lifetime P. For every $p < P$ amongst the infinity of functions which constitute the point A some functions will be dead at the time p while other functions are still alive. Those latter functions, however, will not form a single point in the space $\boldsymbol{R}_p$ but an infinity of points in $\boldsymbol{R}_p$, in connection with the fact that two functions $f(x)$ and $g(x)$ which belong to the same point in $\boldsymbol{R}_1$ need not belong to the same point in $\boldsymbol{R}_p$. This infinity of points in $\boldsymbol{R}_p$ which all originate from one and the same point A in $\boldsymbol{R}_1$ will be called the p-children of the point A. Amongst this infinity of p-children of the point A there is, however, for every $p < P$ a single one which distinguishes itself beyond all the others. In fact, if $p_1 > p$ but still $< P$ all the p_1-children of our point A are descendants of one and the same p-child which distinguished p-child may be called the pregnant p-child or the p-generator. All the other p-children may be called the still-born brothers of the pregnant p-child since they first come into existence as independent beings at the time p itself (while at any previous moment they are all lying in the same point, namely the generator child of this moment) and then immediately die (since no function in any of them will survive the time p). If $P < \infty$ and the point A is

living in its moment of death P there can obviously be no P-generator amongst the infinity of P-children, and generally none of the P-children of our point A will play a distinguished role.

Next, I shall say some words about the (especially interesting) case where our point A from the space $\boldsymbol{R}_1$ belongs to the subspace $\boldsymbol{R}_1^*$, i. e. is an almost periodic point. Here, to every $p < P$ amongst the infinity of p-children of the point A there will be exactly one p-child which is an almost periodic point in the space $\boldsymbol{R}_p$. Further, this single almost periodic p-child turns out just to be the pregnant p-child.

So far, everything said holds as well for the W- as for the B-case. Finally I shall briefly mention an interesting and characteristic difference between the two cases which arises when we consider an almost periodic point A just in its moment of death P (provided it is still alive at this moment). While in the B-case there will always among the P-children be one which distinguishes itself, namely as an almost periodic P-child, it may in the W-case happen that none of the P-children is almost periodic. This difference between the two cases is closely connected with another—in itself very important—difference between the W- and the B-case, namely that the space $\boldsymbol{R}_p^B$ is complete for every $p \geqq 1$, while none of the $\boldsymbol{R}_p^W$-spaces is a complete space.

As to further details I refer to the above cited paper of Følner and myself and to Følner's dissertation which contains a thorough and systematic investigation of the interesting and manifold relations between all the spaces in question, the S-spaces, the W-spaces and the B-spaces as well as their almost periodic subspaces.

MEAN MOTIONS AND ALMOST PERIODIC FUNCTIONS:

By Harald BOHR and Børge JESSEN.

1. Our subject has its starting point in the treatment by Lagrange (1782) of the perturbations of the large planets. Let

$$F(t) = a_0 e^{i\lambda_0 t} + a_1 e^{i\lambda_1 t} + \ldots + a_N e^{i\lambda_N t} \tag{1}$$

be a trigonometric polynomial with complex coefficients and real mutually different exponents λ_n. The problem is to study the variation of the argument of $F(t)$. If $F(t)$ contains a preponderant term it is evident that a continuous branch of the argument will be of the form

$$\arg F(t) = ct + O(1), \tag{2}$$

where c is the exponent of the preponderant term. Thus the argument consists of a secular term ct and a bounded remainder.

Lagrange has proposed the problem to study the variation of the argument also in the general case. In this case it may arrive that $F(t)$ takes arbitrarily small values or even the value zero. In passing a zero of odd order the line joining 0 and $F(t)$ will change its positive direction. In order to speak of a continuous branch of the argument it is therefore necessary to consider the argument mod. π and not as usual mod. 2π.

It is no restriction to take $\lambda_0 = 0$ so that

$$F(t) = a_0 + a_1 e^{i\lambda_1 t} + \ldots + a_N e^{i\lambda_N t}.$$

If here all ratios $\frac{\lambda_j}{\lambda_k}$ are rational, $F(t)$ is periodic, and (2) obviously holds, but now c is generally not one of the exponents. For $N = 2$ and an irrational $\frac{\lambda_1}{\lambda_2}$, Bohl (1909) proved by means of equidis-

tribution mod. 1, introduced by him for this purpose, that

$$\arg F(t) = ct + o(t), \tag{3}$$

but here the remainder is generally unbounded. This means that the limit

$$c = \lim_{T \to \infty} \frac{\arg F(T) - \arg F(0)}{T}$$

exists. The constant c is called the *mean motion* of $F(t)$.

By his famous generalisation of Kronecker's theorem, i. e. by extending the theorem on equidistribution mod. 1 to the case $N > 2$, Weyl could prove (1914) that (3) holds whenever $\lambda_1, \ldots, \lambda_N$ are linearly independent. This, howewer, means no complete solution of Lagrange's problem since for $N > 2$ the exponents may be linearly dependent even if two of them have an irrational ratio. Further results along similar lines were obtained by Hartman, van Kampen, and Wintner (1938) and Weyl (1938-1939).

2. It suggests itself to extend Lagrange's problem from trigonometric polynomials to the more general class of almost periodic functions

$$F(t) \sim a_0 e^{i\lambda_0 t} + a_1 e^{i\lambda_1 t} + \ldots,$$

defined either by their translation properties as continuous functions possessing relatively dense translation numbers $\tau(\varepsilon)$ for every ε or as the closure with respect to uniform convergence of the class of trigonometric polynomials. The series on the right is the Fourier series of the function.

It was conjectured by Wintner (1930) that if $F(t)$ does not take arbitrarily small values, i. e.

$$\inf |F(t)| > 0, \tag{4}$$

then (2) is again valid, and the remainder is almost periodic. This was proved by Bohr (1930) by means of the translation properties and by Jessen (1935) by approximation with trigonometric polynomials. Regarding the value of the mean motion c we have the following results :

I. The mean motion c is always expressible by means of a finite number of the exponents λ_n both in the form

$$c = h_0\lambda_0 + \ldots + h_N\lambda_N,$$

where the coefficients h_n are integers with sum 1, and in the form

$$c = r_0\lambda_0 + \ldots + r_N\lambda_N$$

where the coefficients r_n are non-negative rational numbers with sum 1. This implies that in case of trigonometric polynomials with given exponents $\lambda_0, \ldots, \lambda_N$ (and $\inf |F(t)| > 0$) there are only a finite number of possible values of c. The exact characterization of the set of possible values of c is still an open problem.

II. If the movement $F(t)$ is considered from two different points a and b with

$$\inf |F(t) - a| > 0 \qquad \text{and} \qquad \inf |F(t) - b| > 0$$

the corresponding mean motions c_a and c_b have always a rational ratio. This is a special case of a general theorem of Fenchel and Jessen (1935) to the effect, that an almost periodic movement on any closed surface with negative Euler characteristic may be uniformly transferred into a purely periodic movement. Another generalization has recently been obtained by Tornehave, who has proved, that if $F_1(t), \ldots, F_N(t)$ are almost periodic functions such that $\inf |F_n(t)| > 0$, and if the corresponding mean motions $c_1, \ldots, c_N$ are linearly independent, then the set of arguments $\arg F_1(t), \ldots, \arg F_N(t)$ is mod. 2π everywhere dense. Plainly this is an extension of Kronecker's theorem.

3. If the almost periodic function $F(t)$ does not satisfy (4) the variation of its argument may be very complicated and if the function has zeros it may even be impossible to fix the argument as a continuous function of t. The problem seems then only to be of interest in the case of *analytic* almost periodic functions

$$f(s) = f(\sigma + it) \sim \Sigma a_n e^{\lambda_n s},$$

defined in a vertical strip $\alpha < \sigma < \beta$, which we consider on a vertical line $\sigma = \sigma_0$ in this strip. On such a line the function is

an almost periodic function of the real variable t. The series on the right is the Dirichlet series of the function and is composed of the Fourier series belonging to the different vertical lines.

Previous to the general theory of almost periodic functions the distribution of the values of certain analytic almost periodic functions in vertical strips had been the subject of detailed investigations. These investigations concern in particular the Riemann zeta function $\zeta(s)$ and its logarithm and deal not only with the half plane where the fonctions are almost periodic but also with the strip $\frac{1}{2} < \sigma \leqq 1$ where they retain a certain degree of generalized almost periodicity. Among the results we mention only the following one :

For every vertical strip $\frac{1}{2} < \sigma_1 < \sigma < \sigma_2$ and every value of a there exists a relative frequency of zeros of $\zeta(s) - a$ in the strip, i. e. if $N_a(T)$ denotes the number of zeros of $\zeta(s) - a$ lying in the strip and having ordinates between 0 and T then the limit

$$\lim_{T \to \infty} \frac{N_a(T)}{T}$$

exists, and this limit is 0 for $a = 0$ while it is > 0 for $a \neq 0$ when $\sigma_1 \leqq 1$. For $a = 0$ this theorem is due to Bohr and Landau (1914) : for $a \neq 0$ it was announced by Bohr (1922) and the proof developed by Bohr and Jessen (1932).

It is evident, that for an analytic function the distribution of the zeros in vertical strips will be closely connected with the variation of the argument on vertical lines. It is, therefore, not surprising that in the above-mentioned investigation of the zeta function the Bohl-Weyl extension of Kronecker's theorem was a main tool.

4. For an arbitrary analytic almost periodic function the distribution of the zeros in vertical strips and the variation of the argument on vertical lines was considered by Jessen (1933-1938) and Hartman (1939). The main results are as follows :

For every σ between α and β the mean value

$$\varphi(\sigma) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \log |f(\sigma + it)| \, dt$$

exists, and the function $\varphi(\sigma)$ is continuous and convex in the interval $\alpha < \sigma < \beta$ and hence possesses a derivative $\varphi'(\sigma)$ in all points of the interval except an enumerable set. If $\varphi(\sigma)$ is differentiable at the point σ then the function $f(\sigma + it)$ possesses a mean motion c, which is $= \varphi'(\sigma)$, i. e. if $\arg f(\sigma + it)$ denotes a continuous branch of the argument on the corresponding vertical line we have

$$(5) \qquad \lim_{T \to \infty} \frac{\arg f(\sigma + iT) - \arg f(\sigma)}{T} = \varphi'(\sigma).$$

Moreover, if $\varphi(\sigma)$ is differentiable at the points σ_1 and σ_2, then the relative frequency of zeros of $f(s)$ in the strip $\sigma_1 < \sigma < \sigma_2$ exists and is $= \frac{\varphi'(\sigma_2) - \varphi'(\sigma_1)}{2\pi}$, i. e. if $N(T)$ denotes the number of zeros lying in the strip and having ordinates between o and T then

$$\lim_{T \to \infty} \frac{N(T)}{T} = \frac{\varphi'(\sigma_2) - \varphi'(\sigma_1)}{2\pi}.$$

The linearity intervals of $\varphi(\sigma)$ (if any) correspond to the vertical strips without zeros of $f(s)$.

In the special case of a purely periodic function this formula is equivalent to the classical Jensen formula. The function $\varphi(\sigma)$ is called the *Jensen function* belonging to the almost periodic function $f(s)$.

In the conference there will be commented on the proof of the above results.

5. In continuation of the investigations just mentioned the Jensen function and its connection with mean motions and the distribution of zeros of analytic almost periodic functions has been the object of a detailed and systematic study by Jessen and Tornehave, published in *Acta Mathematica* (1945). The main purpose of the present lecture is to indicate the results obtained in this paper. They fall into three parts.

I. A detailed discussion of the variation of the argument on an arbitrary vertical line.

II. Complete characterization of the convex functions which

may occur as the Jensen function of an analytic almost periodic function.

III. Results on special classes of analytic almost periodic functions, including all exponential polynomials and all ordinary Dirichlet series, and leading among other things to a solution of Lagrange's problem.

6. As to the first problem it is convenient, for every vertical line to introduce a left and a right argument of $f(\sigma + it)$ obtained by encircling the zeros of $f(s)$ on the line to the left or right respectively. From these arguments one obtains by taking lower and upper limits four mean motions, which are called the lower and upper, left and right mean motions and are denoted by $\underline{c}^-$, $\bar{c}^-$, $\underline{c}^+$, $\bar{c}^+$. Formula (5) is then for an arbitrary σ replaced by the inequalities

$$\varphi'(\sigma - 0) \leqq \underline{c}^- \leqq \left\{ \begin{array}{c} \underline{c}^+ \\ \bar{c}^- \end{array} \right\} \leqq \bar{c}^+ \leqq \varphi'(\sigma + 0)$$

connecting the four mean motions with the left and right derivatives of the convex function $\varphi(\sigma)$.

The main result is now that these inequalities are best possible in the sense that to any six numbers satisfying this system of inequalities there exists an almost periodic function $f(s)$ and a σ such that the six numbers are just the two derivatives of $\varphi(\sigma)$ and the four mean motions.

7. As to the second problem we first mention, that in case of a purely periodic function, say with period $2\pi i$, the Jensen function $\varphi(\sigma)$ is a convex polygon for which $\varphi'(\sigma)$ takes only integral values (the vertices lie on the vertical lines through the zeros). Conversely any such polygon is the Jensen function of a periodic function with period $2\pi i$.

In case of a limit periodic function with limit period $2\pi i$, i. e. with rational exponents, it is easily seen that the values of $\varphi'(\sigma)$ in the linearity intervals of $\varphi(\sigma)$ (if any) are rational. It was shown by Buch (1938) that conversely any convex function of this kind

is the Jensen function of a limit periodic function with limit period $2\pi i$.

These results would suggest, that any convex function should be the Jensen function of some analytic almost periodic function, but this is not the case. The necessary and sufficient condition for a convex function $\varphi(\sigma)$, $\alpha < \sigma < \beta$ to be a Jensen function is that to any reduced interval $(\alpha <)\, \alpha_0 < \sigma < \beta_0 (< \beta)$ there exist a finite set of linearly independent numbers $\mu_1, \ldots, \mu_M$ and a number $k > 0$ such that if σ_1 and σ_2, where $\alpha_0 < \sigma_1 < \sigma_2 < \beta_0$, belong to different linearity intervals of $\varphi(\sigma)$, then

$$\varphi'(\sigma_2) - \varphi'(\sigma_1) = r_1 \mu_1 + \ldots + r_M \mu_M,$$

where the coefficients r_m are rational numbers and

$$\frac{r_1 \mu_1 + \ldots + r_M \mu_M}{\sqrt{r_1^2 + \ldots + r_M^2}} \geq k.$$

8. Finally we come to the third problem. In case of a purely periodic function it is easily seen that the mean motion exists for all values of σ and is determined by

$$(6) \qquad \lim_{T \to \infty} \frac{\arg f(\sigma + iT) - \arg f(\sigma)}{T} = \frac{\varphi'(\sigma - 0) + \varphi'(\sigma + 0)}{2}.$$

This is due to the fact that there are no zeros in the immediate neighbourhood of the line. Similarly the frequency of zeros exists for every strip and is determined by

$$(7) \qquad \lim_{T \to \infty} \frac{N(T)}{T} = \frac{\varphi'(\sigma_2 - 0) - \varphi'(\sigma_1 + 0)}{2\pi}.$$

The problem naturally arises to find other classes for which these more precise relations hold.

Since periodicity means, that the exponents λ_n belong to a discrete module $\{ h\mu \}$ where $\mu \neq 0$ and h runs through all integers, one might expect a positive answer in case of other simple modules, such as $\{ r\mu \}$, r rational, or $\{ h_1 \mu_1 + h_2 \mu_2 \}$, where $\frac{\mu_1}{\mu_2}$ is irrational, and h_1 and h_2 run through all integers. This, however, is not the case. In fact, the result for problem I holds good even if we

restrict ourselves to functions $f(s)$ with exponents from any given everywhere dense module.

Nevertheless, the formulae (6) and (7) hold for large classes of functions $f(s)$ for which the exponents belong to a module with an integral base, i. e. are linear combinations with integral coefficients of a finite or infinite number of linearly independent numbers.

To explain the situation let us consider a function $f(s)$ with exponents from a module of the above type $\{h_1\mu_1 + h_2\mu_2\}$, i. e. a module with an integral base of two numbers μ_1 and μ_2. The Dirichlet series may then be written

$$f(s) \sim \sum a_{h_1, h_2} e^{(h_1\mu_1 + h_2\mu_2)s}.$$

Together with this series we consider the class of all series

$$f(s;\, x_1,\, x_2) \sim \sum a_{h_1, h_2} e^{i(h_1 x_1 + h_2 x_2)} e^{(h_1\mu_1 + h_2\mu_2)s},$$

where x_1 and x_2 are real parameters. They are easily seen to be Dirichlet series of analytic almost periodic functions $f(s;\, x_1,\, x_2)$ in the same strip $\alpha < \sigma < \beta$ as $f(s)$, and we have for any real τ

$$f(s + i\tau) = f(s;\, \mu_1\tau,\, \mu_2\tau). \tag{8}$$

Plainly $f(s;\, x_1, x_2)$ is periodic in the parameters x_1 and x_2 with the period 2π. On account of (8) it is called the spatial extension of $f(s)$.

Our result is that the relations (6) and (7) hold *whenever the spatial extension $f(s;\, x_1,\, x_2)$ is regular not only in s but also in the parameters x_1 and x_2*. The same result holds for an arbitrary finite integral base. In particulier (6) and (7) hold for any exponential polynomial

$$f(s) = a_0 e^{\lambda_0 s} + a_1 e^{\lambda_1 s} + \ldots + a_N e^{\lambda_N s}, \tag{9}$$

since in this case there exists a finite integral base (of at most N + 1 numbers) and the spatial extension obviously is regular in all the variables as it contains only a finite number of terms. Since the trigonometric polynomial (1) is obtained by considering the exponential polynomial (9) on the imaginary axis, this shows that the mean motion c exists for an arbitrary trigonometric polynomial $F(t)$ and is given by

$$c = \frac{\varphi'(-0) + \varphi'(+0)}{2},$$

where $\varphi(\sigma)$ is the Jensen function corresponding to $f(s)$. Thus Lagrange's problem is solved, and with a positive answer.

Indications of the proof will be given in the conference. It is based on the Kronecker-Weyl theorem and on Weierstrass '*Vorbereitungssatz*' and also shows that the points of non-differentiability of $\varphi(\sigma)$ have no accumulation points in the interior of the interval $\alpha < \sigma < \beta$.

9. Quite analogous results hold for an analytic almost periodic function with an infinite integral base. The spatial extension here contains infinitely many parameters $x_1, x_2, \ldots$, and the regularity in the variables $s, x_1, x_2, \ldots$, to be assumed is merely regularity in any finite number of the variables for fixed values of the remaining variables. Besides, integration in infinitely many variables is applied in the proof.

An important class of functions with infinite integral base and analytic spatial extension (in the above sense) is formed by *all ordinary Dirichlet series*

$$f(s) = \sum_{1}^{\infty} \frac{a_n}{n^s} = \sum_{1}^{\infty} a_n e^{-(\log n)s},$$

in their half-plane of uniform convergence. As base we may here use the numbers $-(\log p)$, where p runs through the primes, and the exponents are then integral combinations with non-negative coefficients of the numbers in the base. Writing

$$n = p_{n_1}^{\gamma_1} \ \ldots \ p_{n_r}^{\gamma_r} \qquad \text{and} \qquad e^{ix_m} = y_m,$$

the spatial extension takes the form

$$f(s;\, x_1, x_2, \ldots) = \sum_{1}^{\infty} \frac{a_n}{n^s} y_{n_1}^{\gamma_1} \ \ldots \ y_{n_r}^{\gamma_r}.$$

As this is a power series in $y_1, y_2, \ldots$, the regularity in all the variables easily follows. In a slightly different form this power series was used for other purposes by Bohr (1913).

10. At the end of the conference a recent unpublished investigation by Miss Borchsenius and Jessen will be briefly dicussed. It con-

cerns an extension of the results connected with the Jensen function to generalized almost periodic functions and a detailed study of the functions

$$\zeta(s) - a \qquad \text{and} \qquad \log \zeta(s) - a$$

in the half-plane $\sigma > \frac{1}{2}$.

LITERATURE.

B. JESSEN and H. TORNEHAVE. *Mean motions and zeros of almost periodic functions.* (*Acta Math.*, 77, 1945, p. 137-279. This paper contains a full bibliography of the subject.

ON ALMOST PERIODIC FUNCTIONS AND THE THEORY OF GROUPS*

HARALD BOHR, University of Copenhagen

1. Introduction. The subject which I have chosen for my lecture, the theory of the almost periodic functions, has gradually been increased to a comprehensive and extensive theory by the contributions of numerous mathematicians in various countries. Therefore it would be a rather impossible task to try to give in a single lecture even a very cursory survey of the many different problems which have been taken up for treatment within the scope of this theory and its generalizations. The task, then, which I have set myself today is less comprehensive. First I shall try to describe quite briefly what might be called the main problem of the theory, confining myself however to the consideration of functions of a real variable, and to explain some especially important features of its solution. Subsequently I shall explain to you in a few words how this main problem could later on be fitted into a much wider class of problems than was originally the case, that is to say, that it could be considered as a problem in the so-called theory of groups. The points of view which led to this extension were first emphasized by Weyl, whereas the accomplishment of the group theoretical treatment is due to von Neumann.

2. Periodic functions. Before I begin to speak about the almost periodic functions it will be natural and convenient to say first a few words about the theory of the purely periodic functions. We shall start with a very simple, but at the same time very general notion, namely, a quite arbitrary periodic continuous motion in the plane. Let t denote the time, and let us use complex numbers $w=u+iv$ to characterize the points of the plane. Then this motion can be represented by an equation

$$w = w(t) = u(t) + iv(t),$$

where $w(t)$ is a continuous complex function for $-\infty < t < \infty$, periodic with a period p. Such a motion can of course be extremely complicated. Among these motions the most primitive one is certainly a uniform motion on a circle, for instance with its centre at the origin. Such a motion may be represented by the equation

$$w = ae^{i\lambda t}$$

where a is a complex constant and λ a real number. Let $a = re^{i\theta}$. Then r indicates the radius of the circle, and λ is the angular velocity, so that the period is $2\pi/|\lambda|$, whereas θ indicates the phase, *i.e.*, determines where on the circle the point is to be found at the time $t=0$. Such a uniform circular motion, represented by $w = ae^{i\lambda t}$, will be called a pure oscillation. For the present, we shall consider only those pure oscillations which have a given number p as one

* Rouse Ball Lecture, delivered in Cambridge, England, in May, 1946.

C51. *Amer. Math. Monthly* 56 (1949).

of their periods; let us call them mutually harmonic. If for the sake of convenience we choose $p=2\pi$, the circular motions selected in this manner are just the ones represented by $a_n e^{int}$, where n is an arbitrary integer and a_n is a complex constant. Combining an arbitrary finite number of such simple circular motions with the period 2π, *i.e.*, considering an exponential polynomial $s(t)$ of the form

$$s(t) = a_0 + a_1 e^{it} + a_{-1} e^{-it} + \cdots + a_n e^{int} + a_{-n} e^{-int},$$

we get of course again a continuous periodic motion of period 2π, which may, however, look very complicated. Such motions, produced by so-called superposition of mutually harmonic oscillations were, as is well known, not unfamiliar even to ancient Greek astronomers. Of every motion produced in this way we shall say that it can be decomposed into mutually harmonic pure oscillations. But we shall extend the meaning of this notion somewhat further, it being convenient to operate not only with finite sums, but also with infinite sums, that is to say, also to involve a limit transition. We shall here consider only a limit process uniform for all t, as the simplest possible limit transition. Thus more generally we shall say about a function $w(t)$ that it can be decomposed into mutually harmonic oscillations, if the function can be represented as the result of a uniform limit transition on finite sums of the kind in question. From the point of view of pure mathematics as well as of the applications, it is evidently a problem of decisive importance to find out which continuous periodic motions with the period 2π may in this way be composed of uniform circular motions. As is well known, this problem was solved by Weierstrass in his famous theorem that every continuous motion which is periodic with the period 2π allows such a decomposition. In other words any quite arbitrary continuous periodic motion with the period 2π can be approximated for all t by one of our polynomials $s(t)$ with an arbitrarily given degree of approximation. This being the case, the question naturally arises as to the intensity and phase with which a definite one of our oscillations e^{int} occurs in the arbitrary motion $w=w(t)$ under consideration. That is, what value does the coefficient of e^{int} assume? Evidently the coefficients of the single terms of the approximating polynomial $s(t)$ cannot be fixed exactly before we have gone to the limit, *i.e.*, as long as we only consider a polynomial $s(t)$ approximating $w(t)$ with a certain degree. However, $s_v(t)$ being a sequence of polynomials which for $v\rightarrow\infty$ converges uniformly to the given function $w(t)$, the coefficient $a_n^{(v)}$ with which the oscillation e^{int} occurs in the polynomial $s_v(t)$ will for every fixed n converge to a definite value A_n. This number A_n is called the nth Fourier coefficient of the function, and it may be said to indicate the intensity and phase with which the corresponding oscillation e^{int} occurs in the given motion $w(t)$. This nth Fourier coefficient A_n is determined by

$$A_n = \mathfrak{M}\{w(t)e^{-int}\},$$

where $\mathfrak{M}\{w(t)e^{-int}\}$ denotes the mean value $(1/2\pi)\int_0^{2\pi} w(t)e^{-int}dt$. There is

another way, namely, by a more formal consideration, by which we may immediately arrive at these expressions for the Fourier coefficients. Let us write formally $w(t)$ as an infinite series

$$w(t) = \sum_{-\infty}^{\infty} A_n e^{int}$$

and make use of the fact that the system of functions e^{int} forms a normalized orthogonal system in the sense that for any two arbitrary functions $\phi(t) = e^{in_1t}$ and $\psi(t) = e^{in_2t}$ of the system

$$\mathfrak{M}\{\phi(t)\bar{\psi}(t)\} = \mathfrak{M}\{e^{in_1t}e^{-in_2t}\} = \begin{cases} 0 & \text{for} \quad n_1 \neq n_2 \\ 1 & \text{for} \quad n_1 = n_2. \end{cases}$$

Then, by multiplying our infinite series by e^{-int} and integrating term by term we get just the expression given above for the coefficient A_n of e^{int}. We have indicated above how, starting from Weierstrass' approximation theorem and by performing the limit transition, we were led to the Fourier coefficients and thus to the Fourier series of the function $w(t)$. Conversely, however, we can prove Weierstrass' approximation theorem starting from the formally formed Fourier series of the function $w(t)$. Indeed, the exponential polynomials which are directly determined by the partial sums $\sum_{-n}^{n} A_\nu e^{i\nu t}$ of the Fourier series cannot always be used as uniform approximation sums; these partial sums do, it is true, in a certain sense approximate the function best, namely in the so-called mean, but not uniformly, as we claim here; however, starting from the Fourier series we may in different ways by various so-called summation methods form finite sums $S_N(t)$ which for $N \to \infty$ converge uniformly to the given function $w(t)$. From the mere fact that it is possible from the Fourier series of the function $w(t)$ to determine, *i.e.*, to come back to the function $w(t)$, we see in particular that the function $w(t)$ is uniquely determined by its Fourier series, *i.e.*, that two different continuous periodic functions cannot have one and the same Fourier series. This fundamental theorem, the so-called uniqueness theorem, can also be formulated in the following way: The function $w(t)$ identically 0 is the only function the Fourier constants of which vanish altogether; consequently there does not exist any function $w(t)$ which may be added to the system e^{int} so that the extended system again becomes a normalized orthogonal system. This is expressed by saying that the system e^{int} is a complete normalized orthogonal system.

3. Almost periodic functions. I have dwelt at comparative length on the theory of the continuous purely periodic functions and the theory of their Fourier series, and adapted my remarks about them, in order to be able to speak all the more briefly of the corresponding more general theory of the almost periodic functions. Just as before, we start our discussion of the general

theory by considering the simple pure oscillations

$$w = ae^{i\lambda t},$$

but now we include all of them in our considerations, *i.e.*, we do not select a simple mutually harmonic system by considering only those oscillations which have a given period. This gives from the very beginning an essentially different situation in view of the fact that the total number of pure oscillations has the power of the continuum whereas there exists only a denumerable number of pure oscillations with a given period. As before, so also here, we consider all finite sums of our pure oscillations, *i.e.*, all exponential polynomials of the form

$$s(t) = \sum a_n e^{i\lambda_n t}$$

where now, however, the exponents λ_n may be quite arbitrary real numbers and not all of them multiples of one and the same number as before. As formerly, we are interested in the continuous motion which is determined by the function $w=s(t)$. A principal difference, though, is that in this case the composed motion is no longer periodic; certainly the single components $a_n e^{i\lambda_n t}$ are still periodic, but in general they will have no common period, since the exponents may be incommensurable. However, as pointed out by Bohl, who in some very interesting papers studied some classes of continuous functions which include the periodic functions and are contained in the more general class of the almost periodic functions, the motion described by $w=s(t)$ must at any rate present certain periodic-like features, namely, for every $\epsilon>0$ there is an infinite number of so-called almost periods or translations numbers $\tau(\epsilon)$. By a translation number $\tau(\epsilon)$ we understand a number τ which for all t satisfies the inequality

$$|w(t+\tau) - w(t)| \leqq \epsilon.$$

In the study of the general class of motions which may be decomposed into a finite or infinite number of pure oscillations the first important problem is, of course, to find the theorem analogous to Weierstrass' theorem concerning the special case of mutually harmonic oscillations e^{int}. To put it more exactly, we ask in analogy with the former case: Which continuous motions $w=w(t)$ may be represented either by finite sums of pure oscillations or may be uniformly approached by such sums. Evidently these motions need not be periodic, but they are far from being quite arbitrary, since they must certainly present some periodic-like features. The exact solution of the problem is that the function $w(t)$ should be what I have called an almost periodic function. The definition of such a function reads: A function $f(t)$ continuous for all t is called almost periodic if, firstly, for any $\epsilon>0$ it possesses translation numbers $\tau(\epsilon)$ in the sense defined above and if furthermore, the set of translation numbers belonging to a given $\epsilon>0$ is relatively dense, which means that there do not exist arbitrarily great intervals which are free from such translation numbers $\tau(\epsilon)$.

In view of the subsequent group theoretical investigations I should like to insert a remark about another way of characterizing the almost periodic func-

tions, a way which proves to be closely associated with the original definition given above. Already in my early investigations of almost periodic functions I had occasion to use the following theorem: Let $f(x)$ be an almost periodic function and $h_1, h_2, \cdots$ an arbitrary sequence of real numbers. Consider the sequence of functions $f(x+h_1), f(x+h_2), \cdots$ whose elements originate from the given almost periodic function $f(x)$ by the corresponding translations of the independent variable. Then we can always select a subsequence $h_{n_1}, h_{n_2}, \cdots$ so that the new sequence of functions $f(x+h_{n_1}), f(x+h_{n_2}), \cdots$ converges uniformly on the whole x-axis. Later on Bochner found the interesting result that this theorem can be converted so that we have really a new characterization of the very notion of almost periodicity. This new definition reads in exact formulation: A function $f(x)$ continuous for all x is called almost periodic, if from each sequence of functions $f(x+h_1), f(x+h_2), \cdots$ formed from $f(x)$ by translations of the x-axis a sub-sequence may be selected which converges uniformly for all x. This may also be expressed by saying that the set of functions $\{f(x+h)\}$, $-\infty < h < \infty$, formed from $f(x)$ by all possible translations is compact.

It is not very difficult to prove that every function $w(t)$ which can be decomposed into pure oscillations, *i.e.*, can be approximated uniformly by exponential polynomials $s(t)$, is an almost periodic function. The essential difficulty lies in the proof of the converse, *i.e.*, that every almost periodic motion $w(t)$ can be approximated uniformly by sums of pure oscillations. Here let me stress that the oscillations $a_n e^{i\lambda_n t}$ which occur with no quite negligible coefficients in an exponential sum $s(t)$ which approximates the given function $w(t)$ sufficiently closely must have exponents λ_n, which, in contrast to the coefficients a_n, have exactly determined values characteristic of the function $w(t)$ in question. This is intimately associated with the fact that while an oscillation $ae^{i\lambda t}$ is only slightly changed if the coefficient a is changed a little, the least change of the exponent λ means an essential change of the course of the oscillation, since we are interested in this course for all times, *i.e.*, for $-\infty < t < \infty$. In proving that an arbitrary almost periodic function $f(t)$ can really be approximated by finite sums of pure oscillations, we must therefore begin by finding a way to make the given almost periodic function $f(t)$, so to speak, deliver as its oscillation exponents certain numbers Λ_n characteristic of that function. This is obtained by connecting a Fourier series with the function, just as in the case of the periodic functions. This Fourier series, however, has here the general form

$$\sum A_n e^{i\Lambda_n t},$$

where the set of the exponents Λ_n, characteristic of the function, may be any enumerable set of real numbers and not just of integers. Thus, since the oscillation exponents Λ_n of an almost periodic function are first disclosed by its Fourier series, the Fourier series assumes, in a sense, a still more central position in the theory of the almost periodic functions than it does in the more restricted class of the purely periodic functions, where the exponents are given beforehand. In this lecture I am, of course, not able to go further into the structure of the theory,

but shall only say a few words about it. The starting point is that the total non-enumerable system of all pure oscillations $e^{i\lambda t}$, where λ runs over all real numbers, forms a normalized orthogonal system, but now, of course, only if we consider the system on the whole t-axis; for, denoting by $\mathfrak{M}\{f\}$ the mean value over the infinite t-interval, namely,

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(t)dt,$$

we have

$$\mathfrak{M}\{e^{i\Lambda_1 t}e^{-i\Lambda_2 t}\} = \begin{cases}0 & \text{for} \quad \Lambda_1 \neq \Lambda_2,\\ 1 & \text{for} \quad \Lambda_1 = \Lambda_2.\end{cases}$$

Therefore, writing formally our given almost periodic function $f(t)$ as an infinite sum of pure oscillations, that is,

$$f(t) = \sum_1^{\infty} A_n e^{i\Lambda_n t},$$

we can determine the coefficient A_n of the oscillation $e^{i\Lambda_n t}$ by a formal calculation, quite analogous to that in the case of the purely periodic functions. Multiplying the equation by $e^{-i\Lambda_n t}$,and taking the mean value on both sides, we get

$$A_n = \mathfrak{M}\{f(t)e^{-i\Lambda_n t}\}.$$

However, this formal starting point must be seen from a point of view essentially different from that of the purely periodic case, where the exponents Λ_n were numbers given beforehand and where the relation was only to serve to determine the corresponding coefficients A_n. In our case where we do not know the exponents of the function but have to determine them, we proceed in the following way. For an arbitrary real λ we form the mean value

$$\mathfrak{M}\{f(t)e^{-i\lambda t}\} = a(\lambda).$$

The forming of this mean value may be interpreted as a question put to the function: Do you for the given λ contain the oscillation $e^{i\lambda t}$? The result $a(\lambda)=0$ means that the function $f(t)$ answers the question in the negative, whereas the answer $a(\lambda)\neq 0$ means that the function admits that it contains the oscillation $e^{i\lambda t}$ in question, and with a coefficient the value of which is given by the number $a(\lambda)$. Now it is relatively easy to show, and this result is a decisive point in the development of the theory, that the answer is negative for almost all values of λ, *i.e.*, that $a(\lambda)=0$ for all λ apart from an enumerable set, say $\Lambda_1, \Lambda_2, \cdots$. These values of λ are called the Fourier exponents of the function, and the corresponding mean values $A_n = a(\Lambda_n)\neq 0$ are called the Fourier coefficients of the function. With these pairs of numbers Λ_n, A_n we form the infinite series $\Sigma A_n e^{i\Lambda_n t}$ called the Fourier series of the almost periodic function $f(t)$ in question. Thus

we have obtained a first, important, though for the present, formal, starting point of the theory.

About the further development, leading up to the final result that $f(t)$ can be approximated uniformly by finite exponential sums $s(t)$, I shall say only a few words. We have seen how to every almost periodic function $f(t)$ is attached an infinite series $\Sigma A_n e^{i\Lambda_n t}$ with real exponents Λ_n and complex coefficients A_n, the Fourier series of the function. It now becomes a question of decisive importance to ascertain whether conversely the function $f(t)$ is uniquely determined by its Fourier series, or in other words, whether two different almost periodic functions always have two different Fourier series. We may also formulate this question in another way, and ask whether our normalized orthogonal system $e^{i\Lambda t}$ is complete in the set of the almost periodic functions, *i.e.*, whether we may add a further almost periodic function to this system so that the extended system becomes again a normalized orthogonal system. In a more vague formulation we may interpret the question in the following way: Can an almost periodic function really be entirely decomposed into a denumerable number of pure oscillations or does there remain an undecomposable remainder of the function after the pure oscillations given by the Fourier series have been removed? Fortunately the uniqueness theorem saying that every almost periodic function is entirely characterized by its Fourier series is valid. To prove this theorem or other theorems equivalent to it is the central, but also the most difficult point, of the theory. Several proofs exist, varying as to starting point and method. The most simple, and in a certain sense the most elementary one, is de la Vallée Poussin's proof; it may be characterized as a considerable simplification of the original and rather complicated proof of the lecturer. Other proofs were given by Norbert Wiener in connection with his interesting general spectral theory and by Hermann Weyl. Weyl's proof, based on an analogy with the theory of integral equations, has proved to be especially significant for the group-theoretical generalization of the theory about which I shall speak in a moment.

With the uniqueness theorem at our disposal, we may advance in various ways to obtain the main theorem of our theory, the theorem of the uniform approximation by exponential polynomials which is the counterpart and the generalization of Weierstrass' approximation theorem for purely periodic functions. The original proof, given by the lecturer, and generalizing a method of Bohl which was developed for an essentially more restricted class of functions, was based on a theory of Fourier series for the so-called limit-periodic functions of an infinite number of variables. Later on, Bochner succeeded in showing that the approximating exponential sums $s(t)$ in question could also be obtained directly from the Fourier series of the function $f(t)$ without the transition to functions of an infinite number of variables. Other proofs of the approximation theorem were given by Weyl and Wiener among others. Among these proofs Wiener's proof has turned out to be especially suitable for generalization. In one sense or the other all the different proofs may be regarded as summation methods, the application of which to the Fourier series of an almost periodic

function produces finite sums $s(t)$ which converge uniformly to the function.

Before I proceed to place the theory within the scope of the theory of groups, I want briefly to mention that, like the classical theory of Fourier series of purely periodic functions, the theory of the Fourier series of the almost periodic functions has also been generalized in various ways by numerous mathematicians. Of special interest is a generalization due to Besicovitch, who succeeded in obtaining a class of functions almost periodic in a generalized sense, the Fourier series of which may be characterized in an especially simple way, being just the series $\Sigma A_n e^{i\Lambda_n t}$ for which $\Sigma |A_n|^2$ is convergent, while the Λ_n may be quite arbitrary real numbers.

4. The theory of almost periodic functions as a part of the theory of groups. In the remaining part of my lecture I shall try to describe briefly the points of view which made Weyl and von Neumann see the theory of the almost periodic functions (including in particular the classical theory of the purely periodic functions) in a far more general light, *i.e.*, to see it as belonging within the general theory of groups.

We may start our considerations by observing what in the present connection may be said to be the essential properties of the normalized orthogonal system in question, *i.e.*, the system of the pure oscillations $e^{i\lambda x}$. These pure oscillations may, of course, be looked upon in different ways, but their main characteristics may be said to be that they satisfy the simple functional equation

$$\phi(x + y) = \phi(x)\cdot\phi(y).$$

As is well known, this functional equation has an infinite number of solutions, both continuous and discontinuous, the latter being of a rather unpleasant or abnormal character, in fact, not even measurable in the general sense of Lebesgue. As regards the continuous solutions, they simply consist of all functions $e^{\alpha x}$, where α is an arbitrary complex constant. Selecting from among them only those for which the exponent α is a purely imaginary number $i\lambda$, we get only our pure oscillations $e^{i\lambda x}$, which thus may be characterized as the bounded continuous solutions of the functional equation in question. However, a functional equation of the type

$$\phi(x + y) = \phi(x)\cdot\phi(y)$$

is also encountered in quite another discipline, namely, in the general theory of groups, where the independent variable x, however, need not be a number, but may be a symbol of quite a different kind, whereas the values of the function $\phi(x)$ are still ordinary complex numbers. Before going into details I must first say a few words about the general notion "abstract group" which plays such a fundamental part in the mathematics of our day; this importance follows from the fact that in the theory of groups many apparently quite different investigations taken from numerous mathematical disciplines may be comprised. For the sake of simplicity I shall confine myself in what follows to the consideration of

the so-called commutative or Abelian groups. Such a group is a set of a finite or an infinite set of elements $x, y, \cdots, A, B, \cdots$ (the term "element" is used if we have to deal with abstract investigations where we do not want to commit ourselves beforehand to the considered objects being of any definite kind). For these elements a single so-called composition rule is given which is called multiplication or addition, and is denoted by the multiplication sign $\cdot$ or the addition sign $+$, respectively, but which need not have anything to do with ordinary multiplication or addition, for the simple reason that the elements need not be numbers.

If we use the multiplication sign for the composition of the group, the formulas

$$A \cdot B = B \cdot A$$

$$A \cdot (B \cdot C) = (A \cdot B) \cdot C$$

are valid, and, furthermore, we claim that the equation

$$A \cdot X = B,$$

where A and B are two arbitrary given elements, is always satisfied by one and only one element X. If we use the addition sign, these rules read that

$$A + B = B + A$$

$$A + (B + C) = (A + B) + C$$

and that the equation

$$A + X = B$$

has one and only one solution.

Let me give you one or two examples of such Abelian groups, chosen in close connection with our subject.

First, let us consider the set of all rotations of a circle, for instance, the unit circle, about its center. Let the single rotation be characterized by its rotation angle x, where the real number x is determined modulo 2π. Then the composition rule, *i.e.*, the composition of two rotations x and y, is simply expressed by the sum $x+y$, this number being of course only determined modulo 2π. On the other hand, if we characterize the rotation by the point or complex number $X=e^{ix}$ on the unit circle, into which the point 1 is transformed by the rotation, then the composition of two rotations, determined by $X=e^{ix}$ and $Y=e^{iy}$, respectively, is expressed by ordinary multiplication $X \cdot Y=e^{i(x+y)}$.

As the second example I choose the set of the translations of a straight line into itself. Let a single translation be characterized by the (positive or negative) number x, indicating the length of the translation or, what comes to the same thing, by the point (number) x into which the origin is transformed by the translation. Then the composition of two translations given by x and y, respectively, is, of course, expressed by the ordinary sum $x+y$. Both the rotation group and the translation group are so-called topological Abelian groups, which means

that, besides the structure fixed by the given composition, they have also another so-called topological structure, in consequence of which we may, for example, speak about two elements of the group lying near each other or far from each other. We observe that the first group, the rotation group, may in a certain sense be placed under the latter group, the translation group, by identifying those points on the line which differ by a multiple of 2π, or more geometrically expressed, by imagining the straight line twisted around the unit circle.

Besides these two examples of infinite groups, *i.e.*, groups containing an infinite number of elements, I shall mention a classical example of a finite Abelian group, the group of the classes of residues modulo n, where n is a positive integer. As is well known, this group is a dominating factor in an essential part of the elementary theory of numbers. Let us consider, for instance $n=3$; then the group contains only two elements which we may denote by a_1, a_2 corresponding to the two classes of integers which are relatively prime to 3. These two classes consist of the numbers of the form $3n+1$ and $3n+2$, respectively. The composition of the group is given by the following scheme

$$a_1 \cdot a_1 = a_1, \qquad a_1 \cdot a_2 = a_2 \cdot a_1 = a_2, \qquad a_2 \cdot a_2 = a_1.$$

This scheme states that the product of two numbers, both of the form $3n+1$ or both of the form $3n+2$, is a number of the form $3n+1$, whereas the product of two numbers, one of the form $3n+1$ and the other of the form $3n+2$, is of the form $3n+2$.

While generally the elements of an Abelian group are themselves not numbers, but symbols of one kind or another, they can in a natural way be connected with numbers, generally complex numbers, so that the composition rule for two arbitrary elements of the group is reproduced by ordinary multiplication of the numbers attached to these elements. This is obtained by means of the so-called group characters. A character belonging to an Abelian group is a function $\chi(X)$, where the independent variable X ranges over the elements of the group, while the values of the function are ordinary complex numbers, satisfying, for any two elements X and Y of the group, the equation

$$\chi(X \cdot Y) = \chi(X) \cdot \chi(Y)$$

where $X \cdot Y$ determines the element resulting from X and Y by the composition of the group. If the composition of the group is expressed by $X+Y$ instead of by $X \cdot Y$, the equation characteristic of a character reads

$$\chi(X+Y) = \chi(X) \cdot \chi(Y).$$

Thus we see an evident association with the functional equation valid for the exponential function, *i.e.*,

$$\phi(x+y) = \phi(x) \cdot \phi(y)$$

where x and y denote ordinary real numbers. It may be noted that it is not

demanded that a character $\chi(X)$ should take two different values for two different elements X_1 and X_2; thus for any group we have the trivial so-called main character which assumes the value $\chi(X)=1$ for every X. Generally there exist both real characters for which $\chi(X)$ is a real number for every X and complex characters $\chi(X)$ which assume complex values for certain elements X. If $\chi(X)$ is a complex character, the conjugate function $\bar{\chi}(X)$ is obviously again a complex character. Further, as the functional equation provides immediately, the product $\chi_1(X)\chi_2(X)$ of two characters $\chi_1(X)$ and $\chi_2(X)$ is again a character of the group.

I shall speak briefly about the characters of the three particular Abelian groups mentioned above. As regards the group of the classes of residues modulo n with $h=\phi(n)$ elements $X_1, X_2, \cdots, X_h$, where $\phi(n)$ is the Euler function indicating the number of elements among $1, 2, \cdots, n$ which are relatively prime to n, we have at the same time, $h=\phi(n)$ different characters $\chi_1(X), \chi_2(X), \cdots, \chi_h(X)$. For these characters we have the important relations

$$\frac{1}{h}\sum_{m=1}^{h}\chi_\nu(X_m)\bar{\chi}_\mu(X_m) = \begin{cases} 0 & \text{for} \quad \nu \neq \mu, \\ 1 & \text{for} \quad \nu = \mu \end{cases}$$

which may be said to express the fact that the characters form a normalized orthogonal system by a simple formation of mean value. It was the study of the characters of this group of the residues modulo n which was the starting point of Dirichlet's famous proof that every arithmetical progression contains an infinite number of primes.

Concerning the two other groups, the rotation group and the translation group, the elements of which we will denote by x modulo 2π and by x, respectively, where x ranges over the real numbers, we will call attention only to some of their characters, namely, the bounded characters; the unbounded characters are of no importance for our purpose. For a bounded character it is seen readily that $|\chi(x)|=1$ for all x. If furthermore we require that our bounded characters should be continuous functions on the group, (considering the group as a topological group, *i.e.*, continuous functions of the variable x), then, in the case of the rotation group (where a character must be periodic with the period 2π) these characters are only the functions e^{inx}, $n=0, \pm 1, \pm 2, \cdots$, whereas for the translation group we get the much more comprehensive set of characters $e^{i\lambda x}$, where λ is an arbitrary real number. Thus we realize that the mutually harmonic pure oscillations forming the basis of the theory of Fourier series of the purely periodic functions may be characterized from a group theoretical point of view as the bounded continuous characters of the rotation group, while the set of all pure oscillations $e^{i\lambda x}$ forming the basis of the theory of the almost periodic functions may be characterized as the bounded continuous characters of the translation group. Now we understand how the main problem solved in the theory of the purely periodic and of the almost periodic functions may, according to Weyl, be generalized to the following general problem concerning a quite arbitrary Abelian group: Which function $f(X)$ defined on the group, *i.e.*,

the independent variable of which ranges over the elements of the group, can be represented by a linear composition of the bounded characters of the group? Precisely speaking, which functions defined on the given Abelian group can either be represented as a finite sum $s(X)=\sum a_\chi \chi(X)$, where the coefficients a_χ are complex constants, or can be uniformly approached by such sums?

Passing on to an outline of the solution of this general problem I start with the following remark, where for the sake of connection with the first part of my lecture it will be convenient to use an additive and not multiplicative notation for the composition of the group. We consider first a single, arbitrarily chosen bounded character $\chi(X)$ of the group. Let H be a parameter which ranges over the whole group. Then the set of all the functions $\{\chi(X+H)\}$ will, as is seen easily from the functional equation $\chi(X+H)=\chi(X)\chi(H)$, be a compact set, in the sense that from any sequence of functions $\chi(X+H_1)$, $\chi(X+H_2)$, $\cdots$ taken from the set, we can choose a subsequence of functions converging uniformly on the group. Furthermore from this property of any single bounded character we may without difficulty conclude that every function $f(X)$ which can be composed linearly of bounded characters of the group will have the same quality, *i.e.*, for every such function $f(X)$ the set of functions $\{f(X+H)\}$ will be compact. Now, guided by Bochner's formulation of the definition of the notion "almost periodicity" for the functions of a real variable, von Neumann set up the following general definition: A complex function $f(X)$ defined on an arbitrary Abelian group is called almost periodic on the group if the set of functions $\{f(X+H)\}$ is compact in the above sense.

As just mentioned, it is easily seen that every function on the group which can be composed linearly of bounded characters is almost periodic on the group. Von Neumann has shown that the converse theorem is valid for a quite arbitrary Abelian group (and not only for the translation group and the rotation group), so that the functions on an arbitrary Abelian group which can be linearly composed of the bounded characters of the group are exactly the almost periodic functions on the group. In its main ideas, von Neumann's proof of this fundamental theorem follows previous proofs given in the theory of the ordinary almost periodic functions. Corresponding to the theory of the ordinary almost periodic functions, where a Fourier series of the form $\sum A_n e^{i\Lambda_n x}$ was attached to every almost periodic function, there is also, in the general case of an arbitrary Abelian group, attached to any function $f(X)$ almost periodic on the group, a Fourier series, here of the form $\sum A_n \chi_n(X)$, where $\chi_1(X)$, $\chi_2(X)$, $\cdots$ is a denumerable set of bounded group characters characteristic of the function $f(X)$ under consideration. By a generalization of the method which Weyl developed to prove the uniqueness theorem for the ordinary almost periodic functions, it is shown that also in the general group theoretical case the Fourier series determines the function $f(X)$ uniquely. Thus, to two different almost periodic functions belong two different Fourier series. A generalization of the method applied by Wiener in proving the approximation theorem for the ordinary almost periodic functions is further used to accomplish the proof of the main theorem.

This theorem states that also in the general group theoretical case, starting from the Fourier series $\sum A_n\chi_n(X)$, we may form finite sums $s(X)=\sum a_n\chi_n(X)$ which approximate the given function $f(X)$ uniformly on the whole group.

So far, the theory of the almost periodic functions on arbitrary Abelian groups is quite parallel to the theory concerning the special case of the translation group. But before the theory could get started at all and be developed on the lines indicated above, there was a fundamental difficulty to be overcome which would seem to make the whole problem quite unapproachable. In fact the very basis of the formation of the Fourier series of an ordinary almost periodic function was the consideration of the mean value of the function $f(x)$ defined by

$$\lim_{\tau\to\infty}\frac{1}{2\tau}\int_{-\tau}^{\tau}f(x)dx.$$

In the general case, however, when we consider an arbitrary Abelian group without any topological structure, it might seem at first that there is no possibility of defining the notion of a mean value of a function over the group. I have no time to explain the simple and ingenious way in which von Neumann succeeded in defining this notion of the mean value $\mathfrak{M}\{f(X)\}$ of a function $f(X)$ almost periodic on the group. I shall only mention that, as soon as the mean value had been defined, the way was open to the further development of the theory in analogy with the theory of the ordinary almost periodic functions. In particular, as might be expected, the set of the bounded group characters proved to be a normalized orthogonal system, that is,

$$\mathfrak{M}\{\chi_1(X)\bar{\chi}_2(X)\} = \begin{cases} 0 & \text{for} \quad \chi_1 \neq \chi_2, \\ 1 & \text{for} \quad \chi_1 = \chi_2. \end{cases}$$

This being the case we naturally get the Fourier series $\sum A_n\chi_n(X)$ of an almost periodic function $f(X)$ by forming the mean value

$$a(\chi) = \mathfrak{M}\{f(X)\bar{\chi}(X)\}$$

for an arbitrary character $\chi(X)$ and by showing that this mean value is equal to 0 for all χ apart from a denumerable number of characters $\chi_n(X)$ characteristic for the function in question. These $\chi_n(X)$ are of course just the characters which occur in the composition of the almost periodic function $f(X)$ under consideration.

So far I have spoken as if the general theory of the almost periodic functions on an arbitrary Abelian group included in particular the ordinary theory of the purely periodic and of the almost periodic functions of a real variable, if we simply specialize our group to be the rotation group or the translation group, respectively. But as you may have observed, this is not the case. Indeed the pure oscillations forming the basis of the ordinary theory of the purely periodic or the almost periodic functions are not all of the bounded characters of the rota-

tion group or the translation group, but only those bounded characters which are continuous on these particular groups with their natural topology. In the general case of an arbitrary Abelian group, however, we must necessarily treat all the bounded characters of the group on the same footing, as a restriction concerning continuity cannot be formulated at all on a non-topological group. Seen from a general group theoretical point of view, the different bounded characters are therefore all equally simple. In the special case of the translation group, the class of all von Neumann almost periodic functions on this group, which functions were investigated by Ursell prior to and independently of the general theory, is essentially more comprehensive than the class of the ordinary continuous almost periodic functions, as von Neumann does not claim continuity for his class. Fortunately, and this is another beautiful chapter of von Neumann's theory, the theory of the ordinary, *i.e.*, of the continuous almost periodic functions, proves to fit quite naturally into his general theory. I must confine myself to a few words about this point. Let us consider an arbitrary Abelian group, and let it be possible to introduce into this group (as it is possible in the case of the translation group and the rotation group) a topology of some kind or other in order to be able to ascribe any sense at all to the notion of continuous function on the group. Then the main theorem will remain valid if in the whole of the theorem we restrict the functions under consideration by demanding that they shall be continuous on the group considered as a topological group. More precisely, it holds for any topological group that the continuous functions $f(X)$ almost periodic on the group are just those functions which can be composed linearly of the continuous characters of the group.

5. Concluding remarks. In conclusion, I should like to make two remarks in order to emphasize what has been gained by von Neumann's general theory, of which I have given you only a rough outline. In the first place, and this must be said to be a characteristic feature of the mathematics of our day, we have achieved the combination and harmonization of investigations hitherto quite unrelated into one single theory of a general abstract character. Thus, in our case, we have learned about an intimate, hitherto unobserved, connection between the theory of Fourier series of purely periodic and almost periodic functions not only of one variable but of several variables, indeed of an infinite number of variables, and the theory of the group characters of the finite Abelian groups, in particular the group of the classes of residues, which forms the basis of the investigations of the distribution of the prime numbers in the different arithmetical progressions.

As for the other remark, it concerns in particular the translation group and the difference emphasized above between the theory of the almost periodic functions of this group considered as a group without any structure (apart of course from the structure given by the composition itself) and considered as a group topologized by means of the ordinary metric of the straight line. Von Neumann's theory has enabled us to fit into our considerations the total set of all the

bounded characters of this group, *i.e.*, to operate not only with the continuous, but also with the discontinuous solutions of the classical functional equation

$$\phi(x + y) = \phi(x) \cdot \phi(y),$$

and to treat these discontinuous solutions, hitherto disdained, on exactly the same footing as the continuous solutions, *i.e.*, the pure oscillations. Indeed we have seen that in order to obtain the right systematization, it is even necessary to include these discontinuous solutions.

Here, as so often before in the history of mathematics, phenomena which appeared at first to be, so to speak, of a pathological nature, and which therefore from the start had to be excluded by means of protecting definitions, were later recognized, from a more general point of view, to be pertinent, even indispensable, to the subject under consideration.

On limit periodic functions of infinitely many variables.

By HARALD BOHR in Copenhagen.

1. In the sequel the functions to be considered are continuous complex-valued functions of unrestricted real variables. Furthermore, convergence of a sequence of functions is always to be taken in the sense of uniform convergence over the whole range of the variable (or the variables).

2. Among the a.p. (almost periodic) functions of one variable x,

$$F(x) \sim \Sigma A_n e^{i\Lambda_n x},$$

the p.p. (purely periodic) functions $P(x)$ are the simplest ones; the periods of such a function are either all real numbers (in the case of $P(x)$ being constant), or the integral multiples of a real number $p_0 \neq 0$. Another simple, although more general case of a.p functions $F(x)$ are the l.p. (limit periodic) functions $G(x)$ the set of which are obtained from the class of the p. p. functions $\{P(x)\}$ by closing it (with respect to uniform convergence), i. e.

$$\{G(x)\} = \mathrm{Cl}\,\{P(x)\}.$$

As easily seen (II, p. 141), for the l. p. functions $G(x)$ a kind of period still exists since two p.p. functions $P_1(x)$ and $P_2(x)$ which approximate a non-constant l.p. function $G(x)$ sufficiently well, must necessarily have periods with rational ratio. Denoting the class $\{G(x)\} = \mathrm{Cl}\,\{P(x)\}$ of all l. p. functions by C, and by $C_p\,(p \neq 0)$ the closure $\mathrm{Cl}\,\{P_p(x)\}$ of only those periodic functions $P_p(x)$ which have a rational multiple of p as one of their periods, we conclude that the set C is the union of all the sets C_p, i. e.

$$C = \bigcup_p C_p.$$

We may express this in the following way: We get the same set of functions (and not a smaller one) by closing first, for a fixed $p \neq 0$, the set $\{P_p(x)\}$ and then forming the union of all these closures, as we get by closing directly the whole set $\{P(x)\}$. As to the Fourier series of the l. p. functions, these are characterized, among the Fourier series of the a. p. functions, by having exponents Λ_n with mutually rational ratios; more particular, the exponents of a l. p. function from C_p are rational multiples of the number $\frac{2\pi}{p}$.

C 52. *Acta Sci. Math. Szeged* 12 B (1950).
Dedicated to L. Fejér and F. Riesz.

3. In the study of the a. p. functions of one variable the l. p. functions of an infinite number of variables play an important role (II, p. 118—163). We consider the enumerable-dimensional space of points $X=(x_1, x_2, \ldots)$ with arbitrary real coordinates where convergence of a sequence of points $X', X'', \ldots$ simply means convergence in each of the coordinates. A (continuous) function $P(X)=P(x_1, x_2, \ldots)$ is called p p. (purely periodic) with respect to the axis, if there exist non-vanishing real numbers $p_1, p_2, \ldots$ such that for each n the equation

$$P(x_1, x_2, \ldots, x_n+p_n, \ldots)=P(x_1, x_2, \ldots, x_n, \ldots)$$

holds good in the whole space. On account of the continuity of the function $P(X)$ we then also have (II, p. 135)

$$P(x_1+\nu_1 p_1, x_2+\nu_2 p_2, \ldots)=P(x_1, x_2, \ldots)$$

for each choice of the integers $\nu_1, \nu_2, \ldots$. By closing the set of all functions $P(X)$, p. p. with respect to the axis, we get the functions $G(X)$, l. p. with respect to the axis,

$$\{G(X)\}=\mathrm{Cl}\,\{P(X)\}.$$

Furthermore (II, p. 148), denoting for $p_1 \neq 0, p_2 \neq 0, \ldots$ by $C_{p_1, p_2, \ldots}$ the closure of only those of our p. p. functions $P_{p_1, p_2, \ldots}(X)$ which have rational multiples of $p_1, p_2, \ldots$ as periods with respect to the axis, we have just as before

$$C=\bigcup_{p_1, p_2, \ldots} C_{p_1, p_2, \ldots}.$$

Among the Fourier series of the a. p functions $F(X)$ of infinitely many variables

$$F(x_1, x_2, \ldots) \sim \Sigma A_n e^{i(\Lambda_{n,1} x_1+\Lambda_{n,2} x_2+\ldots+\Lambda_{n, m_n} x_{m_n})}$$

studied by Bochner (I), those belonging to l. p. functions of the class $C_{p_1, p_2, \ldots}$ are characterized by having, for each fixed m, all the numbers $\Lambda_{n, m}$ in the exponents equal to rational multiples of $\dfrac{2\pi}{p_m}$. Thus the Fourier series belonging to the l. p. functions of the class $C_{2\pi, 2\pi, \ldots}$ are just those Fourier series of a. p. functions for which the numbers $\Lambda_{n, m}$ are all rational.

4. We now introduce the notion of a substitution in our infinite-dimensional space as a linear one-to-one bicontinuous transformation T of the whole space on the whole space itself. As easily seen (III, p. 11 and V, p. 53) such a substitution may be written in the form

$$\begin{aligned} x_1 &= L_1(Y)=\alpha_{11} y_1+\alpha_{12} y_2+\ldots+\alpha_{1 q_1} y_{q_1} \\ x_2 &= L_2(Y)=\alpha_{21} y_1+\alpha_{22} y_2+\ \ldots\ +\alpha_{2 q_2} x_{q_2} \\ & \ldots\ldots\ldots\ldots \end{aligned}$$

where the linear forms $L_m(Y)$, each containing only a finite number of the coordinates of Y, fulfill the following two conditions. 1°. The $L_m(Y)$ are linearly independent, i. e. there exists no linear relation with constant, not

all vanishing coefficients among any finite number of them. 2°. Each variable y_q $(q=1,2,\ldots)$ may be "isolated", i. e. expressed as a linear combination with constant coefficients of a finite number of the $L_m(Y)$. For a later application we remark that if a (finite or enumerable) set of linear forms satisfies only the condition 1° we may always (III, p. 14) add to the set new linear forms — and even such consisting each of only one coordinate — so that also condition 2° be fulfilled.

5. The notion of almost periodicity of a function $F(X)$ on our infinite-dimensional space is invariant under any substitution T performed on X, i. e. $F(TX)$ is again an a. p. function of X. In fact the set of the a. p. functions $F(X)$ may be characterized as the closure of the set of the trigonometric polynomials $S(X)$, and a trigonometric polynomial $S(X)$ is evidently transformed into a trigonometric polynomial $S(TX)$. Now, applying a substitution T to a function $P(X)$, p. p. with respect to the axis, and denoting again the new variable point by X, instead of Y, we obtain a (continuous) function which we denote by $P_T(X)$ and abbreviatively call a p. p. function with respect to the substitution T (or more correctly with respect to the straight lines into which the coordinate axis are transformed, and which again span the whole space). For a fixed substitution T and all the $P(X)$ we form the class $\{P_T(X)\}$ and its closure $C_T=\mathrm{Cl}\,\{P_T(X)\}$, the functions of which we call l. p. functions with respect to T. Finally, we form the union Γ of all these classes C_T, $\Gamma=\bigcup\limits_{T} C_T$, the functions of which we simply denote as l. p. functions.

6. Before proceeding, it may be illustrating to consider the notion of periodicity in our infinite-dimensional space from a more general point of view. A vector $V=(v_1,v_2,\ldots)$ in our space is called a period of the (continuous) function $F(X)$ if $F(X+V)=F(X)$ for all X. Each function has the trivial period $(0,0,\ldots)$. Obviously, on account of the continuity of $F(X)$ the set of all periods of $F(X)$ is a closed module. Now, according to a simple, but not trivial theorem of E. Følner and myself (IV, p. 30 or V, p. 46) every closed module in our space may be transformed by a substitution T into a module of the simple type $\{(v_1,v_2,\ldots,v_n,\ldots)\}$ where the indices $1,2,\ldots,n,\ldots$ fall into three classes $\{n_r\}$, $\{n_s\}$, $\{n_t\}$ such that the coordinates v_{n_r} independently run through all real numbers, the coordinates v_{n_s} independently run through all integers while the remaining coordinates v_{n_t} are all equal to 0. Here we are only interested in the case in which the last class $\{n_t\}$ is empty, as otherwise the module does not span the whole space (i. e. is lying in a proper subspace). Thus we see that there exists no other function $F(X)$ with a period module which span the whole space than those introduced above, i. e. functions belonging to one of the classes $\{P_T(X)\}$.

7. Returning to the l. p. functions, there exists in case of functions of infinitely many variables a problem which has no analogue for l. p. functions of one variable[1]) and to which B. JESSEN has called my attention, namely *whether the union Γ of all the closed sets $C_T = \mathrm{Cl}\{P_T(X)\}$ is identical with (or only forms a part of) the closure Γ^* of the whole set of all the p. p. functions.* Since Γ^* is closed and Γ obviously contains all the p. p. functions, the problem is, in other words, whether the set Γ is closed. The purpose of this paper is to give the solution of this problem by proving

Theorem. *The set $\Gamma = \bigcup_T C_T$ consisting of all the l. p. functions of infinitely many variables is a closed one.*

8. The proof to be given in the next section depends on the consideration of the Fourier series of the l. p. functions $G(X) = G(x_1, x_2, \ldots)$. From what have been said before it easily follows that a necessary and sufficient condition for a Fourier series of an a. p. function $F(X)$ to be that of one of our l. p. functions is that the linear forms in the exponents

$$M_n(X) = \Lambda_{n,1}x_1 + \Lambda_{n,2}x_2 + \ldots + \Lambda_{n,m_n}x_{m_n}$$

can be obtained from linear forms with mere rational coefficients by subjecting them to some linear substitution. From this characterization of the Fourier series of the l. p. functions we shall deduce the following

Lemma. *A necessary and sufficient condition for a Fourier series $\Sigma A_n e^{iM_n(X)}$ of an a. p. function to belong to an l. p. function is that in any relation which expresses one of the linear forms $M_n(X)$ as a linear combination of a finite number of linearly independent forms of the sequence $M_1(X), M_2(X), \ldots$, the occurring constant coefficients (uniquely determined) shall all be rational.*

That the condition is necessary can immediately be seen. In fact, as a linear substitution does not change linear relations, or linear independance, among the linear forms in the exponents, it suffices to prove that the condition is fulfilled for a function l. p. with respect to the axis, for instance of the special class $C_{2\pi, 2\pi, \ldots}$. But in this case all the occurring coefficients $\Lambda_{n,m}$ are rational numbers and hence the condition is evidently fulfilled since a finite number of ordinary linear equations with rational coefficients and only one solution can only have a solution in rational numbers.

In order to see that the condition is sufficient we proceed in the following manner. From the sequence $M_1(X), M_2(X), \ldots$ we first select (successively) a subsequence of which any finite number of its terms is linearly independent and such that any $M_n(X)$ may be expressed as a linear

[1]) The problem exists also for l. p. functions of a finite number of variables $x_1, x_2, \ldots, x_n$ $(n > 1)$ and the solution given below is also valid in this case. However, in the finite-dimensional case the problem may easily be solved without applying the theory of Fourier series.

combination of a finite number of forms of this subsequence. To the linear forms of this subsequence we may, as remarked before, add new linear forms such that the enlarged (enumerable) set of linear forms may be used as right-hand sides of a linear substitution. By performing this substitution — or rather the inverse one — on our Fourier series $\Sigma A_n e^{iM_n(X)}$ we evidently obtain, from the assumption that the condition be fulfilled, a new Fourier series with mere rational coefficients in the exponents. Thus the corresponding function and hence also the function $F(X)$ before the transformation is a l.p. function.

9. We can now easily prove our theorem, viz. that the set Γ of all l. p. functions $G(X)$ is closed. We have to prove that if $F(X) \sim \Sigma A_n e^{iM_n(X)}$ is an a. p., but not a l.p. function, then $F(X)$ cannot be approximated uniformly by l. p functions. According to our lemma there exists among the linear forms $M_n(X)$ in the exponent of the Fourier series of $F(X)$ a linear relation

$$M_N(X) = b_1 M_{n_1}(X) + b_2 M_{n_2}(X) + \ldots + b_s M_{n_s}(X)$$

with linearly independent $M_{n_1}(X), M_{n_2}(X), \ldots, M_{n_s}(X)$ and not all b's rational. Now, as well-known, uniform convergence of a sequence of a. p. functions towards an (a. p.) function $F(X)$ implies formal convergence of the Fourier series of the functions of the sequence towards the Fourier series $\Sigma A_n e^{iM_n(X)}$ of $F(X)$. Hence in the Fourier series of any a. p. function $H(X)$ which approximates $F(X)$ sufficiently close each of the finite number of linear forms $M_N(X), M_{n_1}(X), \ldots, M_{n_s}(X)$ must necessarily occur as exponents, simply because they occur in the Fourier series of $F(X)$. Consequently, using the lemma once more, we see that the a. p. function $H(X)$ cannot be l. p. This proves the theorem.

References.

I. S. Bochner, Beiträge zur Theorie der fastperiodischen Funktionen. II. Teil, *Math. Annalen*, **96** (1926), pp. 383–409.

II. H. Bohr, Zur Theorie der fastperiodischen Funktionen. II. Teil, *Acta Math.*, **46** (1925), pp. 101—214.

III. H. Bohr, Unendlich viele lineare Kongruenzen mit unendlich vielen Unbekannten, *Kongelige Danske Videnskabernes Selskab, Math.-fys. Meddelelser*, **7** (1926), No. 1.

IV. H. Bohr and E. Følner, Infinite systems of linear congruences with infinitely many variables, *Ibidem*, **24** (1948), No. 12.

V. H. Bohr and E. Følner, On a structure theorem for closed modules in an infinite-dimensional space, *Studies and Essays presented to R. Courant* (New York, 1948), pp. 45–62.

(Received September 21, 1949.)

A SURVEY OF THE DIFFERENT PROOFS OF THE MAIN THEOREMS IN THE THEORY OF ALMOST PERIODIC FUNCTIONS

HARALD BOHR

1. The theory of almost periodic functions, like any other mathematical theory of a somewhat general character, has several connections with different mathematical disciplines. Thus it has important applications to the functions occurring in the analytic theory of numbers, as the Riemann zeta function, the study of which was the very origin of the development of the theory of almost periodicity; and through the work of Weyl, von Neumann, and others it has given rise to a new chapter in the general theory of groups. In my lecture today, in which I shall try to give you a short survey of the different ways in which some of the main theorems of the theory can be established, several such connections with other mathematical fields will appear indirectly. In the following I shall limit myself to considering functions of a real variable x, $f(x) = u(x) + iv(x)$, and moreover only functions continuous for all x; hence, naturally, the notion of uniform convergence, which preserves continuity, will play an essential role. Besides uniform convergence, another predominant notion will be convergence in the mean.

2. It may be convenient first to say a few words about continuous pure periodic functions $w = f(x)$. Among the periodic movements described by such functions the simplest one is certainly a uniform movement on a circle, as given by a so-called pure oscillation $w = ae^{i\lambda x}$ where λ is a real and a a complex number. In the theory of periodic functions one considers only pure oscillations with a given period p, say $p = 2\pi$, i.e., the oscillations $a_n e^{inx}$ $(n = 0, \pm 1, \pm 2, \cdots)$. By superposition of a finite number of such oscillations we get the exponential polynomials

$$s(x) = a_0 + a_1 e^{ix} + a_{-1}e^{-ix} + \cdots + a_n e^{inx} + a_{-n}e^{-inx}.$$

Rounding off the set of all such polynomials $s(x)$ with the help of uniform convergence, we get the closure $\mathrm{Cl}\{s(x)\}$, the elements of which, evidently, are again continuous periodic functions with period 2π. The famous theorem of Weierstrass states that this set $\mathrm{Cl}\{s(x)\}$ contains all continuous periodic functions $P(x)$ with period 2π. Among the different proofs of this theorem, those based on the theory of Fourier series are of main interest for our purpose. Starting from the fact that the pure oscillations e^{inx}, considered in an interval of length 2π, form a normal orthogonal set, i.e.,

$$M\{e^{in_1x}\cdot e^{-in_2x}\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in_1x}\cdot e^{-in_2x}\,dx = \begin{cases}0 \text{ for } n_1 \neq n_2\\ 1 \text{ for } n_1 = n_2\end{cases},$$

III 12 C 53. *Proc. Internat. Congr. Math. Cambridge Mass.* 1950, 1.

one is led to associate with every one of our functions $P(x)$ an infinite series

$$P(x) \sim \sum_{-\infty}^{\infty} A_n e^{inx}$$

as its Fourier series, namely the series with the coefficients $A_n = M\{P(x)e^{-inx}\}$. Further, one sees, with the help of Bessel's inequality, that the series $\sum | A_n |^2$ is convergent to a sum $\leqq M\{| P(x) |^2\}$. A fundamental theorem, the Parseval theorem, states that we always have the sign of equality, i.e., that

$$\sum | A_n |^2 = M\{| P(x) |^2\}$$

which, on account of Bessel's formula, is equivalent with the mean convergence of the Fourier series towards $P(x)$, i.e.,

$$M\left\{\left| P(x) - \sum_{-N}^{N} A_n e^{inx} \right|^2\right\} \to 0 \quad \text{as} \quad N \to \infty.$$

The usual way of proving this mean convergence, or the Parseval equation, is first to prove the much stronger Weierstrass theorem concerning uniform approximation, and this may be done by one or another suitable summation method applied to the—generally divergent—Fourier series of $P(x)$. In particular, as shown by Fejér, the application of the simple kernel

$$K_n(t) = \sum_{\nu=-n}^{n} \left(1 - \frac{| \nu |}{n}\right) e^{i\nu t}$$

will lead to exponential polynomials

$$s_n(x) = \underset{t}{M}\{P(x + t)K_n(t)\} = \sum_{\nu=-n}^{n} \left(1 - \frac{| \nu |}{n}\right) a_\nu e^{i\nu x}$$

which converge uniformly to $P(x)$.

3. We now pass to the main subject, the a.p. functions. As above we start from the pure oscillations $ae^{i\lambda x}$, but now we consider all of them, and not only a denumerable subset having a given number as a common period. Again we consider superpositions, i.e., exponential polynomials

$$s(x) = a_1e^{i\lambda_1 x} + a_2e^{i\lambda_2 x} + \cdots + a_ne^{i\lambda_n x},$$

but this time with arbitrary real exponents—so that the functions are no longer periodic—, and again we form the closure $\mathrm{Cl}\{s(x)\}$ of the set of all these exponential polynomials with respect to uniform convergence for all x. A main problem then presents itself: to characterize the functions of this closure $\mathrm{Cl}\{s(x)\}$ by means of structural properties, in analogy with and as a generalization of the classical Weierstrass theorem for functions with a given period. Let me at once remind you of the answer, which is that the function shall be what I have called "almost periodic", the exact definition being the following: A continuous

function $f(x)$ is called an a.p. function if for every given $\epsilon > 0$ there exists an infinity of translation numbers or almost periods belonging to ϵ, i.e., numbers $\tau = \tau(\epsilon)$ such that

$$|f(x + \tau) - f(x)| \leqq \epsilon \qquad \text{for } -\infty < x < \infty,$$

and, moreover, if for any fixed ϵ the set of these translation numbers $\{\tau(\epsilon)\}$ is relatively dense in the sense that each sufficiently large interval on the real axis contains at least one of these numbers.

That every function in the closure $\mathrm{Cl}\{s(x)\}$ is an a.p. function is fairly easy to prove. The difficulty lies in proving the converse, namely that every a.p. function can really be approximated uniformly by exponential polynomials $s(x)$. Here—in contrast to the pure periodic case where the exponents are given beforehand—the problem arises how to force the given a.p. function to deliver its characteristic exponents, i.e., the exponents to be used in the approximating polynomials. This is done by attaching to every a.p. function a certain infinite series as its Fourier series. In this general case also, the starting point is that the set of our pure oscillations, i.e., here the non denumerable set $\{e^{i\lambda x}\}$ form a normal orthogonal set, but now of course with the mean value taken over the whole infinite interval, i.e.,

$$M\{e^{i\lambda_1 x}\cdot e^{-i\lambda_2 x}\} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e^{i\lambda_1 x}\cdot e^{-i\lambda_2 x}\,dx = \begin{cases} 0 \text{ for } \lambda_1 \neq \lambda_2 \\ 1 \text{ for } \lambda_1 = \lambda_2. \end{cases}$$

For a given a.p. function $f(x)$, and an arbitrary real λ, we then naturally ask whether the pure oscillation $e^{i\lambda x}$ is resonant with $f(x)$, i.e., we form the mean value

$$a(\lambda) = M\{f(x)e^{-i\lambda x}\}$$

which may be shown to exist for every real λ. And rather easily we find the important result that this mean value differs from 0 for at most a denumerable set of values $\lambda = \Lambda_1, \Lambda_2, \cdots, \Lambda_n, \cdots$ (which may very well be everywhere dense on the real axis). With these numbers Λ_n as exponents and the corresponding values $a(\Lambda_n) = A_n$ as coefficients we form the infinite series, the Fourier series of $f(x)$,

$$f(x) \sim \sum A_n e^{i\Lambda_n x}.$$

4. Now, in analogy with the pure periodic case, we naturally ask whether the Fourier series of an a.p. function $f(x)$ also converges in mean to the function $f(x)$, i.e., whether

$$M\left\{\left|f(x) - \sum_1^N A_n e^{i\Lambda_n x}\right|^2\right\} \to 0 \quad \text{for} \quad N \to \infty,$$

and here again this question is equivalent to asking whether the infinite series $\sum |A_n|^2$, which is easily seen to converge to a sum $\leqq M\{|f(x)|^2\}$, is always equal to $M\{|f(x)|^2\}$, i.e., whether the Parseval equation

$$\sum |A_n|^2 = M\{|f(x)|^2\}$$

holds true for every a.p. function. That this is really the case is the clue to the whole theory, and constitutes the so-called fundamental theorem. Briefly I shall try to indicate the several proofs of this theorem given by various authors. Let me add that the Parseval equation immediately implies another fundamental result, the unicity theorem, stating that two different a.p. functions always have two different Fourier series, or, that the only a.p. function whose Fourier series is empty is the function $f(x)$ identically 0. In fact, in case of an empty Fourier series, the Parseval equation states that $M\{|f(x)|^2\} = 0$ and, luckily, on account of the almost periodicity, this implies that $f(x)$ must be identically 0.

5. In the original proof of the Parseval equation for an a.p. function the underlying idea was a very simple one, whereas the details were rather complicated. It consisted in reducing the general case to the special pure periodic case by considering the periodic function $f_T(x)$ with the period T, which coincides with the given a.p. function $f(x)$ in the period interval $0 < x < T$, next applying the Parseval equation to this periodic function $f_T(x)$, and finally—what was of course the difficult part of the proof—carrying out the limit process $T \to \infty$. An essentially simplified proof along these lines (i.e., to start by approximating the given function $f(x)$ by the periodic function $f_T(x)$) was given later on by de la Vallée Poussin. Both my original proof and the proof of de la Vallée Poussin have lately been further simplified by Jessen who, instead of working with the periodic function $f_T(x)$, uses the function $g_T(x)$ which like $f_T(x)$ is equal to $f(x)$ in the interval $0 < x < T$ but vanishes outside this interval. To this function $g_T(x)$ he applies the theory of Fourier integrals, which turns out to be better adapted for the purpose than the Fourier series of the periodic function $f_T(x)$.

6. The greater simplicity of de la Vallée Poussin 's proof, as compared with my own, was due mainly to his application of the important process of convolution of two a.p. functions, a notion implicitly applied by Norbert Wiener in his proof of the Parseval equation, to which I shall return later on. This notion also is the basis of the proof of Weyl, of which I shall speak in a moment. In the special but particularly important case of the convolution of an a.p. function $f(x) \sim \sum A_n e^{i\Lambda_n x}$ with the a.p. function $\bar{f}(-x) \sim \sum \bar{A}_n e^{i\Lambda_n x}$ we form the mean value

$$F(x) = \underset{t}{M}\{f(x+t)\bar{f}(t)\},$$

to be denoted briefly as the "convolution of $f(x)$", which is easily seen again to be a.p. and to have the absolutely convergent Fourier series with positive coefficients

$$F(x) \sim \sum |A_n|^2 e^{i\Lambda_n x}.$$

By means of this convolution process it is easily seen that the unicity theorem is not only a consequence of, but in fact equivalent with, the Parseval equation. Further, we find that the proof of either of these two theorems is again equivalent

to proving that in the above relation the sign $\sim$ may simply be replaced by the sign $=$, i.e., that the (absolutely convergent) Fourier series of $F(x)$ has the function $F(x)$ itself as its sum.

7. Now we turn to Weyl's proof of the fundamental theorem, say of the unicity theorem. The basis of this proof is an important connection between the classical theory of ordinary Fourier series for pure periodic functions on the one hand and the theory of integral equations on the other. This connection could easily be extended to the general case of a.p. functions, the integral equations in question being simply replaced by mean value equations of a quite similar type. His starting point was the following: If in the equation

$$a(\lambda) = M\{f(x)e^{-i\lambda x}\}$$

serving to determine the terms of the Fourier series of the a.p. function $f(x)$, we replace x by $x - y$, taking y as the variable and x as a constant, we get immediately, on account of the functional equation $e^{-i\lambda(x-y)} = e^{-i\lambda x}e^{i\lambda y}$, the mean value equation

$$a(\lambda)e^{i\lambda x} = \underset{y}{M}\{f(x - y)e^{i\lambda y}\}$$

or, if we replace $a(\lambda)$ by γ and $e^{i\lambda x}$ by $\varphi(x)$, the mean value equation

$$\gamma\varphi(x) = \underset{y}{M}\ \{f(x - y)\varphi(y)\}.$$

Hence we see that it is the same thing to say that $A_n e^{i\Lambda_n x}$ is a term of the Fourier series of $f(x)$ as to say that the mean value equation with the kernel $K(x, y) = f(x - y)$ has $\gamma = A_n$ as a characteristic value and $\varphi(x) = e^{i\Lambda_n x}$ as a corresponding characteristic function. Thus the content of the unicity theorem is equivalent with saying that our mean value equation, in case of an a.p. function $f(x)$ not identically 0, has a characteristic value $\neq 0$, and a pure oscillation as a corresponding characteristic function. As could be expected, rather than working with $f(x)$ itself, it is more convenient to work with its convolution $F(x)$, since in case of a non-empty Fourier series, the Fourier coefficients $| A_n |^2$ of $F(x)$ are all real and positive. Thus we consider the equation

$$\gamma\varphi(x) = \underset{y}{M}\{F(x - y)\varphi(y)\}.$$

On account of the almost periodicity of the kernel, this mean value equation may be treated practically as if it were an ordinary integral equation (where the mean value is to be taken over a finite interval). In fact the classical method of Erhard Schmidt could be applied almost unchanged to the mean value equation in question and led to the result that if $f(x)$ (and hence $F(x)$) is not identically 0, there always exists a positive characteristic value $\gamma = \gamma_0$ and a corresponding characteristic a.p. function $\varphi(x) = \varphi_0(x)$ not identically 0. However, this result is not yet the unicity theorem, as it would have been if the characteristic function

$\varphi_0(x)$ were known to be a pure oscillation and not merely an a.p. function. In order to complete the proof we naturally have to use the fact that the kernel $K(x, y) = F(x - y)$ depends only on the difference $x - y$; this immediately gives that together with $\varphi_0(x)$ every translated function $\varphi_0(x + h)$, where h is an arbitrary real number, is again a characteristic function belonging to the characteristic value γ_0. If now the space of all the characteristic functions belonging to γ_0 has the dimension 1, everything is easy; we then have the simple relation $\varphi_0(x + h) = c(h)\varphi_0(x)$ for all h, from which it follows that $\varphi_0(x)$ must be a pure oscillation. If, however, the space of our characteristic functions belonging to γ_0 has a dimension m which, though certainly finite, is > 1 (this will really happen if in the Fourier series of $F(x)$ there is more than one term for which the coefficients $| A_n |^2$ is equal to γ_0), a certain difficulty appears. This is due to the fact that if $\varphi_1(x), \cdots, \varphi_m(x)$ is a normal orthogonal basis for the functional space in question, we cannot conclude that, for each ν, $\varphi_\nu(x + h) = c_\nu(h)\varphi_\nu(x)$, but only that $\varphi_\nu(x + h)$ is a linear combination $\varphi_\nu(x + h) = c_{\nu 1}\varphi_1(x) + \cdots + c_{\nu m}\varphi_m(x)$ of the m functions $\varphi_1(x), \cdots, \varphi_m(x)$. This difficulty was overcome by Weyl by the following simple group-theoretical consideration. Since, together with $\varphi_1(x), \cdots, \varphi_m(x)$, the translated system $\varphi_1(x + h), \cdots, \varphi_m(x + h)$ is, evidently, again a normal orthogonal basis for our functional space, the corresponding matrix $M = \{c_{\nu\mu}\}$ is, for each h, a unitary matrix, and further as these matrices $M = M(h)$ $(-\infty < h < \infty)$ form a commutative group (commutative simply because $h_1 + h_2 = h_2 + h_1$) they may, by a classical theorem of Frobenius, be transformed, simultaneously for all h, into matrices of the diagonal form with mere zeros outside the main diagonal. This means, however, that we may choose the coordinate system in our m-dimensional space, i.e., the basic orthogonal system $\varphi_1(x), \cdots, \varphi_m(x)$, in such a way that for each $\nu = 1, \cdots, m$ we have, just as in the one-dimensional case, a simple relation of the form $\varphi_\nu(x + h) = c_\nu(h)\varphi_\nu(x)$ and thus may conclude that the functions $\varphi_\nu(x)$ are really pure oscillations. It may be added that an interesting variant of Weyl's proof was given by Hammerstein who, in analogy with classical methods in the theory of ordinary integral equations, treated the mean value equation in question by means of the direct methods from the calculus of variations. Further, it may be mentioned that several interesting artifices have been indicated, from various sides, to avoid the explicit use of group-theoretical considerations in the last step of the proof.

8. We now proceed to the interesting proof of the Parseval equation due to Norbert Wiener—the first proof of the fundamental theorem to follow the original one—as a part of his important general theory of harmonic analysis. I shall give a short account of this proof in the simplified and very beautiful form given by Bochner. As in Weyl's proof it is essential to treat not the a.p. function $f(x)$ itself, but its convolution

$$F(x) = \underset{t}{M}\{f(x + t)\bar{f}(t)\}.$$

As emphasized above, the task is to prove that the convergent Fourier series $\sum | A_n |^2 e^{i\Lambda_n x}$ of $F(x)$, with positive coefficients, has the function $F(x)$ as its sum, and this is easily seen to be equivalent with showing that the function $F(x)$ may be expressed by a Stieltjes integral

$$F(x) = \int_{\alpha=-\infty}^{\alpha=\infty} e^{i\alpha x}\, dD(\alpha)$$

where $D(\alpha)$ is a bounded monotonically increasing function which is completely discontinuous, i.e., increases only through a denumerable number of jumps (namely the jumps $| A_n |^2$ in the points Λ_n). Now, more generally, we consider functions $G(x)$ which do not possess (as we shall prove about $F(x)$) only a discontinuous spectrum, but the spectrum of which may partly be discontinuous and partly continuous, i.e., functions $G(x)$ which may be represented as a Stieltjes integral

$$G(x) = \int_{-\infty}^{\infty} e^{i\alpha x}\, dV(\alpha)$$

where $V(\alpha)$ is a quite arbitrary bounded increasing function of α. Splitting $V(\alpha)$, in the usual way, into two monotonic components, $V(\alpha) = D(\alpha) + C(\alpha)$ where $D(\alpha)$ is completely discontinuous and $C(\alpha)$ continuous for all α, we get the function $G(x)$ split into two functions

$$G(x) = \int_{-\infty}^{\infty} e^{i\alpha x}\, dD(\alpha) + \int_{-\infty}^{\infty} e^{i\alpha x}\, dC(\alpha) = G_D(x) + G_C(x).$$

Here $G_D(x)$ has a pure discontinuous spectrum and hence has a strong structural property, namely that of almost periodicity, whereas $G_C(x)$ has a pure continuous spectrum and hence behaves in quite another way, namely has a tendency to tend to 0 for x tending to $\pm\infty$, or, more precisely, satisfies the mean value equation

$$M\{| G_C(x) |^2\} = 0.$$

For an arbitrary function $G(x)$ of the above type it is immediately shown (simply with the help of the functional equation of the exponential function) that it is a so-called positive definite function, a notion first introduced by Mathias. By this is meant that $G(x)$ is a continuous bounded function, of Hermitian character, $\bar{G}(-x) = G(x)$, which satisfies the condition that the quadratic form

$$\sum_{\mu=1}^{m} \sum_{\nu=1}^{m} G(x_\mu - x_\nu)\rho_\mu \bar{\rho}_\nu$$

is $\geqq 0$ for arbitrary real numbers $x_1, \cdots, x_m$ and complex numbers $\rho_1, \cdots, \rho_m$. Now Bochner shows, by means of the theory of Fourier transforms, that the property just mentioned is characteristic of the functions $G(x)$ in question, i.e., that conversely, every positive definite function may be expressed by a Stieltjes integral of the form in question. The proof then proceeds in the following

manner. For a rather large class of functions $g(x)$, including the a.p. functions $f(x)$ as a special case, it is easily shown by direct considerations that their convolutions

$$G(x) = \underset{t}{M}\{g(x+t)\bar{g}(t)\}$$

possess all the properties demanded by the definition of a positive definite function. Hence each of these convolutions $G(x)$ is a function of the kind considered above and therefore may be written as a sum of two functions

$$G(x) = G_D(x) + G_C(x)$$

where $G_D(x)$ is almost periodic while $G_C(x)$ has a purely continuous spectrum. The remainder of the proof follows automatically. In fact, if the function $g(x)$ from which we started is just an a.p. function $f(x)$, we get for its convolution $F(x)$ the expression

$$F(x) = F_D(x) + F_C(x),$$

where the first term $F_D(x)$ is the almost periodic part of $F(x)$. Our task is to show that the second term $F_C(x)$ is identically 0, i.e., that the spectrum of $F(x)$ contains no continuous part. But this is evident, as on the one hand, as mentioned above, $M\{|F_C(x)|^2\}$ is $= 0$ while on the other hand $F_C(x)$, as the difference between the two a.p. functions $F(x)$ and $F_D(x)$, is itself an a.p. function, and an a.p. function for which the mean value of its numerical square is 0 must necessarily, as emphasized above, be identically 0.

9. Although the fundamental theorem dealing with the mean convergence of the Fourier series may be said to be the clue to the whole theory, it is not really the main theorem—at any rate not as long as we consider only the ordinary continuous a.p. functions. As the main theorem must naturally be considered the approximation theorem dealing with uniform convergence since it gives the exact characterization of our class of functions. Before finishing my lecture with a short review of one more interesting proof of the fundamental theorem, due to Bogolioùboff, I should like to say some few words concerning the various proofs of the approximation theorem. The starting point of my original proof was a consideration of the Fourier exponents of the function from an arithmetical point of view, namely, their representation by means of a so-called basis, i.e., as linear combinations of rationally independent numbers, generally infinitely many, and generally with rational (and not just integral) coefficients. Led by the Kronecker theorem on Diophantine approximation, and in generalization of ideas already used by Bohl in the special case of a finite integral basis, and by the lecturer in his studies of the value-distribution of Dirichlet series, a function of more variables was introduced, the "spatial extension" of $f(x)$, which is generally a so-called limit periodic function $L(x_1, x_2, \cdots)$ of infinitely many variables $x_1, x_2, \cdots$, and from which the given function $f(x)$ itself could again be obtained by considering the function $L(x_1, x_2, \cdots)$ on the main diagonal $x_1 = x_2 = \cdots = x$ of the infinite-dimensional space. For these limit periodic

functions $L(x_1, x_2, \cdots)$ a theory of Fourier series was then established, and in particular an approximation theorem, based on Fejér kernels in several variables, was established, leading to exponential polynomials $s(x_1, x_2, \cdots)$ of more and more variables which tended uniformly to the function $L(x_1, x_2, \cdots)$; finally, exponential polynomials $s(x)$ in one variable approximating the given a.p. function $f(x)$ were then obtained by considering the approximating polynomials $s(x_1, x_2, \cdots)$ on the main diagonal of the space. A very interesting turn and a substantial simplification of this proof of the approximation theorem on arithmetic lines was given by Bochner who realized that one could arrive at the approximating polynomials $s(x)$ in question without the spatial extension of the function $f(x)$, and thus completely avoiding the use of functions of infinitely many variables which are, however, indispensable for the treatment of other problems in the theory of almost periodic functions. Bochner constructed composed kernels simply through multiplication of suitably chosen Fejér kernels, and then in the usual way formed the convolution $s(x) = M\{f(x + t)K(t)\}$. Another proof of the approximation theorem, on quite other lines, was given by Weyl who constructed kernels not, as Bochner, with the help of the exponents of the Fourier series, but by means of the translation numbers of the function. Another very beautiful proof on these lines was later given by Wiener who, by the construction of his kernels, made use of the so-called translation function of the given a.p. function, a function implicitly introduced already in the original proof of the fundamental theorem and studied in detail by Bochner. As to the two purest proofs of the approximation theorem, the Bochner proof and the Wiener proof, they may be considered as being of rather opposite character—and each of them having its particular interest—so far as the first is a general summation method, the same for all a.p. functions with the same set of Fourier exponents, while the latter is of a more individual character, especially adjusted to every single a.p. function.

10. And now, finally, some few words about Bogolioùboff's proof of the fundamental theorem, the latest one to appear and in some respect perhaps the most elementary one. It consists in directly showing that the translation numbers of an a.p. function possess some kind of arithmetical properties (connected with the Fourier exponents of the function) which hitherto could be deduced only by use of the fundamental theorem but of which it was early recognized that their establishment would be sufficient for the proof of the latter theorem. In Bogolioùboff's proof of these properties of the translation numbers the following very interesting and rather surprising general lemma, belonging to the additive theory of numbers, is the main tool. Let E be a quite arbitrary relatively dense set of integers and let E^* be the set of all integers of the form $n_1 + n_2 - n_3 - n_4$ where the n's belong to E; then E^* will always contain a "Diophantine set", i.e., a set consisting of all integral solutions of a finite number of Diophantine inequalities of the form

$$|\lambda_\nu x| < \delta \pmod 1 \qquad (\nu = 1, \cdots, N).$$

It may be interesting to observe that Bogolioùboff's proof of this beautiful lemma—which has such an important application to the theory of a.p. functions and their Fourier series—is itself based on the theory of Fourier series, but in its most elementary, so to say embryonic, state where the variable runs only over a finite number of values and everything, including the Parseval equation, the unicity theorem, the process of convolution, etc., is quite elementary since no limit process whatsoever is involved.

11. I am now at the end of my lecture. I am quite aware that I have not given you any deeper insight into the different proofs in question since, for instance, all the more difficult parts of the proofs have had to be omitted. The task I have set myself is a more modest one, only to try to give you some general ideas about the essentially different ways in which the theory in question may be established.

University of Copenhagen,
Copenhagen, Denmark.

C 53. *Proc. Internat. Congr. Math. Cambridge Mass.* 1950, 1.

ON THE DEFINITION OF ALMOST PERIODICITY

By

Harald Bohr

in Copenhagen, Denmark.

Introduction

By a function we shall, in the following, always mean a real function of the real variable x, defined in the whole interval $-\infty<x<\infty$ and continuous for all x.

For a given function $f(x)$ and a given number $\varepsilon>0$, the number $\tau=\tau(\varepsilon)=\tau_f(\varepsilon)$ is said to be a *translation number* of $f(x)$ belonging to ε if

$$\sup_{-\infty<x<\infty} |f(x+\tau)-f(x)| \leqq \varepsilon .$$

By $E(\varepsilon)=E_f(\varepsilon)$ we shall denote the set $\{\tau_f(\varepsilon)\}$ of all translation numbers of $f(x)$ belonging to ε. Evidently, $E(\varepsilon)$ contains the number $\tau=0$, and together with τ also contains $-\tau$; further, $E(\varepsilon_1)\subseteqq E(\varepsilon_2)$ if $\varepsilon_1<\varepsilon_2$.

A function $f(x)$ is called *almost periodic* if, for every fixed $\varepsilon>0$, the set $E(\varepsilon)$ is *relatively dense*, i. e. if there exists a length $L\,(=L(\varepsilon))$ such that every interval of the length L contains a number of the set $E(\varepsilon)$.

Already in the first paper on almost periodic functions, "*Zur Theorie der fastperiodischen Funktionen*, I", *Acta Mathematica* 45, 1924, the author discussed, in an appendix, a much more comprehensive class of so-called *periodiclike* ("periodenartige") functions, characterized by the simple claim that, for each $\varepsilon>0$, the set $E(\varepsilon)$ should contain some *arbitrarily large* numbers τ. A certain general type of functions was constructed which was periodiclike without being almost periodic, and by means of these functions it could be shown—in contrast to what is the case for the well-rounded class of the almost periodic functions — that the class of the periodiclike functions is not invariant, say to the simple operation of addition, i.e. the sum of two periodiclike functions need not again be periodiclike.

C54. *J. Anal. Math.* 1 (1951).

Now, as pointed out by Jessen, it may be followed, as a byresult, from a very interesting general investigation of Bogolioùboff, "*Sur quelques propriétés arithmétiques des presquepériodes*", *Ann. Phys. Math. Kiew* 4, 1939, that in the definition of an almost periodic function $f(x)$ the claim that each set $E(\varepsilon)$ shall be relatively dense may be replaced by an apparently much weaker claim, namely that $E(\varepsilon)$, for every ε, shall have "positive upper density" (in a very weak sense, to be defined below, which, roughly speaking, only demands the existence of some arbitrarily large intervals containing rather many translation numbers $\tau(\varepsilon)$). Later on, Følner, in a paper in Danish, "*Bemaerkning om Naestenperiodicitetens Definition*", *Mat. Tidsskrift* B, 1944, has given a simple direct proof of this interesting result, his proof being based on a general Lemma on point sets. In §1 of the present paper, we shall give the exact definition of the notion of a set with "positive upper density" and a reproduction — in a slightly simplified form—of the proof of Følner's Lemma and of its application to the sets $E_f(\varepsilon)$.

The question now naturally arises whether the result, mentioned above, may be improved, i.e. whether in the definition of an almost periodic function the claim on the sets $E(\varepsilon)$ as to their density may be given a still weaker form. The purpose of this paper is to show that, in a certain well-defined sense, this is *not the case*, so that the above result may be said to be a "best-possible" one. The exact formulation of a theorem to this effect is given in §2; it is based on the notion of certain subclasses C_φ of the general class of periodiclike functions. Furthermore, §2 contains the formulation of a theorem concerning addition of two functions belonging to one and the same class C_φ, a theorem analogous to, and generalizing, the above mentioned result concerning the addition of two periodiclike functions.

§3 describes a certain method of constructing types of periodiclike functions which are especially suited to serve as "counter-examples". The leading idea of this construction has already been applied in the appendix to my paper in *Acta Mathematica* cited above; for the present applications, however, the method had to be essentially generalized.

Finally, in §4, we give the proofs of the theorems formulated in § 2.

§ 1. The notion of upper density. Fölner's Lemma and its application.

In the following, we shall be concerned with sequences of real numbers $\{t_r\}$ $(r = 1, 2, \ldots)$ which satisfy the condition

$$\inf_{r \neq s} |t_r - t_s| > 0\,, \tag{1}$$

i.e. $|t_r - t_s| > \alpha > 0$ for $r \neq s$ (where the positive constant α may depend on the sequence). A sequence $\{t_r\}$ satisfying (1) we shall denote as a "discrete sequence". With an arbitrary discrete sequence $\{t_r\}$ we associate a function M_l $(0 < l < \infty)$ determined in the following manner. We start with an arbitrary interval $x_0 \leqq x < x_0 + l$ of length l and denote by $m = m(x_0, l)$ the number of points t_r situated in the interval; then, for a fixed $l > 0$, the number M_l is simply defined by

$$M_l = \max_{-\infty < x_0 < \infty} m(x_0, l)\,.$$

Next we build the fraction

$$\frac{M_l}{l} \left(< \frac{1}{l}\left(\frac{l}{\alpha} + 1\right) = \frac{1}{\alpha} + \frac{1}{l}\right).$$

From the two obvious inequalities,

$$M_l \leqq M_{l'} \text{ for } l' > l \quad \text{and} \quad M_{nl} \leqq nM_l \ (n = 1, 2, \ldots)\,,$$

we may conclude, in a well-known manner, the existence of the limit

$$\varrho = \lim \frac{M_l}{l}$$

(where, evidently, ϱ is $\geqq 0$, but $< \infty$, indeed $\leqq \frac{1}{\alpha}$); in fact, for fixed $l > 0$, $l' > 0$, putting $l' = nl - d$ $(0 \leqq d < l)$, we get

$$M_{l'} \leqq M_{nl} \leqq nM_l\,, \quad \text{i. e.} \quad \frac{M_{l'}}{l'} \leqq \frac{M_l}{l} \cdot \frac{nl}{l'}\,;$$

letting, for fixed l, l' tend to ∞ in a suitable manner, we get

$$\overline{\lim_{l' \to \infty}} \frac{M_{l'}}{l'} \leqq \frac{M_l}{l}\,,$$

and from this last inequality we get, now letting l tend to ∞ in a suitable manner,

$$\overline{\lim_{l'\to\infty}} \frac{M_{l'}}{l'} \leq \underline{\lim_{l\to\infty}} \frac{M_l}{l},$$

which just shows the existence of the limit $\varrho = \lim \frac{M_l}{l}$. This number ϱ $(0 \leqq \varrho < \infty)$ we call *the upper density of the discrete sequence* $\{t_r\}$.

In the following, we are mainly interested in the discrete sequences $\{t_r\}$ with *positive upper density* (i.e. $\varrho > 0$).

Let now E be an *arbitrary set* of real numbers. We shall say that *E has a positive upper density* if E contains a discrete sequence $\{t_r\}$ with positive upper density.

Evidently, a set E which is relatively dense has positive upper density, while a set with positive upper density (even if it is symmetric with respect to the point 0, as are the sets $E(\varepsilon)$) need not be relatively dense. The rather surprising fact which could be deduced from Bogolioùboff's investigation is, as mentioned in the introduction, that if we start from a function $f(x)$, and not from a set E, and consider *all* the sets $E(\varepsilon)$ arising from this function, then *the claim put to all these sets $E(\varepsilon)$ that they shall have positive upper density involves that they must necessarily all be relatively dense.* Following Fólner, we shall here give a simple direct proof of this interesting fact. As the set $E(\varepsilon)$ certainly contains all the differences between two arbitrary numbers belonging to $E\left(\frac{\varepsilon}{2}\right)$, our assertion that the positive upper density of each of the sets $E(\varepsilon)$ implies that they are all relatively dense is an immediate corollary of the following general Lemma (when it is applied to a discrete sequence $\{t_r\}$ with positive density taken from $E\left(\frac{\varepsilon}{2}\right)$ and not from $E(\varepsilon)$ itself).

Fólner's Lemma. *Let $\{t_r\}$ be an arbitrary discrete sequence with positive upper density. Then the set $\{t_r - t_s\}$ consisting of all differences between two numbers of the sequence $\{t_r\}$ will be relatively dense.*

Proof. Indirectly, we have to show that a discrete sequence $\{t_r\}$ for which the "difference-set" $\{t_r - t_s\}$ is *not* relatively dense must have the upper density $\varrho = 0$.

For an arbitrary fixed positive integer m, we divide the real axis in consecutive intervals $I^{(m)}$ each of the length 2^m,

$$I_n^{(m)}: \; n \cdot 2^m \leqq x < (n+1) \cdot 2^m \quad , \quad (n = 0, \pm 1, \pm 2, \ldots)$$

and denote by $M^*_{2^m}$ the largest number of points t_r contained in any of these intervals. In the definition of the upper density ϱ of the sequence $\{t_r\}$,

$$\varrho = \lim_{l \to \infty} \frac{M_l}{l},$$

we let l tend to ∞ through the sequence 2^m $(m = 1, 2, \ldots)$; thus, putting

$$\varrho_m = \frac{M_{2^m}}{2^m},$$

we have to show that $\varrho_m \to 0$ for $m \to \infty$. Here, obviously, instead of the sequence ϱ_m, we may equally well consider the (non-increasing) sequence

$$\varrho^*_m = \frac{M^*_{2^m}}{2^m};$$

in fact, as $M^*_{2^m} \leqq M_{2^m} \leqq 2M^*_{2^m}$, the two sequences ϱ_m and ϱ^*_m will simultaneously tend to zero. We prove that $\varrho^*_m \to 0$ by showing that to each positive integer m we may find a positive integer $q > m$ such that

$$\varrho^*_q < \frac{3}{4} \varrho^*_m,$$

(where instead of $\frac{3}{4}$ we could have chosen any other number between $\frac{1}{2}$ and 1). The proof is based on the simple fact that if x' is a point in $I_{n'}^{(m)}$ and x'' a point in $I_{n''}^{(m)}$, and if we build the differences $x' - x'' = x_0$, and $n' - n'' = n_0$, then the point x_0 will certainly lie, if not in $I_{n_0}^{(m)}$ itself, then at any rate in one of the three intervals $I_{n_0-1}^{(m)}$, $I_{n_0}^{(m)}$, $I_{n_0+1}^{(m)}$. As, per assumption, the set $\{t_r - t_s\}$ is not relatively dense, we may find an interval J of length $3 \cdot 2^m$ consisting of three consecutive intervals $I_{n_m-1}^{(m)}$, $I_{n_m}^{(m)}$, $I_{n_m+1}^{(m)}$ such that *no* point of $\{t_r - t_s\}$ belongs to J. Hence, if t_s is an arbitrary point of our sequence $\{t_r\}$ situated, say, in the interval $I_n^{(m)}$, we may be sure that *no* point in the interval $I_{n+n_m}^{(m)}$ may belong to $\{t_r\}$ (as, otherwise, if there were a point t_r in $I_{n+n_m}^{(m)}$, we would have a difference $t_r - t_s$ belonging to J). This fact, i.e. the fact that every interval $I_n^{(m)}$ containing a point of $\{t_r\}$ by a translation of the fixed length $n_m \cdot 2^m$ goes over in an

interval containing no point of $\{t_r\}$, evidently involves that, roughly speaking, in any sufficiently large interval there must be (at least) "nearly" so many intervals $I^{(m)}$ which do not contain points of $\{t_r\}$ as intervals $I^{(m)}$ which do contain such points; exactly speaking, we may certainly choose an integer $q > m$ so large that in any of the intervals $I_n^{(q)}$ $(n = 0, \pm 1, \ldots)$ of length 2^q, consisting each of 2^{q-m} intervals $I^{(m)}$, the number of those of these 2^{q-m} intervals $I^{(m)}$ which contain points of $\{t_r\}$ is less than, say, $\frac{3}{4}2^{q-m}$. Now the proof is immediately finished. Indeed, as an interval $I^{(m)}$ which contains points of $\{t_r\}$ can contain at most $\varrho^*_m \cdot 2^m$ such points, and as in any of the intervals $I^{(q)}$ there are less than $\frac{3}{4}2^{q-m}$ intervals $I^{(m)}$ containing points of $\{t_r\}$, the total number of points of $\{t_r\}$ in an arbitrary interval $I_n^{(q)}$ must be less than $\frac{3}{4}2^{q-m} \cdot \varrho^*_m 2^m = \frac{3}{4}\varrho^*_m 2^q$, so that ϱ^*_q is really $< \frac{3}{4}\varrho^*_m$.

§ 2. Definition of the classes C_φ and formulation of the theorems A and B.

In § 1, we have seen that if for a (periodiclike) function $f(x)$ *each* $E(\varepsilon)$ *has positive upper density* — i.e. contains a discrete sequence $\{t_r\}$ with positive upper density — then these sets $E(\varepsilon)$ must necessarily be relatively dense, i.e. the function $f(x)$ is *almost periodic.*

As mentioned in the introduction, the problem we shall treat in the present paper is to investigate *whether the result stated above is a "best possible" one*, i.e. whether we may still conclude that a (periodiclike) function $f(x)$ is almost periodic when the sets $E(\varepsilon)$ of its translation numbers are submitted to conditions like those above $(\varrho > 0)$ but of a weaker kind. When trying to weaken the condition that each set $E(\varepsilon)$ shall contain a discrete sequence $\{t_r\}$ satisfying the inequality

$$(2) \qquad \overline{\lim_{l\to\infty}} \frac{M_l}{l} > 0 ,$$

we, naturally, replace the denominator l in the fraction $\frac{M_l}{l}$ by some other positive function $\varphi(l)$. As the new fraction thus obtained, $\frac{M_l}{\varphi(l)}$, needs not — as $\frac{M_l}{l}$ — to tend to a (finite or infinite) limit for $l \to \infty$, we further replace "lim" by "$\overline{\lim}$", and thus get the condition

$$\overline{\lim_{l \to \infty}} \frac{M_l}{\varphi(l)} > 0. \tag{3}$$

Before proceeding, it will be convenient to introduce the following notion.

Definition of the class C_φ. *Let $\varphi(l)$ be an arbitrary given positive function of the positive variable l. Then we shall say about an (eo ipso periodiclike) function $f(x)$ that it belongs to the class C_φ if, for every $\varepsilon > 0$, the set $E_f(\varepsilon)$ contains a discrete sequence $\{t_r\}$ satisfying the inequality* (3) *where M_l, as above, denotes the maximal number of points t_r contained in any interval $x_0 \leqq x < x_0 + l$ of length l.*

Before giving the exact formulation of our problem, we shall make some simple introductory remarks.

Remark 1. As, for any discrete sequence $\{t_r\}$, it holds that $\frac{M_l}{l}$ tends to a finite limit ($\geqq 0$), it is obvious that it has no sense to consider functions $\varphi(l)$ for which $\frac{\varphi(l)}{l} \to \infty$ for $l \to \infty$; in fact, for such a function $\varphi(l)$, and a quite arbitrary discrete sequence $\{t_r\}$, it will hold that $\frac{M_l}{\varphi(l)} \to 0$, so that the condition (3) cannot be fulfilled for *any* discrete sequence at all, and hence the class C_φ is empty. In other words, it makes only sense to consider functions $\varphi(l)$ for which

$$\underline{\lim_{l \to \infty}} \frac{\varphi(l)}{l} < \infty. \tag{4}$$

Remark 2. For every function $\varphi(l)$ which satisfies this condition (4), it is evident that the class C_φ is not empty, indeed that it *contains the whole class of the almost periodic functions.* In fact, if $\varphi(l)$ satisfies (4), it is clear that each relatively dense set E contains a discrete sequence $\{t_r\}$ satisfying (3). The following other way of argumentation — based on the "new" (apparently weaker) definition of almost periodicity — may, however, be useful with regard to the next remark: For an arbitrary discrete sequence $\{t_r\}$ it follows from the identity

$$\frac{M_l}{l} = \frac{M_l}{\varphi(l)} \cdot \frac{\varphi(l)}{l},$$

(where the left-hand side tends to a limit for $l \to \infty$), that

$$\varrho = \lim \frac{M_l}{l} = \overline{\lim} \frac{M_l}{\varphi(l)} \cdot \underline{\lim} \frac{\varphi(l)}{l} \tag{5}$$

(which last equation, by the way, tells nothing about ϱ in case one of the two factors to the right is 0, the other $+\infty$). Hence, as the second factor to the right is, per assumption, $<\infty$, it follows that if $\varrho > 0$, then also the first factor to the right must be > 0, i. e. if the condition (2) is fulfilled, then also (3) is fulfilled.

Remark 3. Among the non-empty classes C_φ, i.e. the classes associated with a function $\varphi(l)$ satisfying (4), there are some for which it is quite trivial that they are *identical with the class associated with* $\varphi(l) = l$, i.e. with the class of the *almost periodic functions*, namely the classes C_φ for which $\varphi(l)$ satisfies the inequality

$$(6) \qquad \underline{\lim} \frac{\varphi(l)}{l} > 0 .$$

In fact, for a function $\varphi(l)$ satisfying (4) and (6), the above equation (5) immediately shows that it comes to the same thing to say of a discrete sequence $\{t_r\}$ that it satisfies $\varrho = \lim \frac{M_l}{l} > 0$ as to say that it satisfies $\overline{\lim} \frac{M_l}{\varphi(l)} > 0$.

On account of the remarks above, it is clear that the problem with which we are concerned — i.e. whether in the definition of almost periodicity we can weaken the condition (2) in the indicated manner — has only a real sense in case of functions $\varphi(l)$ which satisfy (4) but do not satisfy (6), i. e. for which

$$(7) \qquad \underline{\lim} \frac{\varphi(l)}{l} = 0 .$$

And now, finally, we may give an exact formulation of our problem.

Problem. *Does there exist any positive function* $\varphi(l)$, *satisfying (7), for which the class* C_φ *is not wider than the class of the almost periodic functions?*

The answer is, as already stated in the introduction, a negative one, and is given in the following main theorem.

Theorem A. *For any positive function* $\varphi(l)$ *satisfying (7), the class* C_φ *contains functions which are not almost periodic.*

Furthermore, we shall prove the following theorem which generalizes an old result on periodiclike functions mentioned in the introduction.

Theorem B. *None of the classes mentioned in Theorem A is invariant with respect to addition, i.e. in each class* C_φ *for which* $\varphi(l)$ *satisfies (7) may be found two functions* $g(x)$ *and* $h(x)$ *such that their sum* $g(x)+h(x)$ *does not belong to* C_φ*, indeed so that* $g(x)+h(x)$ *is not even a periodiclike function.*

Obviously, as the sum of two almost periodic functions is again almost periodic, Theorem B contains Theorem A. The proof of Theorem B may, however, be performed on the same lines as that of Theorem A.

§ 3. Construction of a general type of periodiclike functions.

In this section, we shall describe the construction of a certain general type of periodiclike functions on which, in the next section, the proofs of our Theorems A and B shall be based.

We begin with some preliminary remarks: Naturally, we need not consider translation numbers belonging to an arbitrary $\varepsilon > 0$; it will suffice to consider *some sequence of* ε*-values tending to zero*; we choose the values

$$\varepsilon_\nu = \frac{1}{\nu}, \quad (\nu = 2, 3, \ldots).$$

Moreover, the translation numbers τ to be considered will always be *positive integers*, so that any sequence of translation numbers to be applied will eo ipso be a *discrete* one.

As the starting point of our construction, we may take some or other simple figure; we choose a "top", i.e. an isosceles triangle standing with its base on the x-axis. We have to use infinitely many such tops, all with the same length of their bases, say the length 2, but of different heights. A top with the height h will be called a h-top; it will be convenient also to speak of a 0-top, meaning simply an interval (of the length 2) lying on the x-axis. If a top has its midpoint at x_0 we shall say that the top is placed on the point x_0. By the construction of our function $f(x)$ we shall form, successively, a sequence of more and more complicated (continuous) functions $f_1(x)$, $f_2(x)$, ..., each being zero outside a finite interval, and finally we shall carry out the limit process

$$f(x) = \lim_{\nu\to\infty} f_\nu(x).$$

This limit process will involve no difficulty — and the limit function $f(x)$ will again be continuous — as in any finite interval all the functions $f_\nu(x)$ will be identical from a certain step onwards.

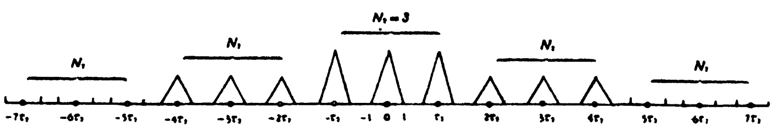
And now to the explicit construction.

1st step. We start from the simple function

$$f_1(x) = \begin{cases} 1 - |x| & \text{for } |x| \leqq 1 \\ 0 & \text{for } |x| > 1\,, \end{cases}$$

i.e. from a single 1-top placed at the point 0. As the "length" of this first step we shall mean the length $\lambda_1 = 2$ of the base of the top.

2nd step. (See the figure). We choose an arbitrary integer $\tau_2 > \lambda_1 = 2$ (i.e. larger than the base of a top) and an arbitrary odd integer $N_2 = 2n_2 + 1$ (in the figure, we have taken $N_2 = 3$, and for the sake of convenience used



different units on the two axis). On each of the N_2 points, with equal distance τ_2,

$$n\tau_2 \quad (n = -n_2, -n_2+1, \ldots, n_2-1, n_2)\,,$$

we place a 1-top. Next on each of the N_2 points to the right and on each of the N_2 points to the left,

$$(N_2+n)\tau_2 \text{ and } (-N_2+n)\tau_2 \quad (n = -n_2, \ldots, 0, \ldots, n_2)\,,$$

we place an $^1/_2$-top. And finally on each of the N_2 points to the right and on each of the N_2 points to the left,

$$(2N_2+n)\tau_2 \text{ and } (-2N_2+n)\tau_2 \quad (n = -n_2, \ldots, 0, \ldots, n_2)\,,$$

we place a 0-top. Thus we have got a chain of altogether $5N_2$ tops placed on the points $n\tau_2$ where n runs from $-2N_2-n_2$ to $2N_2+n_2$. Expressed by a formula, we have

$$f_2(x) =$$
$$0 \,.\, F_2(x+2N_2\tau_2) + \frac{1}{2}F_2(x+N_2\tau_2) + F_2(x) + \frac{1}{2}F_2(x-N_2\tau_2) + 0 \,.\, F_2(x-2N_2\tau_2),$$

where

$$F_2(x) = \sum_{p=-n_2}^{n_2} f_1(x+p\tau_2) .$$

Evidently (see the figure) the function $f_2(x)$ has as a translation number belonging to $\varepsilon_2 = \frac{1}{2}$ each of the N_2 numbers

$$\tau_2,\ 2\tau_2,\ 3\tau_2,\ \ldots,\ N_2\tau_2 .$$

As the length of this second step, we shall mean the total length λ_2 of the interval $[-(2N_2+n_2)\tau_2-1,\ (2N_2+n_2)\tau_2+1]$ on which the whole chain of our $5N_2$ tops is standing, i. e.

$$\lambda_2 = 2((2N_2+n_2)\tau_2+1) = (5N_2-1)\tau_2+2 .$$

3rd step. To the function $f_2(x)$ we now apply a similar procedure as that carried out in step 2 on the function $f_1(x)$. We start from the whole chain of $5N_2$ tops arrived at in step 2. By repeating this chain, unaltered, a certain number of times, we form a new, longer chain; and next we translate this new chain both to the right and to the left after having diminished its ordinates, not as above by the factors $\frac{1}{2}$ and 0, but now by the factors $\frac{2}{3}, \frac{1}{3}$, 0. Exactly speaking, we choose an arbitrary integer $\tau_3 > \lambda_2$ and a quite arbitrary odd integer $N_3 = 2n_3+1$, and then, firstly, build the function

$$F_3(x) = \sum_{p=-n_3}^{n_3} f_2(x+p\tau_3)$$

and next the function

$$f_3(x) = \sum_{q=-3}^{3} \left(1-\frac{|q|}{3}\right) F_3(x+qN_3\tau_3).$$

Thus the function $f_3(x)$ is built up of altogether $7N_3 . 5N_2$ individual tops and is composed of 7 consecutive groups, each consisting of $N_3 . 5N_2$ tops, of which the group in the middle consists of N_3 exemplars of the chain of tops from step 2, while the groups to the right and to the left are gradually diminished.

Evidently, the function $f_3(x)$ has as a translation number belonging to

$\varepsilon_3 = \frac{1}{3}$ each of the N_3 numbers

$$\tau_3,\ 2\tau_3,\ \ldots,\ N_3\tau_3\,.$$

Moreover, the function $f_3(x)$ has obviously maintained the property of the function $f_2(x)$ of having each of the numbers $\tau_2,\ 2\tau_2,\ \ldots,\ N_2\tau_2$ as a translation number belonging to $\varepsilon_2 = \frac{1}{2}$.

As the length of this 3rd step we mean the total length λ_3 of the interval on which the whole chain of tops (altogether $7N_3 \,.\, 5N_2$ tops) is standing, i.e.

$$\lambda_3 = 2(3N_3 + n_3)\tau_3 + \lambda_2 = (7N_3 - 1)\tau_3 + (5N_2 - 1)\tau_2 + 2\,.$$

. . . .

νth step. Having obtained by the $(\nu-1)$th step a function $f_{\nu-1}(x)$ consisting of a chain of altogether $5N_2 \,.\, 7N_3 \ldots (2\nu-1)N_{\nu-1}$ individual tops (with heights varying from 0 to 1) standing on an interval with midpoint at $x = 0$ and of total length

$$\lambda_{\nu-1} = ((2\nu-1)N_{\nu-1} - 1)\,\tau_{\nu-1} + \ldots + (7N_3 - 1)\tau_3 + (5N_2 - 1)\tau_2 + 2$$

and possessing as translation numbers belonging to

$$\varepsilon_\mu = \frac{1}{\mu} \qquad (\mu = 2,\ 3,\ \ldots,\ \nu-1)$$

each of the N_μ numbers

$$\tau_\mu,\ 2\tau_\mu,\ \ldots,\ N_\mu\tau_\mu\,,$$

we now pass to the construction of the function $f_\nu(x)$. We choose an arbitrary integer $\tau_\nu > \lambda_{\nu-1}$ and a quite arbitrary odd integer $N_\nu = 2n_\nu + 1$ and build, first, by repeating the chain of tops from the $(\nu-1)$th step n_ν times to the right and n_ν times to the left, the function

$$F_\nu(x) = \sum_{p=-n_\nu}^{n_\nu} f_{\nu-1}(x + p\tau_\nu)\,,$$

and next, by translating the new, longer chain (of altogether $N_\nu \,.\, 5N_2 \,.\, 7N_3 \ldots (2\nu-1)N_{\nu-1}$ tops), successively diminishing its ordinates by $\frac{\nu-1}{\nu},\ \frac{\nu-2}{\nu},\ \ldots,\ \frac{1}{\nu},\ \frac{0}{\nu}$, the function

$$f_\nu(x) = \sum_{q=-\nu}^{\nu} \left(1 - \frac{|q|}{\nu}\right) F_\nu(x+qN_\nu\tau_\nu) .$$

This function $f_\nu(x)$ thus consists of altogether $5N_2 . 7N_3 \dots (2\nu+1)N_\nu$ individual tops and stands on an interval with midpoint 0 and of total length

$$\lambda_\nu = 2(\nu N_\nu + n_\nu)\tau_\nu + \lambda_{\nu-1}$$
$$= ((2\nu+1)N_\nu - 1)\tau_\nu + ((2\nu-1)N_{\nu-1} - 1)\tau_{\nu-1} + \dots + (5N_2-1)\tau_2 + 2 .$$

Evidently, the function $f_\nu(x)$ possesses as translation numbers belonging to $\varepsilon_\nu = \frac{1}{\nu}$ each of the N_ν numbers

$$\tau_\nu ,\ 2\tau_\nu ,\ \dots,\ N_\nu\tau_\nu ,$$

and furthermore, it has preserved the property of $f_{\nu-1}(x)$ of having, for each $\mu = 2, \dots, \nu-1$, each of the numbers $\tau_\mu,\ 2\tau_\mu,\ \dots,\ N_\mu\tau_\mu$ as a translation number belonging to $\varepsilon_\mu = \frac{1}{\mu}$.

The limit process. Finally, we arrive at the desired function $f(x)$ by carrying out the limit process

$$f(x) = \lim_{\nu\to\infty} f_\nu(x) .$$

This last process involves, as mentioned above, no difficulty whatsoever, as, for each fixed ν, all the functions $f_\nu(x)$, $f_{\nu+1}(x)$, ... are equal in the interval $\left(-\frac{\lambda_\nu}{2}, \frac{\lambda_\nu}{2}\right)$ where $\lambda_\nu \to \infty$ as $\nu \to \infty$. The limit function $f(x)$ is continuous, and for every $\nu = 2, 3, \dots$, possesses each of the numbers $n\tau_\nu$ $(n = 1, 2, \dots, N_\nu)$ as a translation number belonging to $\varepsilon_\nu = \frac{1}{\nu}$, i. e. we have

$$|f(x + n\tau_\nu) - f(x)| \leqq \frac{1}{\nu} ;$$

this follows immediately from the inequality

$$|f_\mu(x + n\tau_\nu) - f_\mu(x)| \leqq \frac{1}{\nu} \qquad (\mu \geqq \nu ,\ 1 \leqq n \leqq N_\nu)$$

by letting $\mu \to \infty$ while ν, x and n are kept fixed.

Summarizing, we have constructed a periodiclike function, depending on an enumerable pair of parameters $\tau_2, N_2;\ \tau_3, N_3;\ \tau_4, N_4; \dots$ and possessing as a *translation number belonging to* $\varepsilon_\nu = \frac{1}{\nu}$ $(\nu = 2, 3, \dots)$ each of the N_ν numbers $\tau_\nu, 2\tau_\nu, \dots, N_\nu\tau_\nu$ and hence also (as $\varepsilon_\mu < \varepsilon_\nu$ for $\mu > \nu$)

each of the *infinitely many numbers*

$$\tau_\mu,\ 2\tau_\mu,\ \ldots,\ N_\mu\tau_\mu \quad (\mu=\nu,\ \nu+1,\ \ldots).$$

In the choice of these parameters, we have a great freedom; in fact we may choose the odd positive numbers $N_2, N_3, \ldots$ completely arbitrary, while for the τ's we have only to take care that each τ_ν satisfies the inequality $\tau_\nu > \lambda_{\nu-1}$ where $\lambda_{\nu-1}$ depends solely on $\tau_2, N_2, \tau_3, N_3, \ldots, \tau_{\nu-1}, N_{\nu-1}$.

We add that in each of the intervals $I_\nu: \frac{1}{2}\lambda_{\nu-1} < x < \tau_\nu - \frac{1}{2}\lambda_{\nu-1}$ of the length $\tau_\nu - \lambda_{\nu-1}$ the function $f(x)$, obviously, has no tops. In the following applications it will be of importance to secure the existence of *arbitrarily large intervals in which $f(x)$ is equal to zero.* This we may simply obtain by claiming that τ_ν shall be chosen $> \Lambda_{\nu-1} = \lambda_{\nu-1} + \nu$ instead of only $> \lambda_{\nu-1}$; in fact, then the length of the interval I_ν will be $> \nu$ and hence $\to \infty$ as $\nu \to \infty$.

§ 4. The proofs of the theorems A and B.

In this final section we shall, by means of periodiclike "counter-examples" of the type constructed in § 3, give the proofs of the two theorems A and B stated in § 2. Due to the great freedom in the choice of the parameters entering in our construction of $f(x)$ (the N_ν's completely arbitrary positive odd numbers, the τ_ν's arbitrary integers subjected only to the condition $\tau_\nu > \Lambda_{\nu-1}$ where $\Lambda_{\nu-1}$ depends solely on $\tau_1, N_1, \ldots, \tau_{\nu-1}, N_{\nu-1}$) these proofs may now easily be established.

Proof of Theorem A. Our task is to construct a function $f(x)$ which belongs to the class C_φ but is *not* almost periodic. Here the given positive function $\varphi(l)$, determining the class C_φ, is only subjected to the one condition

$$\underline{\lim}_{l\to\infty} \frac{\varphi(l)}{l} = 0, \tag{7}$$

i.e. the only thing we know about $\varphi(l)$ is that to an arbitrarily given positive constant δ we may find an arbitrarily large positive l such that

$$\varphi(l) < \delta . l . \tag{8}$$

We consider a function $f(x)$ of the type constructed in § 3 where

the parameters $\tau_2, N_2, \tau_3, N_3, \ldots$ are determined, successively, in the following manner. For τ_2 we choose an arbitrary integer $> \Lambda_1$ $(=4)$. In order to determine N_2 we proceed in the following way; we take an $l_2 > \tau_2$ such that

$$\varphi(l_2) < \frac{1}{\tau_2} l_2$$

(this is possible according to (8), taking $\delta = \frac{1}{\tau_2}$), and then we simply determine the odd integer N_2 so large that $l_2 < N_2\tau_2$. Generally we choose the integer $\tau_\nu > \Lambda_{\nu-1}$, and an $l_\nu > \tau_\nu$ such that

$$\varphi(l_\nu) < \frac{1}{\tau_\nu} l_\nu, \tag{9}$$

and then take the odd integer N_ν so large that $l_\nu < N_\nu\tau_\nu$.

Thus, besides the parameters $\tau_2, N_2, \tau_3, N_3, \ldots$ determining the function $f(x)$ we have introduced the auxiliary numbers $l_2, l_3, \ldots$. Here l_ν $(> \tau_\nu)$ tends to ∞ for $\nu \to \infty$. For each $\nu \geqq 2$ we denote by $m_\nu\tau_\nu$ the least multiple of τ_ν which is $\geqq l_\nu$, i. e.

$$(m_\nu - 1)\tau_\nu < l_\nu \leqq m_\nu\tau_\nu;$$

as $\tau_\nu < l_\nu < N_\nu\tau_\nu$ we have $1 < m_\nu \leqq N_\nu$.

We shall show that the function $f(x)$ thus obtained fulfills the conditions of theorem A.

1. The function $f(x)$ *belongs to the class* C_φ. As translation numbers τ belonging to a fixed $\frac{1}{\nu_0}$ $(\nu_0 \geqq 2)$ it has certainly all the numbers of the increasing sequence of positive integers

$$\tau_{\nu_0}, 2\tau_{\nu_0}, \ldots, N_{\nu_0}\tau_{\nu_0}; \ \tau_{\nu_0+1}, 2\tau_{\nu_0+1}, \ldots, N_{\nu_0+1}\tau_{\nu_0+1}; \ \ldots;$$
$$\tau_\nu, 2\tau_\nu, \ldots, N_\nu\tau_\nu; \ \ldots$$

In order to show that this sequence $\{t_r\}$ satisfies the condition

$$\overline{\lim} \frac{M_l}{\varphi(l)} > 0$$

we consider the fraction $\frac{M_l}{\varphi(l)}$ where l runs through the sequence of positive numbers l_ν $(\nu = \nu_0, \nu_0+1, \ldots)$ which tends to ∞ for $\nu \to \infty$. We shall see that for these values of l the fraction remains greater than a positive constant, indeed that

$$\frac{M_{l_\nu}}{\varphi(l_\nu)} > 1 \quad \text{for all } \nu \geqq \nu_0 .$$

This, however, is an immediate consequence of the inequality (9), holding for all $\nu \geqq 2$. Indeed, since, for every $\nu \geqq \nu_0$, the interval $\tau_\nu \leqq x < \tau_\nu + l_\nu$ $(> m_\nu \tau_\nu)$ of length l_ν contains the m_ν $(\leqq N_\nu)$ numbers $\tau_\nu, 2\tau_\nu, \ldots, m_\nu \tau_\nu$ of our sequence $\{t_r\}$ we have — according to the very definition of the function M_l — the inequality $M_{l_\nu} \geqq m_\nu$, and thus we get, for every $\nu \geqq \nu_0$ the desired inequality

$$\frac{M_{l_\nu}}{\varphi(l_\nu)} > \frac{m_\nu}{\frac{1}{\tau_\nu} \cdot l_\nu} = \frac{m_\nu \tau_\nu}{l_\nu} \geqq 1 .$$

2. The function $f(x)$ is *not an almost periodic function.* In fact, for *no* $\varepsilon < 1$ the set $E(\varepsilon)$ of translation numbers $\tau(\varepsilon)$ is relatively dense since, on account of $f(0) = 1$, every translation number $\lambda = \tau(\varepsilon)$ must satisfy the inequality

$$f(\tau) \geqq 1 - \varepsilon > 0$$

while, by choosing $\tau_\nu > \Lambda_{\nu-1}$, we have—see the last remark in §3—secured the existence of arbitrary long intervals in which $f(x)$ is identically zero.

Proof of Theorem B. Here we shall make use of the following simple remark. When in the course of the construction, in the proof above, of a function $f(x)$ belonging to the class C_φ we have already determined the parameters $\tau_2, N_2, \tau_3, N_3, \ldots, \tau_{\nu-1}, N_{\nu-1}$ we may certainly choose the next pair of parameters τ_ν, N_ν so that all the new tops occurring in the νth step of the construction (i.e., the tops which belong to $f_\nu(x)$ but not to $f_{\nu-1}(x)$) are lying *outside an arbitrarily given interval*; in fact, we need only take τ_ν sufficiently large.

Obviously, the Theorem B will be proved if, by means of our process $f(x) = \lim\limits_{\nu \to \infty} f_\nu(x)$, we can construct two functions in the class C_φ — let us denote them by $g(x) = \lim g_\nu(x)$ and $h(x) = \lim h_\nu(x)$ and their parameters by $\tau'_2, N'_2, \ldots,$ and $\tau''_2, N''_2, \ldots$ respectively — such that *no top of the one function collides with any top of the other* (by which we mean that the bases of the two tops have no point in common), except of course

as regards their common 1-top placed at the point 0. In fact, if this is the case, their sum $g(x) + h(x)$ will certainly *not be a periodiclike function*, since it has the value 2 at the point $x = 0$ and is $\leqq 1$ outside the interval $-1 < x < 1$ so that *no number* $\tau > 1$ can possibly be a translation number of $g(x) + h(x)$ belonging to an $\varepsilon < 1$. To ascertain, however, that no top of $g(x)$ collides with any top of $h(x)$ (except as regards the common top at $x = 0$) causes no difficulty whatsoever on account of the great freedom we still have in the choice of the entering parameters; in fact, the only conditions put to the parameters are that they shall be chosen "sufficiently large". We need only choose the pairs of parameters (τ, N) of the functions $g(x)$ and $h(x)$ alternately. We begin by choosing τ'_2, N'_2 arbitrarily (satisfying, of course, the conditions given in the proof of Theorem A). Next we choose τ''_2, N''_2 so that all the tops of $h_2(x)$, except the 1-top of $h_1(x)$, are lying outside the interval on which the tops of $g_2(x)$ are standing. Then we turn again to the construction of $g(x)$ and choose the parameters τ'_3, N'_3 such that all the tops occurring in $g_3(x)$ but not in $g_2(x)$ lie outside the interval on which the tops of the function $h_2(x)$ are standing. And next we turn again to $h(x)$ and determine τ''_3, N''_3 such that all the tops in $h_3(x)$ but not in $h_2(x)$ lie outside the interval on which the tops of $g_3(x)$ are standing. Proceeding in this manner (and, of course, taking care that the conditions on the N's and the τ's, imposed on them in the proof of Theorem A are satisfied), we obviously arrive at two functions $g(x)$ and $h(x)$ of the class C_φ with no "top-collision" and hence with a sum $g(x) + h(x)$ which is not a periodiclike function.

(Received October 13, 1950.)

LINEAR CONGRUENCES
DIOPHANTINE APPROXIMATIONS

ANOTHER PROOF OF KRONECKER'S THEOREM

By H. BOHR.

[Received March 31st, 1922.—Read April 20th, 1922.]

MANY different proofs of Kronecker's Theorem have been given, among which the proof of Dr. Lettenmeyer, based on simple geometrical considerations, is a particularly beautiful one. In this note I give one more. This, like Weyl's proof of his extension of Kronecker's Theorem*, is based on the use of the function $e^{2\pi i x}$, the "invariant of the number x to modulus 1"; and has a simple connection with the applications I have made of Kronecker's Theorem to the theory of Dirichlet's series.†

Kronecker's Theorem may be stated as follows. Suppose that $\lambda_1 \dots \lambda_N$ are N linearly independent real numbers, $\phi_1 \dots \phi_N$ are N arbitrary real numbers, and ϵ is positive. Then there exist N integers $h_1 \dots h_N$ and a real number t such that

$$|t\lambda_n - \phi_n - h_n| < \epsilon \quad (n = 1, 2, \dots, N),$$

i.e. such that the N complex numbers

$$e^{2\pi i (t\lambda_n - \phi_n)} \quad (n = 1, 2, \dots, N)$$

all differ by less than ϵ_1 from $e^0 = 1$. In other words, the upper limit L_F of the numerical value of the function

$$F(t) = 1 + e^{2\pi i (t\lambda_1 - \phi_1)} + e^{2\pi i (t\lambda_2 - \phi_2)} + \dots + e^{2\pi i (t\lambda_N - \phi_N)},$$

for all real values of t, is equal to the upper limit $L_G = N+1$ of the numerical value of the function

$$G(x_1, x_2, \dots, x_N) = 1 + e^{2\pi i x_1} + e^{2\pi i x_2} + \dots + e^{2\pi i x_N},$$

---

* H. Weyl, "Über die Gleichverteilung von Zahlen mod. Eins", *Math Ann.*, Vol. 77 (1916), pp. 313–352.

† See, for example, H. Bohr, "Über die Bedeutung der Potenzreihen unendlich vieler Variabeln in der Theorie der Dirichletschen Reihen $\Sigma a_n n^{-s}$", *Göttinger Nachrichten* (1913) pp. 441–488.

where $x_1, x_2, \ldots, x_N$ vary *independently* through the interval $0 \leqslant x \leqslant 1$. Evidently it is enough to prove that $L_F \geqslant L_G = N+1$.

We consider the power

$$(1) \qquad \{F(t)\}^p = \{1+e^{2\pi i(t\lambda_1-\phi_1)}+\ldots+e^{2\pi i(t\lambda_N-\phi_N)}\}^p,$$

where p is an arbitrary positive integer. In the polynomial development of (1) no two terms fall together, since—on account of the linear independence of the λ's—no two terms contains the same exponential $e^{2\pi i at}$. In other words, the development (1) contains the same number of terms as the polynomial development of

$$(2) \qquad \{G(x_1, \ldots, x_N)\}^p = \{1+e^{2\pi i x_1}+\ldots+e^{2\pi i x_N}\}^p,$$

and these terms have the same numerical coefficients. Therefore, by a classical argument, the two mean values

$$F_p = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |\{F(t)\}^p|^2\, dt$$

and

$$G_p = \int_0^1\int_0^1 \ldots \int_0^1 |\{G(x_1, x_2, \ldots, x_N)\}^p|^2\, dx_1\, dx_2 \ldots dx_N$$

are equal [their common value being the sum of the squares of the numerical coefficients in the development of (1) and (2)].

Now evidently (for reasons of continuity)

$$G_p^{1/2p} \to L_G (= N+1),$$

when $p \to \infty$, since the function $|G(x_1, x_2, \ldots, x_N)|$ really assumes its upper limit L_G in the point $(0, 0, \ldots, 0)$ of the N-dimensional unit-cube $0 \leqslant x_1 \leqslant 1, \ldots, 0 \leqslant x_N \leqslant 1$; and since $F_p = G_p$, we have

$$F_p^{1/2p} \to L_G.$$

But it is evident that $F_p^{1/2p} \leqslant L_F$ for all p; and therefore

$$L_F \geqslant L_G,$$

which proves the theorem.

It would of course be possible to prove that $F_p^{1/2p} \to N+1$ more directly, by a calculation based on the actual values of the coefficients; but the argument above avoids all calculation and brings out the real point more clearly.

Det Kgl. Danske Videnskabernes Selskab.
Mathematisk-fysiske Meddelelser. **VI,** 8.

NEUER BEWEIS EINES ALLGEMEINEN KRONECKER'SCHEN APPROXIMATIONSSATZES

VON

HARALD BOHR

KØBENHAVN
HOVEDKOMMISSIONÆR: ANDR. FRED. HØST & SØN, KGL. HOF-BOGHANDEL
BIANCO LUNOS BOGTRYKKERI
1924

III 14 D 2. *Dan. Math. Fys. Medd.* 6, no. 8 (1924).

Einleitung.

Dafür, dass die N diophantischen Ungleichungen[1]

$$(1)\quad \begin{cases} |\alpha_{11}t_1+\alpha_{12}t_2+\ldots+\alpha_{1M}t_M-\beta_1|<\varepsilon \quad (\text{mod. } 1)\\ |\alpha_{21}t_1+\alpha_{22}t_2+\ldots+\alpha_{2M}t_M-\beta_2|<\varepsilon \quad (\text{mod. } 1)\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ |\alpha_{N1}t_1+\alpha_{N2}t_2+\ldots+\alpha_{NM}t_M-\beta_N|<\varepsilon \quad (\text{mod. } 1), \end{cases}$$

wo die α_{nm} und β_n reelle Grössen bedeuten, bei jedem beliebig kleinen $\varepsilon>0$ eine Lösung in den M reellen Variablen $t_1, t_2, \ldots, t_M$ besitzen, ist die folgende Bedingung offenbar notwendig:

Bei jedem System von N ganzen Zahlen $g_1, \ldots, g_N$, für welches der Ausdruck

$$g_1(\alpha_{11}t_1+\ldots+\alpha_{1M}t_M)+\ldots+g_N(\alpha_{N1}t_1+\ldots+\alpha_{NM}t_M)$$

identisch in den M Variablen $t_1, \ldots, t_M$ verschwindet, d. h. welches die M Gleichungen

$$(2)\quad g_1\alpha_{1m}+g_2\alpha_{2m}+\ldots+g_N\alpha_{Nm}=0 \qquad (m=1,\ldots,M)$$

befriedigt, muss die Zahl

$$(3)\qquad g_1\beta_1+g_2\beta_2+\ldots+g_N\beta_N$$

eine ganze Zahl sein.

In der Tat folgt ja für beliebige (nicht sämtlich verschwindende) ganze Zahlen $g_1, \ldots, g_N$ aus dem Bestehen

[1] Unter der Schreibweise $|a|<b$ (mod. 1), wo a und $b>0$ reelle Zahlen sind, verstehen wir, dass eine ganze Zahl g derart existiert, dass $|a-g|<b$ ist.

D 2. *Dan. Math. Fys. Medd.* 6, no. 8 (1924).

der Ungleichungen (1) die neue diophantische Ungleichung

$$|g_1(\alpha_{11}t_1+\ldots+\alpha_{1M}t_M-\beta_1)+\ldots+g_N(\alpha_{N1}t_1+\ldots+\alpha_{NM}t_M-\beta_N)|$$
$$< \varepsilon(|g_1|+\ldots+|g_N|) \quad (\text{mod. } 1),$$

also, falls die g_n den Gleichungen (2) genügen, die Ungleichung

$$|g_1\beta_1+\ldots+g_N\beta_N| < \varepsilon(|g_1|+\ldots+|g_N|) \quad (\text{mod. } 1),$$

und hieraus ergibt sich sofort, da ja ε beliebig klein gewählt werden kann, dass die (von ε unabhängige) Zahl $g_1\beta_1+\ldots+g_N\beta_N$ eine ganze Zahl sein muss.

Ein bekannter allgemeiner Satz von Kronecker besagt, *dass die obige notwendige Bedingung auch eine hinreichende Bedingung dafür darstellt, dass die N diophantischen Ungleichungen* (1) *bei beliebig kleinem ε eine Lösung in $t_1, \ldots, t_M$ besitzen*, oder anders ausgedrückt, dass es für die Lösbarkeit der N diophantischen Ungleichungen (1) hinreichend ist, dass sie keinen »offenkundigen« Widerspruch aufweisen.

In dem speziellen Falle, wo alle α_{nm} mit $m > 1$ gleich 0 sind, und ausserdem noch die Grössen $\alpha_{11}, \ldots, \alpha_{N1}$ linear unabhängig sind d. h. keine Relation der Form

$$g_1\alpha_{11}+\ldots+g_N\alpha_{N1} = 0$$

in ganzen (nicht sämtlich verschwindenden) Zahlen $g_1, \ldots, g_N$ befriedigen, geht dieser Satz in den sogenannten »kleinen« Kronecker'schen Satz über, welcher besagt, dass N diophantische Ungleichungen der Form

$$|\alpha_n t-\beta_n| < \varepsilon \quad (\text{mod. } 1) \qquad (n = 1, \ldots, N),$$

wo die Koeffizienten $\alpha_1, \ldots, \alpha_N$ linear unabhängig sind, bei beliebiger Wahl der Zahlen $\beta_1, \ldots, \beta_N$ und beliebig kleinem ε stets eine Lösung in t besitzen. Für diesen letzten Satz, welcher in verschiedenen neueren Untersuchungen eine fundamentale Rolle spielt, habe ich vor einigen Jahren

einen neuen einfachen Beweis gegeben.[1] Das Ziel dieser Note ist zu zeigen, dass meine damalige Methode auch im Stande ist, den oben genannten allgemeinen Kronecker'schen Satz mit einem Schlage zu beweisen.

Beweis des Satzes.

Es seien also NM reelle Grössen α_{nm} beliebig gegeben, und es seien $\beta_1, \ldots, \beta_N$ reelle Grössen derart, dass für jedes System von ganzen Zahlen $g_1, \ldots, g_N$, welches die M Gleichungen (2) befriedigt, die Zahl (3) ganz ausfällt. Wir setzen zur Abkürzung

$$\alpha_{n1} t_1 + \alpha_{n2} t_2 + \ldots + \alpha_{nM} t_M = y_n = y_n(t_1, t_2, \ldots, t_M)$$

und haben alsdann zu beweisen, dass es möglich ist, solche Werte der M Variablen $t_1, t_2, \ldots, t_M$ zu bestimmen, dass die N reellen Zahlen $y_n - \beta_n$, modulo 1 betrachtet, alle beliebig klein werden, d. h. dass die N komplexen Zahlen

$$e^{2\pi i(y_n - \beta_n)} \qquad (n = 1, \ldots, N)$$

alle um beliebig wenig von $e^0 = 1$ abweichen. Die Behauptung lautet mit anderen Worten, *dass die obere Grenze L_F des absoluten Wertes der Funktion*

$$F(t_1, t_2, \ldots, t_M) = 1 + \sum_{n=1}^{N} e^{2\pi i(y_n - \beta_n)},$$

wo der Punkt $(t_1, \ldots, t_M)$ den ganzen M-dimensionalen Raum durchläuft, gleich der oberen Grenze $L_G = N + 1$ des absoluten Wertes der Funktion

[1] H. Bohr, Another Proof of Kronecker's Theorem [Proceedings of the London Mathematical Society, Ser. II, Bd. XXI (1923), S. 315—316].

$$G(x_1, x_2, \ldots, x_N) = 1 + \sum_{n=1}^{N} e^{2\pi i x_n}$$

ist, wo die N Variablen $x_1, \ldots, x_N$ *unabhängig von einander das Intervall* $0 \leq x < 1$ *durchlaufen.* Hierbei genügt es offenbar

$$L_F \geq L_G$$

zu beweisen.

Wir betrachten, bei einem beliebigen positiven ganzen R, die Potenzen

$$\left\{F(t_1, \ldots, t_M)\right\}^R = \left\{1 + \sum_{n=1}^{N} e^{2\pi i (y_n - \beta_n)}\right\}^R \tag{4}$$

und

$$\left\{G(x_1, \ldots, x_N)\right\}^R = \left\{1 + \sum_{n=1}^{N} e^{2\pi i x_n}\right\}^R \tag{5}$$

und vergleichen die beiden Polynomialentwickelungen von (4) und (5). Da die x_n von einander unabhängige Variable bedeuten, können natürlich in der Polynomialentwickelung von (5) keine zwei Glieder zusammengezogen werden; dies kann aber sehr wohl in der Polynomialentwickelung von (4) passieren — weil ja mehrere Glieder denselben Exponentialfaktor $e^{i(\lambda_1 t_1 + \cdots + \lambda_M t_M)}$ enthalten können — und zwar werden offenbar zwei Glieder mit den Faktoren

$$e^{2\pi i (p_1 y_1 + \cdots + p_N y_N)} \quad \text{bezw.} \quad e^{2\pi i (q_1 y_1 + \cdots + q_N y_N)}$$

dann und nur dann zu einem Gliede zusammengefasst werden können, wenn die ganzen Zahlen

$$g_1 = p_1 - q_1, \ \ldots, \ g_N = p_N - q_N$$

den M Gleichungen (2) genügen. In diesem Falle unterscheiden sich aber, weil (3) nach Voraussetzung eine ganze Zahl wird, die beiden Grössen

$$p_1 \beta_1 + \ldots + p_N \beta_N \quad \text{und} \quad q_1 \beta_1 + \ldots + q_N \beta_N$$

um eine ganze Zahl, d. h. es sind die Amplituden der Koeffizienten der beiden betreffenden Glieder in der Polynomialentwickelung von (4) gleich gross. Hieraus folgt aber, *dass die Summe der Quadrate der absoluten Werte der Koeffizienten in der* (durch Zusammenfassung von Gliedern mit demselben Exponentialfaktor zusammengezogenen) *Entwickelung von* (4) *gewiss* $\geqq$ *der Summe der Quadrate der absoluten Werte der Koeffizienten in der Entwickelung von* (5) *ist*; denn für komplexe Zahlen $c_1, c_2, \ldots, c_l$ mit derselben Amplitude gilt ja die Ungleichung

$$|c_1 + c_2 + \ldots + c_l|^2 \geqq |c_1|^2 + |c_2|^2 + \ldots + |c_l|^2.$$

Nun ist aber, nach einem klassischen Verfahren[1], die Summe der Quadrate der absoluten Werte der Koeffizienten in der Entwickelung (4) bezw. (5) gleich dem Mittelwerte

$$F_R = \lim_{T_1, T_2, \ldots, T_M \to \infty} \frac{1}{T_1 T_2 \cdots T_M} \int_0^{T_1} dt_1 \int_0^{T_2} dt_2 \ldots \int_0^{T_M} \left| \left\{ F(t_1, t_2, \ldots, t_M) \right\}^R \right|^2 dt_M$$

bezw.

$$G_R = \int_0^1 dx_1 \int_0^1 dx_2 \ldots \int_0^1 \left| \left\{ G(x_1, x_2, \ldots, x_N) \right\}^R \right|^2 dx_N,$$

und es gilt daher, bei jedem positiven ganzen R, die Ungleichung

$$F_R \geqq G_R,$$

also auch die Ungleichung

(6) $$\sqrt[2R]{F_R} \geqq \sqrt[2R]{G_R}.$$

Wir führen nunmehr den Grenzübergang $R \to \infty$ aus. Hier-

[1] Man hat nur das Quadrat $|F|^2 = F \cdot \overline{F}$ bezw. $|G|^2 = G \cdot \overline{G}$ auszurechnen und von jedem der (endlich vielen) Glieder den Mittelwert zu nehmen, wodurch alle Glieder ausser dem konstanten Gliede wegfallen.

durch erhalten wir sofort (aus Stetigkeitsgründen) die Limesgleichung

$$\lim_{R \to \infty} \sqrt[2R]{G_R} = L_G \quad (= N+1),$$

weil die Funktion $|G(x_1, \ldots, x_N)|$ tatsächlich ihre obere Grenze in einem Punkte des N-dimensionalen Einheitskubus, nämlich dem Punkte $(0, \ldots, 0)$, annimmt. Also gilt, wegen (6), die Limesungleichung

$$\limsup_{R \to \infty} \sqrt[2R]{F_R} \geqq L_G.\text{[1]} \tag{7}$$

Nun ist aber offenbar bei jedem R der Mittelwert $\sqrt[2R]{F_R} \leqq$ der oberen Grenze L_F, und wir können daher aus (7) sofort die gewünschte Ungleichung

$$L_F \geqq L_G$$

folgern, womit der Satz bewiesen ist.

[1] Übrigens auch $\liminf_{R \to \infty} \sqrt[2R]{F_R} \geqq L_G$.

Forelagt paa Mødet den 12. December 1924.
Færdig fra Trykkeriet den 15. December 1924.

Det Kgl. Danske Videnskabernes Selskab.
Mathematisk-fysiske Meddelelser. **VII**, 1.

UNENDLICH VIELE LINEARE KONGRUENZEN MIT UNENDLICH VIELEN UNBEKANNTEN

VON

HARALD BOHR

KØBENHAVN
HOVEDKOMMISSIONÆR: ANDR. FRED. HØST & SØN, KGL. HOF-BOGHANDEL
BIANCO LUNOS BOGTRYKKERI
1925

D 3. *Dan. Math. Fys. Medd.* 7, no. 1 (1925).

Einleitung.

Bei der Untersuchung der Klasseneinteilung fastperiodischer Funktionen[1] bin ich auf das folgende Problem gestossen:

Es sei ein System von unendlich[2] vielen Linearformen in unendlich vielen Variablen

$$(1) \qquad r_{n,1}x_1 + r_{n,2}x_2 + \cdots + r_{n,q_n}x_{q_n} \qquad (n = 1, 2, \cdots)$$

vorgelegt, wo jede einzelne Linearform nur endlich viele der Variablen $x_1, x_2, \cdots$ enthält, und wo die Koeffizienten r alle rationale Zahlen sind ($r_{n,q_n} \neq 0$).[3] Es bezeichne Π_1 die Menge aller Punkte $(\theta_1, \theta_2, \cdots)$ des unendlich-dimensionalen Raumes, für welche die unendlich vielen linearen Kongruenzen in den unendlich vielen Unbekannten $x_1, x_2, \cdots$

$$(2) \quad r_{n,1}x_1 + r_{n,2}x_2 + \cdots + r_{n,q_n}x_{q_n} \equiv \theta_n \pmod{1} \quad (n = 1, 2, \cdots)$$

eine simultane Lösung $(x_1, x_2, \cdots) = (x_1^*, x_2^*, \cdots)$ besitzen, und es bezeichne Π_2 die Menge aller Punkte $(\theta_1, \theta_2, \cdots)$ des unendlich-dimensionalen Raumes, für welche bei jedem festen positiven ganzen N die N ersten der Kon-

[1] H. Bohr: Zur Theorie der fastperiodischen Funktionen II (Anhang 1, § 3), Acta Mathematica 1925.

[2] Wir gebrauchen das Wort »unendlich« überall im Sinne »abzählbar unendlich«.

[3] Es enthält also jede der unendlich vielen Linearformen mindestens eine der Variablen $x_1, x_2, \cdots$. Dagegen brauchen nicht alle unendlich vielen Variablen tatsächlich vorzukommen.

gruenzen (2) eine Lösung[1] $(x_1, x_2, \cdots) = (x_1^{(N)}, x_2^{(N)}, \cdots)$ haben.[2]

Es ist klar, dass sowohl Π_1 wie Π_2 den Anfangspunkt $(0, 0, \cdots)$ des unendlich-dimensionalen Raumes enthalten, und ferner *dass*

$$\Pi_1 < \Pi_2$$

ist[3]; denn eine simultane Lösung $(x_1^*, x_2^*, \cdots)$ der sämtlichen unendlich vielen Kongruenzen (2) wird ja zugleich jede endliche Anzahl der Kongruenzen befriedigen.

Unser Problem lautet nun: *Wann sind die beiden Mengen Π_1 und Π_2 mit einander identisch,* d. h. wie soll das System (1) von Linearformen beschaffen sein, damit es, bei beliebig gegebenen Werten $\theta_1, \theta_2, \cdots$, für die Existenz einer simultanen Lösung der unendlich vielen Kongruenzen (2) nicht nur notwendig, sondern auch hinreichend sei, dass die N ersten dieser Kongruenzen bei jedem N eine Lösung besitzen?

Es gibt einen einfachen Fall, wo man sofort sieht, dass $\Pi_1 = \Pi_2$ ist. In der Tat gilt

Satz 1. *Falls die rationalen Koeffizienten r der unendlich vielen Linearformen (1) sämtlich* **ganze** *Zahlen sind, fallen die beiden Mengen Π_1 und Π_2 zusammen.*

Beweis. Es sei $(\theta_1, \theta_2, \cdots)$ ein beliebiger Punkt der

[1] Es ist für unsere Zwecke bequem unter einer »Lösung« der N ersten Kongruenzen (2) einen Punkt $(x_1, x_2, \cdots)$ des unendlich-dimensionalen Raumes zu verstehen, obwohl in diesen N Kongruenzen nur endlich viele der Variablen x vorkommen, und es somit ganz belanglos ist, welche Werte die Koordinaten x_m von einer gewissen Stelle an haben.

[2] In der Definition der Menge Π_2 hätten wir natürlich statt »bei jedem N die N ersten der Kongruenzen« ebensogut »jede beliebige endliche Anzahl der Kongruenzen« schreiben können.

[3] Unter $A < B$ (oder $B > A$), wo A und B zwei Punktmengen desselben Raumes sind, verstehen wir, dass die Punktmenge A in der Punktmenge B enthalten ist.

Menge Π_2, d. h. es haben bei jedem festen N die N ersten der Kongruenzen (2) eine Lösung $(x_1^{(N)}, x_2^{(N)}, \cdots)$. Hierbei können wir uns diese Lösung so gewählt denken, dass jede Koordinate x_m zwischen 0 (incl.) und 1 (excl.) gelegen ist; denn da die Koeffizienten der N Kongruenzen alle ganz sind, darf ja in einer Lösung jede Koordinate x_m um eine beliebige ganze Zahl geändert werden. Die aus diesen Lösungen

$$(x_1^{(N)}, x_2^{(N)}, \cdots) \qquad (N = 1, 2, \cdots)$$

gebildete Punktmenge des unendlich-dimensionalen Raumes hat daher gewiss mindestens einen Haüfungspunkt $(x_1^*, x_2^*, \cdots)$, in dem Sinne, dass es eine Folge von wachsenden positiven ganzen Zahlen $N_1, N_2, \cdots, N_p, \cdots$ gibt, so dass bei jedem festen $m = 1, 2, \cdots$ die Limesgleichung

$$\lim_{p \to \infty} x_m^{(N_p)} = x_m^*$$

besteht. Dieser Häufungspunkt $(x_1^*, x_2^*, \cdots)$ wird alsdann eine simultane Lösung der sämtlichen unendlich vielen Kongruenzen (2) sein. In der Tat, falls n_0 eine beliebige positive ganze Zahl ist, wird $(x_1^*, x_2^*, \cdots)$ aus Stetigkeitsgründen die n_0^{te} Kongruenz befriedigen, weil in dieser Kongruenz nur endlich viele der Variablen x vorkommen, und der Punkt $(x_1^{(N_p)}, x_2^{(N_p)}, \cdots)$ bei jedem $N_p \geqq n_0$ eine Lösung der n_0^{ten} Kongruenz darstellt. Es gehört somit der Punkt $(\theta_1, \theta_2, \cdots)$ auch zur Menge Π_1, d. h. es fallen die beiden Mengen Π_1 und Π_2 zusammen.

Das Ziel der vorliegenden Abhandlung ist zu zeigen, dass der im Satze 1 betrachtete Fall »wesentlich« der einzige ist, wo $\Pi_1 = \Pi_2$ ist. Wir werden nämlich beweisen, *dass es für das Zusammenfallen der beiden Mengen Π_1 und Π_2 nicht nur hinreichend, sondern auch notwendig ist,*

dass das gegebene System (1) durch eine »lineare Substitution« in ein neues System von Linearformen mit lauter **ganzen** *Koeffizienten übergeführt werden kann.*

Wir teilen die Untersuchung in fünf Paragraphen ein.

§ 1 enthält einige Bemerkungen über Systeme von linearen Gleichungen mit unendlich vielen Unbekannten, wo (wie bei den obigen Kongruenzen) in jeder einzelnen Gleichung nur endlich viele Unbekannte auftreten. Für ein derartiges Gleichungsystem wurde bekanntlich von Toeplitz in einer interessanten Abhandlung[1] eine erschöpfende Theorie gegeben, in welcher gezeigt wurde, dass die Verhältnisse bei einem solchen System fast ebenso einfach liegen, wie bei einem System von nur endlich vielen Gleichungen. Wir brauchen aus dieser Theorie nur ein einzelnes Resultat, für welches wir einen direkten aüsserst einfachen Beweis geben.

In § 2 wird, mit Hülfe des Satzes von § 1, der Begriff einer linearen Substitution in unendlich vielen Variabeln erörtert.

§ 3 bringt eine einfache Reduktion der gestellten Aufgabe, indem gezeigt wird, dass man sich ohne Beschränkung der Allgemeinheit auf die Betrachtung solcher Systeme (1) beschränken kann, wo jede einzelne der Variablen x_1, $x_2, \cdots$ durch lineare Kombination endlich vieler der Linearformen isoliert werden kann.[2]

[1] O. Toeplitz: Über die Auflösung unendlichvieler linearer Gleichungen mit unendlichvielen Unbekannten, Palermo Rendiconti Bd. 28 (1909), S. 88—96.

[2] Durch diese Reduktion sichern wir uns einerseits, dass jede der Variablen x_m tatsächlich vorkommt, und andererseits, dass nicht gewisse Variable, z. B. x_3 und x_5, überall in einer festen Kombination, etwa

In § 4 leiten wir einige einfache, auch an sich ganz interessante Kriterien dafür her, dass ein (reduziertes) System (1) durch eine lineare Substitution in ein neues System mit lauter ganzen Koeffizienten übergeführt werden kann.

Schliesslich wird im § 5 der oben genannte Hauptsatz über die notwendige und hinreichende Bedingung für das Zusammenfallen der beiden Mengen Π_1 und Π_2 bewiesen.

§ 1.

Unendlich viele lineare Gleichungen mit unendlich vielen Unbekannten.

Für unsere späteren Überlegungen brauchen wir den

Satz 2. *Damit ein System von unendlich vielen linearen Gleichungen der Form*

$$(3)\qquad \varrho_{n,1}x_1+\varrho_{n,2}x_2+\cdots+\varrho_{n,q_n}x_{q_n}=\vartheta_n\qquad (n=1,\,2,\cdots),$$

wo die Koeffizienten ϱ und die Konstanten ϑ_n beliebige reelle Zahlen sind, eine simultane Lösung $(x_1^*,\,x_2^*,\cdots)$ besitze, ist nicht nur notwendig, sondern auch hinreichend, dass die Gleichungen keinen »offenkundigen Widerspruch« aufweisen, d. h. dass es kein solches System von endlich vielen reellen Zahlen $\sigma_1,\,\sigma_2,\cdots,\,\sigma_N$ gibt, dass durch lineare Kombination der N ersten Gleichungen mit den respektiven Multiplikatoren $\sigma_1,\,\sigma_2,\cdots\sigma_N$ eine Gleichung der Form

$$0x_1+0x_2+\cdots+0x_L=k$$

mit $k\neq 0$ erhalten wird.

Beweis. Es handelt sich darum, aus der Annahme, dass

$\frac{1}{3}x_2+\frac{2}{7}x_5$ auftreten (in welchem Falle es ja natürlicher ist, diese zwei Variablen x_2 und x_5 durch eine einzige neue Variable $z=\frac{1}{3}x_2+\frac{2}{7}x_5$ zu ersetzen).

kein offenkundiger Widerspruch zwischen den Gleichungen vorliegt, die Existenz mindestens einer (simultanen) Lösung zu beweisen. Hierzu betrachten wir zunächst die Variable x_1 und unterscheiden zwischen den beiden folgenden Möglichkeiten:

1^0 Entweder ist x_1 »isolierbar«; d. h. es lässt sich durch lineare Kombination endlich vieler der Gleichungen (3) eine Gleichung der Form $x_1 = c_1$ (c_1 konstant) ableiten. Vielleicht lässt sich x_1 in mehreren verschiedenen Weisen isolieren; das Resultat (d. h. der Wert der Konstanten c_1) muss aber immer dasselbe sein, weil sonst die Gleichungen (3) im offenkundigen Widerspruch mit einander wären.

2^0 Oder x_1 ist nicht isolierbar.

Im Falle 1^0 setzen wir $x_1 = x_1^*$, wo x_1^* die bei der Isolation von x_1 auftretende Konstante c_1 bedeutet, und im Falle 2^0 geben wir x_1 einen ganz beliebig gewählten, von nun an festzuhaltenden Wert x_1^*.

Wir setzen nun für die Variable x_1 ihren somit bestimmten Wert x_1^* in das Gleichungsystem (3) ein (und ziehen die entsprechenden Glieder auf die rechten Seiten der Gleichungen über). Das hierdurch entstandene neue Gleichungsystem in den Variablen $x_2, x_3, \cdots$ weist offenbar, wie das ursprüngliche Gleichungsystem in den Variablen $x_1, x_2, \cdots$, keinen offenkundigen Widerspruch auf, d. h. falls $\sigma_1, \cdots, \sigma_N$ solche reelle Zahlen sind, dass durch lineare Kombination der N ersten der neuen Gleichungen mit den Multiplikatoren $\sigma_1, \cdots, \sigma_N$ eine Gleichung der Form

$$0x_2 + 0x_3 + \cdots + 0x_L = K$$

entsteht, die Zahl K auf der rechten Seite notwendigerweise gleich 0 sein muss. In der Tat entsteht durch Kom-

bination der N ersten Gleichungen des ursprünglichen Systems mit denselben Multiplikatoren $\sigma_1, \cdots, \sigma_N$ eine Gleichung der Form

$$\alpha x_1 + 0 x_2 + 0 x_3 + \cdots + 0 x_L = \beta, \tag{4}$$

wo die Zahlen K, α, β durch die Relation $K = \beta - \alpha x_1^*$ verbunden sind (weil das neue System aus dem alten System durch Einsetzung von $x_1 = x_1^*$ hervorgegangen ist), und aus der Gleichung $K = \beta - \alpha x_1^*$ erhellt sofort, dass $K = 0$ ist; denn im Falle $\alpha = 0$ muss auch $\beta = 0$ sein (da das ursprüngliche Gleichungsystem (3) sonst einen offenkundigen Widerspruch aufweisen würde), und im Falle $\alpha \neq 0$ bedeutet die Gleichung (4) ja eine »Isolation« von x_1 aus dem ursprünglichen Gleichungsystem, und es ist alsdann der Wert x_1^* gerade gleich der Konstanten $\frac{\beta}{\alpha}$ gewählt.

Wir können daher das Verfahren fortsetzen und bestimmen nunmehr x_2 aus dem neuen Gleichungsystem in genau derselben Weise wie wir x_1 aus dem ursprünglichen System bestimmt haben, d. h. falls x_2 isoliert werden kann, etwa $x_2 = c_2$, setzen wir $x_2 = x_2^* = c_2$, und falls x_2 nicht isoliert werden kann, geben wir x_2 einen beliebig gewählten Wert x_2^*.

Danach setzen wir $x_2 = x_2^*$ in das Gleichungsystem ein, bestimmen $x_3 = x_3^*$, u. s. w.

Durch abzählbar viele Schritte erhalten wir in dieser Weise ein Wertsystem $(x_1^*, x_2^*, \cdots)$, welches offenbar eine simultane Lösung der unendlich vielen Gleichungen (3) darstellt; in der Tat kommen in jeder dieser Gleichungen, etwa der N^{ten}, nur endlich viele Unbekannte vor, etwa $x_1, \cdots, x_M$, und aus der Bestimmungsweise der Zahlen x^* folgt, dass diese Gleichung nach Einsetzung von $x_1 = x_1^*, \cdots, x_M = x_M^*$ keinen offenkundigen Widerspruch aufweisen darf, also sich auf $0 = 0$ reduzieren muss.

Aus dem Satze 2 folgt, dass die Verhältnisse in bezug auf die in der Einleitungen angegebene Fragestellung ganz anders bei den Gleichungen als bei den Kongruenzen liegen. In der Tat ergibt sich für die Gleichungen der Satz:

Es sei ein System von unendlich vielen Linearformen der Form

$$\varrho_{n,1} x_1 + \varrho_{n,2} x_2 + \cdots + \varrho_{n,q_n} x_{q_n} \qquad (n = 1, 2, \cdots)$$

gegeben, und es bezeichne M_1 *die Menge der Punkte* $(\vartheta_1, \vartheta_2, \cdots)$ *des unendlich-dimensionalen Raumes, für welche die Gleichungen*

$$\varrho_{n,1} x_1 + \varrho_{n,2} x_2 + \cdots + \varrho_{n,q_n} x_{q_n} = \vartheta_n \qquad (n = 1, 2, \cdots)$$

eine simultane Lösung $(x_1^*, x_2^*, \cdots)$ *haben, und* M_2 *die Menge der Punkte* $(\vartheta_1, \vartheta_2, \cdots)$, *für welche bei jedem festen* N *die* N *ersten dieser Gleichungen eine Lösung* $(x_1^{(N)}, x_2^{(N)}, \cdots)$ *besitzen. Dann ist immer* $\mathrm{M}_1 = \mathrm{M}_2$. Für die simultane Lösbarkeit sämtlicher Gleichungen ist also nicht nur notwendig, sondern auch hinreichend, dass jede endliche Anzahl der Gleichungen eine Lösung besitzen.

In der Tat, falls bei jedem N die N ersten Gleichungen eine Lösung haben, kann das Gleichungsystem gewiss keinen offenkundigen Widerspruch aufweisen, und nach dem Satze 2 genügt dies um schliessen zu können, dass eine simultane Lösung der sämtlichen Gleichungen vorhanden ist.[1]

[1] Der Grund, weshalb die Verhältnisse bei den Gleichungen so ganz anders — und zwar viel einfacher — als bei den Kongruenzen liegen, kann darin gesucht werden, dass es bei einem System von Gleichungen ohneweiteres erlaubt ist (wie wir es oben bei der successiven Isolierungen der Variablen getan haben) durch beliebige lineare Kombinationen der gegebenen Gleichungen neue Gleichungen zu bilden, während die Bildung solcher linearer Kombinationen bei den Kongruenzen nur dann erlaubt ist, falls die verwendeten Multiplikatoren sämtlich ganze Zahlen sind.

§ 2.

Lineare Substitutionen in unendlich vielen Variablen.

Es sei ein System von unendlich vielen Gleichungen der Form

$$(5) \qquad y_n = \alpha_{n,1} x_1 + \alpha_{n,2} x_2 + \cdots + \alpha_{n,u_n} x_{u_n} \qquad (n = 1, 2, \cdots)$$

gegeben, wo die Koeffizienten α beliebige reelle Zahlen bedeuten. Durch dieses Gleichungsystem wird jedem gegebenen Punkte $(x_1, x_2, \cdots)$ des unendlich-dimensionalen x-Raumes ein eindeutig bestimmter Punkt $(y_1, y_2, \cdots)$ des unendlich-dimensionalen y-Raumes zugeordnet. Wir nennen das System (5) eine *lineare Substitution*, falls es eine ein-eindeutige Abbildung des x-Raumes auf den y-Raum bewerkstelligt, falls also die Gleichungen (5) bei jedem gegebenen Wertesystem von $y_1, y_2, \cdots$ eine und nur eine Lösung in den Variablen $x_1, x_2, \cdots$ besitzen.

Satz 3. *Damit das System (5) eine lineare Substitution bilde, ist notwendig und hinreichend:*

1. *dass keine Dependenzen zwischen den Linearformen $L_1, L_2, \cdots$ auf den rechten Seiten der Gleichungen (5) bestehen* (d. h. es darf keine lineare Kombination endlich vieler dieser Linearformen mit nicht sämtlich verschwindenden Multiplikatoren geben, welche identisch in den x verschwindet), *und*

2. *dass jede der Variablen x_m durch lineare Kombination endlich vieler der Linearformen $L_1, L_2, \cdots$ isoliert werden kann.*[1]

[1] Diese zwei Bedingungen können offenbar auch als nur eine Bedingung formuliert werden, nämlich: *es soll jede der Variablen x_m in einer und nur einer Weise aus den Linearformen $L_1, L_2, \cdots$ isoliert werden können*, d. h. zu jeder Variablen x_m soll es ein und nur ein System von endlich vielen, $N = N(m)$, reellen Zahlen $\sigma_1, \sigma_2, \cdots, \sigma_N$ mit $\sigma_N \neq 0$ so geben, dass in $\sigma_1 L_1 + \sigma_2 L_2 + \cdots + \sigma_N L_N$ die Variable x_m den Koeffizienten 1 und alle übrigen Variablen x die Koeffizienten 0 besitzen.

Beweis. 1. Wir zeigen zunächst, dass die angegebenen Bedingungen notwendig sind, dass sie also gewiss erfüllt sind, falls (5) eine Substitution bildet. Zunächst ist klar, dass keine Dependenz, etwa $\mu_1 L_1 + \cdots + \mu_N L_N = 0$, vorhanden sein kann; denn hieraus würde folgen, dass das Gleichungsystem (5) für einen Punkt $(y_1, y_2, \cdots)$, welcher nicht die entsprechende Bedingung $\mu_1 y_1 + \cdots + \mu_N y_N = 0$ erfüllte, gewiss keine Lösung in den x haben könnte. Und aus dem Beweise des Satzes 2 in § 1 geht ferner hervor, das jede der Variablen x_m isoliert werden kann. In der Tat, da die Gleichungen (5) bei gegebenen Werten von den y nach Voraussetzung eine Lösung besitzen (und also keinen offenkundigen Widerspruch aufweisen), können wir durch das dort angegebene Verfahren eine Lösung bestimmen; wäre nun eine der Variablen nicht isolierbar, könnten wir ohne Beschränkung der Allgemeinheit annehmen, dass gerade die »erste« Unbekannte x_1 nicht isolierbar wäre (indem wir sonst die Unbekannten einfach umnummerierten), und wir könnten alsdann bei unserem Lösungsverfahren dieser Unbekannten x_1 einen ganz beliebigen Wert x_1^* geben, im Widerspruch zu der Voraussetzung, dass nur eine Lösung existiert.

2. Danach zeigen wir, dass die beiden Bedingungen hinreichend sind. Hierzu schliessen wir zunächst aus der Annahme, dass keine Dependenzen vorhanden sind, dass die Gleichungen (5), wie auch die Werte der y gewählt werden, niemals einen offenkundigen Widerspruch aufweisen können, und daher (nach dem Satze 2) bei beliebig gegebenen $y_1, y_2, \cdots$ mindestens eine Lösung haben. Und dass die Gleichungen nicht mehr als eine Lösung haben können, folgt natürlich sofort daraus, dass jede Variable x isoliert werden kann.

Indem wir an die letzte Bemerkung des obigen Beweises anknüpfen, fügen wir hinzu, dass — falls (5) eine Substitution bildet — die Werte der Variablen x, bei gegebenen Werten der y, durch Isolierung der x bestimmt werden können. Diese Isolation (die, wie wir wissen, in einer und nur einer Weise möglich ist) verläuft aber unabhängig davon, welche Werte den Variablen y auf den linken Seiten von (5) zugeteilt sind, und wir erhalten daher die Lösung in der Form

$$x_n = \beta_{n,1} y_1 + \beta_{n,2} y_2 + \cdots + \beta_{n,\nu_n} y_{\nu_n} \qquad (n = 1, 2, \cdots), \tag{6}$$

wo die β Konstanten sind, die nur von den Konstanten α in (5) abhängen. Dieses neue System (6) bestimmt offenbar dieselbe ein-eindeutige Abbildung des x-Raumes auf den y-Raum wie (5), und bildet daher ebenfalls eine Substitution, die wir als die zu (5) inverse Substitution bezeichnen. Natürlich ist umgekehrt (5) die inverse Substitution zu (6).

Beispiel. Als einfachstes Beispiel einer linearen Substitution nennen wir ein System von Gleichungen der Form

$$\left\{\begin{array}{l} y_1 = \alpha_{1,1} x_1 \\ y_2 = \alpha_{2,1} x_1 + \alpha_{2,2} x_2 \\ \cdots\cdots\cdots\cdots \\ y_n = \alpha_{n,1} x_1 + \alpha_{n,2} x_2 + \cdots + \alpha_{n,n} x_n \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{array}\right. \qquad (\alpha_{n,n} \neq 0 \text{ für alle } n). \tag{7}$$

Dass dieses System (7) eine Substitution bildet, sieht man sofort sowohl daraus, dass die in der Definition geforderte Ein-eindeutigkeit der Abbildung vorhanden ist, als auch daraus, dass die im Satze 3 angegebenen Bedingungen des Nicht-Vorhandenseins von Dependenzen und der Isolierbarkeit der x erfüllt sind. Wir

finden hier sofort durch successive Bestimmung von $x_1, x_2, \cdots$ die inverse Substitution, welche dieselbe Form

$$(8) \quad \begin{cases} x_1 = \beta_{1,1}\, y_1 \\ x_2 = \beta_{2,1}\, y_1 + \beta_{2,2}\, y_2 \\ \dots\dots\dots\dots\dots \\ x_n = \beta_{n,1}\, y_1 + \beta_{n,2}\, y_2 + \cdots + \beta_{n,n}\, y_n \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \end{cases} \qquad \left(\beta_{n,n} = \frac{1}{\alpha_{n,n}}\right)$$

erhält.

Wir schliessen diesen Paragraphen mit einigen für das Folgende nützlichen Bemerkungen:

Bemerkung I. (»Verlängerung« eines Systems). Es sei $L_1, L_2, \cdots$ eine Folge von unendlich (oder vielleicht nur endlich) vielen Linearformen in den unendlich vielen Variablen $x_1, x_2, \cdots$, wo jede der Linearformen wie immer nur endlich viele der Variablen x enthält, und es sei vorausgesetzt, dass *keine Dependenzen zwischen den Linearformen bestehen.* Daraus folgt natürlich nicht, dass das Gleichungsystem

$$(9) \qquad y_1 = L_1, \; y_2 = L_2, \cdots,$$

welches jeden Punkt des x-Raumes auf einen Punkt des y-Raumes abbildet, eine Substitution bildet; wir können wohl aus dem Satze 2 des § 1 folgern, dass die Gleichungen (9) bei beliebig gegebenen y mindestens eine Lösung in den x besitzen, aber im Allgemeinen wird es mehrere (unendlich viele) solche Lösungen geben, weil wir nicht vorausgesetzt haben, dass jede der Variablen x aus den Linearformen L isoliert werden kann.

Wir werden aber zeigen, dass das System (9) (falls es nicht zufällig schon eine Substitution bildet) immer in einfachster Weise *zu einer Substitution »verlängert« werden kann,* indem der Raum $y_1, y_2, \cdots$ durch Hinzufügung neuer Va-

riabeln $\eta_1, \eta_2, \cdots$ erweitert wird, und dass diese »Verlängerung« einfach so vorgenommen werden kann, *dass dem System (9) eine (endliche oder unendliche) Folge von Gleichungen der Form*

$$\eta_\nu = x_{n_\nu} \qquad (\nu = 1, 2, \cdots)$$

hinzugefügt wird.[1] Hierzu gehen wir folgendermassen vor: Es sei x_{n_1} die erste Variable x (d. h. die mit dem kleinsten Index), welche aus den gegebenen Linearformen L nicht isoliert werden kann. Wir fügen dann dem Gleichungsystem (9) die neue Gleichung $\eta_1 = x_{n_1}$ zu, und behaupten, dass in dem so erweiterten System von Linearformen (welches also aus den L und der neuen Form $L'_1 = x_{n_1}$ besteht) ebenfalls keine Dependenzen vorkommen. Denn, falls die lineare Kombination

$$\mu_1 L_1 + \mu_2 L_2 + \cdots + \mu_N L_N + \mu' x_{n_1}$$

identisch in den x verschwindet, muss erstens $\mu' = 0$ sein (weil x_{n_1} sonst aus den L isolierbar wäre), und danach folgt weiter, dass auch $\mu_1, \mu_2, \cdots, \mu_N$ gleich 0 sein müssen (weil das L-System nach Voraussetzung keine Dependenzen enthält). Wir haben nun einfach in dieser Weise fortzusetzen, d. h. wir fügen nun unserem neuen Gleichungsystem

$$y_1 = L_1, \; y_2 = L_2, \cdots; \quad \eta_1 = x_{n_1}$$

die Gleichung $\eta_2 = x_{n_2}$ hinzu, wobei x_{n_2} $(n_2 > n_1)$ die erste der Variablen x bezeichnet, welche aus den rechten Seiten der Gleichungen nicht isoliert werden kann; bei dieser Hinzufügung können, nach dem obigen, gewiss keine Dependenzen auftauchen. Durch unendlich (oder vielleicht nur endlich)

[1] Hierbei denken wir uns natürlich, dass die neuen Gleichungen $\eta_\nu = x_{n_\nu}$ dem ursprünglichen Gleichungsystem (9) so eingefügt werden, dass das Gesamtsystem in der Form einer (durch die Zahlen 1, 2, 3, $\cdots$ numerierten) Folge von Gleichungen erscheint.

viele Schritte gelangen wir dann zu einem Gleichungsystem, das offenbar eine Substitution zwischen dem x-Raum und dem y-η-Raum bildet; denn einerseits besteht keine Dependenz zwischen den Linearformen $L_1, L_2, \cdots, x_{n_1}, x_{n_2}, \cdots$, und andererseits kann jede der Variablen x aus diesen Linearformen isoliert werden.

Bemerkung II. (»Verkürzung« eines Systems). Es sei

$$(5) \qquad y_n = \alpha_{n,1} x_1 + \alpha_{n,2} x_2 + \cdots + \alpha_{n,u_n} x_{u_n} \qquad (n = 1, 2, \cdots)$$

eine Substitution, welche den unendlich-dimensionalen x-Raum auf den unendlich-dimensionalen y-Raum abbildet, und es sei $n_1, n_2, \cdots$ eine beliebig gegebene Folge von (endlich oder unendlich vielen) positiven ganzen Zahlen, welche nur nicht gerade die gesamte Zahlenreihe $1, 2, \cdots$ ausmachen. Wir wollen in (5) *die Variablen $y_{n_1}, y_{n_2}, \cdots$ fortlassen und doch immer eine Substitution zurück behalten*, und zwar werden wir versuchen, dies einfach dadurch zu erreichen, dass *eine gewisse Folge $x_{m_1}, x_{m_2}, \cdots$ von den Variablen x ebenfalls aus (5) gestrichen wird.*

Hierzu streichen wir zunächst aus dem Gleichungsystem (5) die Gleichungen mit den Nummern $n_1, n_2, \cdots$, so dass die erwähnten Variablen $y_{n_1}, y_{n_2}, \cdots$ nicht mehr vorkommen. In den zurück gebliebenen Gleichungen sind die Linearformen $l_1, l_2, \cdots$ auf den rechten Seiten gewiss dependenzfrei (weil ja sogar das Gesamtsystem $L_1, L_2, \cdots$ dependenzfrei ist). Dagegen wird nicht mehr jede der Variablen x isolierbar sein.[1] Es sei nun x_{m_1} die erste der Variablen x,

[1] In der Tat muss für mindestens eine der Variablen x, welche in der Linearform L_{n_1} vorkommt, gelten, dass bei ihrer (nur in einer Weise möglichen) Isolierung aus den Linearformen $L_1, L_2, \cdots$, diese Linearform L_{n_1} tatsächlich verwendet wird, weil sonst L_{n_1} durch eine lineare Kombination von anderen der Linearformen L ausgedrückt werden könnte (und das L-System also nicht dependenzfrei wäre).

welche nicht aus den zurückgebliebenen Linearformen $l_1, l_2, \cdots$ isoliert werden kann. Wir streichen dann einfach x_{m_1} aus allen diesen Linearformen $l_1, l_2, \cdots$ und bezeichnen die hierdurch entstandenen Linearformen in den übrigen Variablen x mit $l'_1, l'_2, \cdots$. Es ist klar, einerseits, dass jedes x, welches aus den Linearformen l_1, l_2 isoliert werden kann, ebenfalls aus den neuen Linearformen $l'_1, l'_2 \cdots$ isoliert werden kann, und zwar durch dieselbe Kombination, d. h. unter Benutzung der entsprechenden Linearformen und derselben Multiplikatoren (weil die Formen l' aus den Formen l einfach durch Weglassung der Glieder mit x_{m_1} entstanden sind), und andererseits, dass das neue System l'_1, $l'_2, \cdots$ ebenfalls keine Dependenzen enthält; in der Tat führt die Annahme der Existenz einer solchen Dependenz, etwa $\mu_1 l'_1 + \cdots + \mu_N l'_N = 0$, sofort zu einem Widerspruch, weil alsdann $\mu_1 l_1 + \cdots + \mu_N l_N = c x_{m_1}$ sein müsste, und dies offenbar unmöglich ist (denn $c = 0$ würde bedeuten, dass eine Dependenz zwischen den l stattfände, und $c \neq 0$, dass x_{m_1} aus den l isolierbar wäre). Wir setzen nun in dieser Weise fort, d. h. streichen aus den Linearformen l' die erste Variable x_{m_2} $(m_2 > m_1)$, welche nicht aus den l' isolierbar ist, u. s. w. Durch unendlich (oder vielleicht nur endlich) viele Schritte gelangen wir offenbar zu einer Substitution zwischen den zurückgeblieben x und den zurückgebliebenen y; denn zwischen den endgültig erhaltenen Linearformen auf den rechten Seiten bestehen keine Dependenzen, und jede der zurückgebliebenen x ist aus diesen Linearformen isolierbar.

Wir fügen noch hinzu, dass falls die gegebene Folge $n_1, n_2, \cdots$ aus sämtlichen positiven ganzen Zahlen bis auf endlich viele besteht, also nur endlich viele y zurückgelassen werden, auch nur endlich viele x (und zwar

genau ebensoviele) zurückbleiben können; wir erhalten also eine »gewöhnliche« Substitution zwischen zwei Systemen von endlich vielen Variablen.

Bemerkung III. Falls in einer Substitution (5) die Koeffizienten α alle rationale Zahlen sind, werden die Koeffizienten β der inversen Substitution ebenfalls sämtlich rational sein. In der Tat folgt aus einem Satz der elementaren Algebra[1], dass eine Variable x_m, welche aus einem System von Linearformen mit rationalen Koeffizienten isoliert werden kann, gewiss auch mit Hülfe lauter rationaler Multiplikatoren isoliert werden kann, und bei den Linearformen einer Substitution, wo die Isolierung der Variablen nur in einer Weise möglich ist, müssen also gewiss die zu verwendenden Multiplikatoren, d. h. gerade die Koeffizienten β der inversen Substitution, alle rational ausfallen.

Ferner bemerken wir, dass bei der in Bemerkung I bezw. II betrachteten »Verlängerung« bezw. »Verkürzung« eines Systems von Linearformen, die Koeffizienten des neuen Systems alle rational werden, falls die Koeffizienten des ursprünglichen Systems rational waren.

Im Folgenden werden wir überall, wo von einer linearen Substitution oder einem System von Linearformen die Rede ist, stillschweigend voraussetzen, dass die Koeffizienten sämtlich rationale Zahlen sind.

[1] Nämlich aus dem Satze, dass, wenn ein System von endlich vielen linearen Gleichungen mit endlich vielen Unbekannten, in welchem alle auftretenden Konstanten rationale Werte haben, überhaupt lösbar ist, es gewiss auch eine Lösung in lauter rationalen Zahlen hat.

§ 3.
Eine einfache Reduktion des gegebenen Systems von Linearformen.

Wir nennen zwei Systeme von Linearformen

$$(1) \qquad r_{n,1}x_1 + r_{n,2}x_2 + \cdots + r_{n,q_n}x_{q_n} \qquad (n = 1, 2, \cdots)$$

und

$$(10) \qquad s_{n,1}y_1 + s_{n,2}y_2 + \cdots + s_{n,p_n}y_{p_n} \qquad (n = 1, 2, \cdots)$$

unter einander *aequivalent*, wenn das erste System aus dem zweiten durch eine lineare Substitution (5) (und damit das zweite aus dem ersten durch die inverse Substitution (6)) hervorgeht. Es ist klar, dass die Anwendung einer linearen Substitution die beiden in der Einleitung definierten Punktmengen Π_1 und Π_2 unberührt lässt, d. h. *dass zu zwei aequivalenten Systemen sowohl dieselbe Menge Π_1 als auch dieselbe Menge Π_2 gehören.* In der Tat, falls $(x_1, x_2, \cdots)$ eine Lösung der sämtlichen (bezw. der N ersten) der Kongruenzen

$$r_{n,1}x_1 + r_{n,2}x_2 + \cdots + r_{n,q_n}x_{q_n} \equiv \theta_n \pmod{1} \quad (n = 1, 2, \cdots)$$

ist, wird der (durch die Substitution bestimmte) Bildpunkt $(y_1, y_2, \cdots)$ eine Lösung der sämtlichen (bezw. der N ersten) Kongruenzen

$$s_{n,1}y_1 + s_{n,2}y_2 + \cdots + s_{n,p_n}y_{p_n} \equiv \theta_n \pmod{1} \quad (n = 1, 2, \cdots)$$

sein.

Wir werden ein System (1) »*ganzartig*« nennen, falls *es mit einem System (10) mit lauter ganzen Koeffizienten s aequivalent ist.* Da die Zusammensetzung zweier Substitutionen wieder eine Substitution ergibt, wird, falls von zwei aequivalenten Systemen das eine ganzartig ist, das andere ebenfalls ganzartig sein. Wie in der Einleitung angegeben, ist

das Ziel der vorliegenden Abhandlung zu beweisen, dass es für das Zusammenfallen der beiden zu einem System (1) gehörigen Mengen Π_1 und Π_2 notwendig und hinreichend ist, dass das System ganzartig ist. Aus dem Obigen folgt unmittelbar, dass es beim Beweise dieses Satz ohne weiteres erlaubt ist, statt des gegebenen Systems (1), ein beliebiges anderes mit (1) aequivalentes System zu betrachten; denn der Übergang zu einem aequivalenten System ändert ja weder die Mengen Π_1 und Π_2 noch die Ganzartigkeit oder Nicht-Ganzartigkeit des Systems. Mit Hilfe dieser Bemerkung werden wir in diesem Paragraphen zeigen, dass wir uns im folgenden *auf die Betrachtung solcher Systeme beschränken können, in welchen jede der Variablen (in mindestens einer Weise) isoliert werden kann.* Wir gehen hierbei schrittweise vor, indem wir zunächst von dem gegebenen System (1) zu einem mit (1) äquivalenten System übergehen, in welchem jede der Variablen welche tatsächlich vorkommt (d. h. nicht überall den Koefficienten 0 hat) isoliert werden kann, und dann nachher das so gewonnene System in ein System der gewünschten Art überführen, in welchem überhaupt jede der Variablen isoliert werden kann.

1°. Wir bezeichnen zur Abkürzung die m^{te} Linearform in (1) mit L_m, und setzen $L_{m_1} = y_1$, wo $m_1 = 1$ ist. Danach ersetzen wir jede der Linearformen $L_{m_1+1}, L_{m_1+2}, \cdots$, welche in der Form $R_1 L_{m_1}$ mit einem konstanten (rationalen) Faktor R_1 geschrieben werden kann, durch $R_1 y_1$. Wir setzen alsdann $L_{m_2} = y_2$, wo L_{m_2} die erste der Linearformen L_{m_1+1}, $L_{m_1+2}, \cdots$ bezeichnet, welche nicht diese Form $R_1 L_{m_1}$ hat, und ersetzen jede dieser Linearformen, welche in der Form $R_1 L_{m_1} + R_2 L_{m_2}$ mit konstanten (rationalen) R_1, R_2 ($R_2 \neq 0$) geschrieben werden kann, durch $R_1 y_1 + R_2 y_2$. Danach setzen wir $L_{m_3} = y_3$, wo L_{m_3} die erste der Linearformen

L_{m_2+1}, $L_{m_2+2}, \cdots$ bedeutet, welche nicht diese Form $R_1 L_{m_1} + R_2 L_{m_2}$ hat, und ersetzen jede dieser Linearformen, welche in der Form $R_1 L_{m_1} + R_2 L_{m_2} + R_3 L_{m_3}$ (mit $R_3 \neq 0$) geschrieben werden kann, durch $R_1 y_1 + R_2 y_2 + R_3 y_3$, u. s. w. Nach unendlich (oder vielleicht nur endlich) vielen Schritten,

$$(11) \qquad L_{m_1} = y_1, \; L_{m_2} = y_2, \; L_{m_3} = y_3, \cdots,$$

erhalten wir somit aus dem System (1) ein neues System von Linearformen in den Variablen $y_1, y_2, \cdots$, welches wir mit

$$(12) \qquad t_{n,1} y_1 + t_{n,2} y_2 + \cdots + t_{n,p_n} y_{p_n} \qquad (n = 1, 2, \cdots)$$

bezeichnen. *In diesem System (12) kann offenbar jede der Variablen y isoliert werden*; es kommen sogar unter den Linearformen (12) die einzelnen Variablen $y_1, y_2, \cdots$ isoliert vor.

Wir können aber nicht unmittelbar behaupten, dass dieses System (12) mit dem Ausgangsystem (1) aequivalent ist; denn es brauchen ja die Gleichungen (11) keine Substitution zu bilden; in der Tat folgt aus der Wahl der Linearformen $L_{m_1}, L_{m_2}, \cdots$ nur, dass sie keine Dependenzen aufweisen, aber nicht, dass jede der Variablen x aus ihnen isoliert werden kann. Über diese anscheinende Schwierigkeit können wir aber sofort mit Hülfe der Bemerkung I in § 2 hinwegkommen; in der Tat können wir nach dieser Bemerkung das Gleichungsystem (11) durch Hinzufügung neuer Gleichungen der Form

$$x_{n_1} = \eta_1, \quad x_{n_2} = \eta_2, \cdots$$

zu einer Substitution »verlängern«, welche den Raum der Variablen x auf den durch die Variablen $y_1, y_2, \cdots$ und $\eta_1, \eta_2, \cdots$ gebildeten Raum abbildet, und durch Anwendung dieser Substitution auf das System (1) ergibt sich

ja gerade das obige System (12), indem die neuen Variablen $\eta_1, \eta_2, \cdots$ gar nicht zum Vorschein kommen.

2°. Man könnte versucht sein zu glauben, dass wir jetzt am Ziele wären. In der Tat, falls das neue System (12) mit dem gegebenen System (1) aequivalent ist, können wir ja ebensogut das System (12) wie das System (1) bei der Behandlung unserer Aufgabe zu Grunde legen. Man muss aber bedenken, dass das System (12) nur dann mit dem System (1) aequivalent ist, wenn (12) als ein System in den Variablen $y_1, y_2, \cdots$ und $\eta_1, \eta_2, \cdots$ betrachtet wird, und dass es ja nur die y sind, welche aus dem System isoliert werden können, und nicht auch die η (welche überhaupt nicht in den Linearformen auftreten). Es muss daher noch bewiesen werden, dass es für unsere Aufgabe gleichgültig ist, ob wir das System (12) als ein System in den Variablen y und η, oder als ein System in den Variablen y allein betrachten. Es ist unmittelbar klar, das die beiden Mengen Π_1 und Π_2 davon unabhängig sind, welchen von diesen beiden Gesichtspunkten wir anlegen, und es handelt sich daher nur darum zu zeigen, dass auch die eventuelle Ganzartigkeit des Systems (12) nicht davon abhängt, ob wir die y allein, oder die y und η als die Variablen betrachten. Hierzu bemerken wir zunächst, dass, falls das System als Funktion der y allein betrachtet ganzartig ist — also durch eine Substitution $(y_1, y_2, \cdots) \rightarrow (z_1, z_2, \cdots)$ in ein System in $z_1, z_2, \cdots$ mit ganzen Koeffizienten übergeführt werden kann — es eo ipso auch als Funktion von y und η betrachtet ganzartig ist; wir haben ja nur der benutzten Substitution $(y_1, y_2, \cdots) \rightarrow (z_1, z_2, \cdots)$ die Gleichungen $\eta_1 = \xi_1, \eta_2 = \xi_2, \cdots$, wo die ξ neue Variablen bedeuten, hinzuzufügen, um eine Substitution $(y_1, y_2, \cdots; \eta_1, \eta_2, \cdots) \rightarrow$

$(z_1, z_2, \cdots; \xi_1, \xi_2, \cdots)$ zu erhalten, welche unser System (12) in ein System mit lauter ganzen Koeffizienten in den Variablen $z_1, z_2, \cdots; \xi_1, \xi_2, \cdots$ überführt. Wir haben danach zu beweisen, dass, falls das System (12) als Funktion der y und η betrachtet ganzartig ist — also durch eine Substitution, welche den y-η-Raum auf einen z-Raum abbildet, in ein System in $z_1, z_2, \cdots$ mit ganzen Koeffizienten übergeführt werden kann —, das System (12) auch als Funktion von den y allein betrachtet ganzartig ist. Dies folgt aber sofort aus der Bemerkung II in § 2 über die »Verkürzung« einer Substitution. In der Tat können wir nach dieser Bemerkung aus der benutzten Substitution $(y_1, y_2, \cdots; \eta_1, \eta_2, \cdots) \rightarrow (z_1, z_2, \cdots)$ durch einfache Streichung der Variablen $\eta_1, \eta_2, \cdots$ und gewisser Variablen $z_{n_1}, z_{n_2}, \cdots$ eine neue Substitution erhalten, welche den y-Raum auf einen »Unterraum« des z-Raumes abbildet, und durch diese Substitution geht gewiss das System (12) in ein System mit lauter ganzen Koeffizienten über, nämlich in dasjenige System, welches aus dem obigen System in den Variabeln $z_1, z_2, \cdots$ dadurch hervorgeht, dass die Variabeln $z_{n_1}, z_{n_2}, \cdots$ überall gestrichen werden.

3°. Wir können somit statt des gegebenen Systems (1) ebensogut das neue System

$$(12) \qquad t_{n,1}\, y_1 + t_{n,2}\, y_2 + \cdots + t_{n,p_n}\, y_{p_n} \qquad (n = 1, 2 \cdots)$$

(als Funktion der Variablen $y_1, y_2, \cdots$ allein betrachtet) zu Grunde legen, aus welchem jede der Variablen $y_1, y_2, \cdots$ isoliert werden kann. Hierbei ist aber noch zu bedenken, dass wir jetzt nach dieser Reduktion zwei verschiedene Fälle unterscheiden müssen, nämlich denjenigen, wo die Variablen $y_1, y_2, \cdots$ eine unendliche Folge bilden, und denjenigen, wo es nur endlich viele Variabeln, etwa

$y_1, \cdots, y_M$ gibt. Um Wiederholungen zu vermeiden, ziehen wir aber vor, statt den letzteren (wesentlich einfacheren) Fall für sich zu behandeln, lieber gleich zu zeigen, dass dieser Fall auf den von unendlich vielen Variabeln direkt zurückgeführt werden kann. In der Tat brauchen wir nur dem System (12) in den Variabeln $y_1, \cdots, y_M$ die (unendlich vielen) neuen Linearformen

$$L' = y_{M+1}, \quad L'' = y_{M+2}, \quad L''' = y_{M+3}, \cdots$$

hinzuzufügen, um ein System in den unendlich vielen Variabeln $y_1, y_2, \cdots$ zu erhalten, welches für unsere Aufgabe mit (12) ganz gleichwertig ist. Denn falls die Gleichung $\Pi_1 = \Pi_2$ für das System (12) besteht, wird sie offenbar auch für das erweiterte System gelten (und umgekehrt). Und ferner werden die beiden Systeme auch gleichzeitig ganzartig sein; in der Tat, falls das System (12) in den Variabeln $y_1, \cdots, y_M$ ganzartig ist, wird das erweiterte System eo ipso auch ganzartig sein (wie durch Erweiterung der benutzten Substitution $(y_1, \cdots, y_M) \rightarrow (z_1, \cdots, z_M)$ mit den Gleichungen $y_{M+1} = z_{M+1}, y_{M+2} = z_{M+2}, \cdots$ unmittelbar hervorgeht), und umgekehrt, falls das erweiterte System ganzartig ist, wird auch das System (12) ganzartig sein (wie durch »Verkürzung« der benutzten Substitution $(y_1, y_2, \cdots) \rightarrow (z_1, z_2, \cdots)$ durch Streichung der Variablen $y_{M+1}, y_{M+2}, \cdots$ und entsprechender der Variablen z sofort zu sehen ist).

Wir können also im folgenden ohne Beschränkung der Allgemeinheit annehmen, *dass das gegebene System (1) tatsächlich alle die unendlich vielen Variabeln $x_1, x_2, \cdots$ enthält, und dass jede dieser Variablen aus dem System isoliert werden kann.* Ein solches System werden wir als ein »*reduziertes*« System in den unendlich vielen Variabeln $x_1, x_2, \cdots$ bezeichnen.

§ 4.
Notwendige und hinreichende Bedingungen für die „Ganzartigkeit" eines (reduzierten) Systems von Linearformen.

Die gesuchten Kriterien der Ganzartigkeit eines reduzierten Systems in unendlich vielen Variablen

$$(1) \qquad r_{n,1}x_1 + r_{n,2}x_2 + \cdots + r_{n,q_n}x_{q_n} \qquad (n = 1, 2, \cdots)$$

beruhen auf die Betrachtung der unendlich vielen »*Nullkongruenzen*«

$$(13) \; r_{n,1}x_1 + r_{n,2}x_2 + \cdots + r_{n,q_n}x_{q_n} \equiv 0 \,(\mathrm{mod.} 1) \; (n = 1,2,\cdots),$$

welche aus den unendlich vielen Linearformen (1) entstehen, wenn jede dieser Formen $\equiv 0$ (mod. 1) gesetzt wird. Es bezeichne Γ die Punktmenge des unendlich-dimensionalen Raumes, welche aus den sämtlichen simultanen Lösungen $(x_1, x_2, \cdots)$ dieser unendlich vielen Nullkongruenzen besteht; diese Punktmenge Γ ist gewiss nicht leer, da sie jedenfalls den Anfangspunkt des Raumes $(0, 0, 0, \cdots)$ enthält. Ferner bezeichne Γ_m bei jedem festen $m = 1, 2, \cdots$ die Projektion der Punktmenge Γ auf den m-dimensionalen Unterraum $x_1, \cdots, x_m$, d. h. es gehöre zu Γ_m jeder Punkt $(x_1, \cdots, x_m)$, welcher durch Hinzufügung passend gewählter unendlich vieler Koordinaten $x_{m+1}, x_{m+2}, \cdots$ zu einem Punkte der Menge Γ »ergänzt« werden kann. Es ist klar, dass Γ_m auch die Projektion jeder der Punktmengen $\Gamma_{m+1}, \Gamma_{m+2}, \cdots$ auf den m-dimensionalen Unterraum darstellt.

Wir beweisen zunächst den

Satz 4. *Es ist die Menge Γ eindeutig bestimmt, wenn man*

ihre sämtlichen Projektionen $\Gamma_1, \Gamma_2, \cdots$ kennt[1], *und zwar ist Γ die »grösste« Menge, welche die Mengen Γ_m $(m = 1, 2, \cdots)$ als Projektionen besitzt; d. h. damit ein Punkt $(x_1, x_2, \cdots)$ zu Γ gehöre, ist nicht nur notwendig, sondern auch hinreichend, dass bei jedem m der »abgeschnittene« Punkt $(x_1, \cdots, x_m)$ in Γ_m liegt.*

Beweis. In der Tat, es sei $(x_1, x_2, \cdots)$ ein beliebiger solcher Punkt, dass bei jedem m der abgeschnittene Punkt $(x_1, \cdots, x_m)$ in Γ_m liegt. Wir haben zu beweisen, dass dieser Punkt $(x_1, x_2, \cdots)$ zu Γ gehört, also dass er eine (simultane) Lösung der Kongruenzen (13) darstellt, d. h. bei beliebig gewähltem N die N^{te} Kongruenz befriedigt. Es bezeichne hierzu M den grössten Index eines x, welches in dieser N^{ten} Kongruenz vorkommt. Nach Voraussetzung gehört der abgeschnittene Punkt $(x_1, \cdots, x_M)$ zu Γ_M, d. h. er kann zu einem Punkte $(x_1, \cdots, x_M, \overset{*}{x}_{M+1}, \overset{*}{x}_{M+2}, \cdots)$ ergänzt werden, welcher zu Γ gehört und somit die sämtlichen Kongruenzen (13), also auch die N^{te} Kongruenz erfüllt. Hieraus folgt aber, dass auch der ursprüngliche Punkt $(x_1, x_2, \cdots, x_M, x_{M+1}, \cdots)$ die N^{te} Kongruenz erfüllt, weil ja bei einer Lösung dieser Kongruenz ganz gleichgültig ist, welche Werte die Koordinaten nach der M^{ten} Stelle besitzen.

Wir werden nunmehr die Struktur der Punktmengen Γ_m näher untersuchen. Hierzu beweisen wir zunächst den

Satz 5. *Es bildet bei jedem $m = 1, 2, \cdots$ die Punktmenge*

[1] Im Allgemeinen ist eine Punktmenge Ω des unendlich-dimensionalen Raumes nicht eindeutig bestimmt, wenn man bei jedem $m = 1, 2, \cdots$ ihre Projektion Ω_m auf den m-dimensionalen Unterraum $x_1, \cdots, x_m$ kennt. So haben z. B. die beiden Punktmengen Ω' und Ω'', wo Ω' aus den sämtlichen Punkten des unendlich-dimensionalen Raumes besteht, und Ω'' aus allen solchen Punkten, unter deren Koordinaten unendlich viele Nullen vorkommen, bei jedem m dieselbe Projektion $\Omega'_m = \Omega''_m$, nämlich beide Mal den ganzen m-dimensionalen Raum.

Γ_m ein »Gitter«, d. h. es gibt eine ganze Zahl p ($0 \leqq p \leqq m$) und p linear unabhängige Punkte

$$(x'_1, \cdots, x'_m),\ (x''_1, \cdots, x''_m),\ \cdots,\ (x^{(p)}_1, \cdots x^{(p)}_m)$$

derart, dass die Punktmenge Γ_m gerade aus den ganzzahligen Kombinationen dieser Punkte besteht, d. h. aus allen Punkten der Form

$$(x_1, \cdots, x_m) =$$
$$n_1(x'_1, \cdots, x'_m) + n_2(x''_1, \cdots, x''_m) + \cdots + n_p(x^{(p)}_1, \cdots, x^{(p)}_m),$$

wo $n_1, n_2, \cdots, n_p$ unabhängig von einander alle ganzen Zahlen durchlaufen[1]. Hierbei heisst p die Dimension des Gitters.

Beweis. Damit eine gegebene Punktmenge des m-dimensionalen Raumes ein Gitter bilde, ist bekanntlich nicht nur notwendig, sondern auch hinreichend, 1) dass, falls $(x'_1, \cdots, x'_m)$ und $(x''_1, \cdots, x''_m)$ zwei beliebige (verschiedene oder gleiche) Punkte der Menge bedeuten, der »Differenzpunkt« $(x'_1 - x''_1, \cdots, x'_m - x''_m)$ ebenfalls zur Menge gehört, und 2) dass der Anfangspunkt $(0, 0, \cdots, 0)$ kein Häufungspunkt der Menge ist. Diese beiden Forderungen sind aber gewiss für unsere Menge Γ_m erfüllt. In der Tat:

1. Falls $(x'_1, \cdots, x'_m)$ und $(x''_1, \cdots, x''_m)$ zwei Punkte der Menge Γ_m sind, können sie durch Hinzufügung passend gewählter Werte der Koordinaten $x_{m+1}, \cdots$, zu zwei Punkten

$$(x'_1, \cdots, x'_m, x'_{m+1}, \cdots) \quad \text{und} \quad (x''_1, \cdots, x''_m, x''_{m+1}, \cdots)$$

der Menge Γ ergänzt werden. Aus der Definition der Menge Γ (als der Menge der Lösungen der Nullkongruenzen (13)) folgt aber, dass der Differenzpunkt

$$(x'_1 - x''_1, \cdots, x'_m - x''_m, x'_{m+1} - x''_{m+1}, \cdots)$$

[1] Für $p = 0$ besteht die Punktmenge nur aus dem einzigen Punkte $(x_1, \cdots, x_m) = (0, \cdots, 0)$.

dann ebenfalls zu Γ gehört; die Projektion dieses Punktes auf den m-dimensionalen Unterraum, d. h. der Punkt $(x'_1 - x''_1, \cdots, x'_m - x''_m)$, ist also ein Punkt von Γ_m.

2. Dass der Anfangspunkt $(0, 0, \cdots, 0)$ kein Häufungspunkt von Γ_m ist, ergibt sich sofort daraus, dass das System (1) reduziert ist, und also jede der Variablen x_l aus den gegebenen Linearformen (1) isoliert werden kann. In der Tat, falls G_l den Hauptnenner der (endlich vielen) rationalen Multiplikatoren bezeichnet, welche bei einer Isolierung von x_l verwendet werden, können wir durch lineare Kombination endlich vieler der Nullkongruenzen (13) mit ganzzahligen Multiplikatoren die neue Nullkongruenz

$$G_l x_l \equiv 0 \pmod{1}$$

ableiten, so dass in jeder Lösung $(x_1, x_2, \cdots)$ des Kongruenzensystems (13), d. h. in jedem Punkte $(x_1, x_2, \cdots)$ der Menge Γ, auf der l^{ten} Stelle eine rationale Zahl der Form $x_l = \frac{n}{G_l}$ (n ganz) stehen muss, womit natürlich gezeigt ist, dass der Anfangspunkt des m-dimensionalen Raumes kein Häufungspunkt von Γ_m ist.

Wir sagen von einem Gitter des m-dimensionalen Raumes, dass es ein echtes Gitter bildet, falls die Dimension p des Gitters mit der Dimension m des Raumes übereinstimmt. Damit ein Gitter (wie unser Γ_m), welches aus lauter Punkten mit rationalen Koordinaten besteht, ein echtes Gitter sei, ist bekanntlich nicht nur hinreichend, sondern auch notwendig, dass jede der m Koordinatenachsen mindestens einen vom Anfangspunkte verschiedenen Punkt des Gitters enthalte, also dass es m von 0 verschiedene Zahlen $\gamma_1, \gamma_2, \cdots, \gamma_m$ derart gibt, dass die m Punkte

$$(\gamma_1, 0, 0, \cdots, 0), \ (0, \gamma_2, 0, \cdots, 0), \cdots, \ (0, 0, \cdots, 0, \gamma_m)$$

alle zum Gitter gehören.

Wir gelangen nunmehr zu dem wichtigen

Satz 6. *Damit das System (1) ganzartig sei, ist notwendig und hinreichend, dass bei jedem* $m = 1, 2, \cdots$ *die Projektion* Γ_m *der (aus den simultanen Lösungen der Nullkongruenzen (13) bestehenden) Menge* Γ *ein echtes Gitter des m-dimensionalen Raumes bildet.*

Beweis. 1. Wir zeigen zunächst, dass, falls das System

$$(1) \qquad r_{n,1}x_1 + r_{n,2}x_2 + \cdots + r_{n,q_n}x_{q_n} \qquad (n = 1, 2, \cdots)$$

ganzartig ist, d. h. durch eine lineare Substitution

$$(6) \qquad x_n = \beta_{n,1}y_1 + \beta_{n,2}y_2 + \cdots + \beta_{n,v_n}y_{v_n} \qquad (n = 1, 2, \cdots)$$

in ein neues System

$$(14) \qquad g_{n,1}y_1 + g_{n,2}y_2 + \cdots + g_{n,p_n}y_{p_n} \qquad (n = 1, 2, \cdots)$$

mit lauter ganzen Koeffizienten g übergeführt werden kann, die Menge Γ_m bei jedem m ein echtes Gitter bildet, dass also auf jeder der Koordinatenachsen des m-dimensionalen Raumes ein vom Anfangspunkte verschiedener Punkt existiert, welcher zu Γ_m gehört. Hierzu betrachten wir gleichzeitig mit den Nullkongruenzen

$$(13) \quad r_{n,1}x_1 + r_{n,2}x_2 + \cdots + r_{n,q_n}x_{q_n} \equiv 0 \pmod{1} \quad (n = 1, 2, \cdots),$$

deren simultane Lösungen $(x_1, x_2, \cdots)$ die Punktmenge Γ bilden, das aequivalente System von Nullkongruenzen

$$(15) \quad g_{n,1}y_1 + g_{n,2}y_2 + \cdots + g_{n,p_n}y_{p_n} \equiv 0 \pmod{1} \quad (n = 1, 2, \cdots).$$

Da die Linearformen auf den linken Seiten von (15) aus den Linearformen auf den linken Seiten von (13) durch die lineare Substitution (6) hervorgegangen sind, ist

unmittelbar klar, dass die aus den simultanen Lösungen $(y_1, y_2, \cdots)$ dieses neuen Kongruenzensystems (15) bestehende Menge Γ^* gerade die mit Hülfe der Substitution (6) gebildete Bildmenge der Menge Γ ist. Zu dieser Menge Γ^* des unendlich-dimensionalen y-Raumes gehört aber gewiss — weil die Koeffizienten g der Kongruenzen (15) alle ganz sind — jeder Punkt $(y_1, y_2, \cdots)$, dessen Koordinaten sämtlich ganze Zahlen sind. Somit gehört gewiss zur Menge Γ jeder Punkt des unendlich-dimensionalen x-Raumes, welcher (durch unsere Substitution) Bildpunkt eines Punktes des y-Raumes mit lauter ganzzahligen Koordinaten ist. Es bezeichne nun l den grössten Index eines y, welches in den m ersten Gleichungen der Substitution (6) vorkommt, und G den Hauptnenner aller Koeffizienten α, welche in den l ersten Gleichungen der inversen Substitution

$$(5) \qquad y_n = \alpha_{n,1}x_1 + \alpha_{n,2}x_2 + \cdots + \alpha_{n,u_n}x_{u_n} \qquad (n = 1, 2, \cdots)$$

auftreten. Wir werden zeigen, das bei jedem $\nu = 1, \cdots m$ der Punkt des m-dimensionalen Raumes $(x_1^{(\nu)}, x_2^{(\nu)}, \cdots x_m^{(\nu)})$ mit den Koordinaten

$$x_\nu^{(\nu)} = G, \quad x_\mu^{(\nu)} = 0 \text{ für } \mu \neq \nu \, (1 \leq \mu \leq m)$$

zur Menge Γ_m gehört. Hierzu setzen wir in den oben genannten l ersten Gleichungen der Substitution (5) $x_\nu = G$, $x_n = 0$ für alle $n \neq \nu$, und bezeichnen die dabei herauskommenden, nach der Bestimmung von G gewiss ganzzahligen Werte von $y_1, \cdots, y_l$ mit $Y_1, \cdots, Y_l$. Der Punkt $(Y_1, Y_2, \cdots, Y_l, 0, 0, \cdots)$ des unendlich-dimensionalen y-Raumes gehört wegen der Ganzzahligkeit aller seiner Koordinaten zur Menge Γ^*, und sein Bildpunkt $(X_1, X_2, \cdots)$ des x-Raumes muss daher ein Punkt von Γ sein. In diesem letzten Punkte $(X_1, X_2, \cdots)$ müssen aber auf den m ersten

Stellen gerade die Zahlen $x_1^{(\nu)}, x_2^{(\nu)}, \cdots, x_m^{(\nu)}$ stehen — womit also gezeigt ist, dass der Punkt $(x_1^{(\nu)}, x_2^{(\nu)} \cdots, x_m^{(\nu)})$ tatsächlich zur Projektion Γ_m gehört —; denn es sind (nach der Bestimmungsweise von l) die m ersten Koordinaten $x_1, \cdots, x_m$ eines x-Punktes eindeutig bestimmt, wenn man die l ersten Koordinaten $y_1, \cdots, y_l$ des entsprechenden y-Punktes kennt, und wir wissen ja (nach der Bestimmung von $Y_1, \cdots, Y_l$) dass es einen Punkt des x-Raumes mit den m ersten Koordinaten $x_1^{(\nu)}, \cdots, x_m^{(\nu)}$ gibt (nämlich den Punkt $(x_1^{(\nu)}, \cdots, x_m^{(\nu)}, 0, 0, \cdots)$), für welchen der entsprechende y-Punkt gerade mit den l Koordinaten $Y_1, \cdots, Y_l$ anfängt.

2. Wir zeigen danach, dass, falls bei jedem m die Menge Γ_m ein echtes Gitter bildet, das System (1) ganzartig ist. Hierzu bestimmen wir das aus rationalen Zahlen bestehende Zahlenschema

$$\begin{array}{llllllll}
\beta_{1,1}, 0, & 0, & 0, & \cdots 0, & 0, & 0, \cdots & \\
\beta_{2,1}, \beta_{2,2}, 0, & 0, & & \cdots 0, & 0, & 0, \cdots & \\
\beta_{3,1}, \beta_{3,2}, \beta_{3,3}, 0, & & & \cdots 0, & 0, & 0, \cdots & (\beta_{n,n} \neq 0 \text{ für alle } n) \\
\cdot & & & & & & \\
\cdot & & & & & & \\
\beta_{n,1}, \beta_{n,2}, \beta_{n,3}, \beta_{n,4} & \cdots \beta_{n,n}, & 0, & 0, \cdots & & & \\
\cdots\cdots\cdots & & & & & &
\end{array}$$

durch das folgende Verfahren: Die Zahl $\beta_{1,1}$ ist ein (beliebig gewählter) Punkt $\neq 0$ in dem echten Gitter Γ_1. Da Γ_1 die Projektion von Γ_2 ist und sowohl $x_1 = \beta_{1,1}$ wie $x_1 = 0$ Punkte in Γ_1 sind, können wir $\beta_{2,1}$ und $\beta_{2,2}$ so wählen, dass $(\beta_{1,1}, \beta_{2,1})$ und $(0, \beta_{2,2})$ Punkte in Γ_2 darstellen, und da Γ_2 ein echtes Gitter ist, können wir hierbei $\beta_{2,2} \neq 0$ wählen. Danach bestimmen wir die drei Zahlen $\beta_{3,1}, \beta_{3,2}, \beta_{3,3}$ folgendermassen: da

$$(\beta_{1,1}, \beta_{2,1}), \quad (0, \beta_{2,2}), \quad (0, 0)$$

Punkte in Γ_2 sind und Γ_2 die Projektion von Γ_3 ist, können wir sie durch Hinzufügung passend gewählter Zahlen $\beta_{3,1}, \beta_{3,2}, \beta_{3,3}$ zu drei Punkten

$$(\beta_{1,1}, \beta_{2,1}, \beta_{3,1}), \quad (0, \beta_{2,2}, \beta_{3,2}), \quad (0, 0, \beta_{3,3})$$

ergänzen, welche zu Γ_3 gehören, und hierbei können wir $\beta_{3,3} \neq 0$ wählen, da Γ_3 ein echtes Gitter bildet (und also Punkte mit $x_3 \neq 0$ auf der x_3-Achse enthält). In dieser Weise setzen wir fort, bis das ganze Zahlenschema nach unendlich vielen Schritten bestimmt ist. Es ist hierbei klar, dass jeder Punkt des unendlich-dimensionalen Raumes, dessen Koordinaten durch die in einer senkrechten Spalte des gewonnenen Schemas stehenden Zahlen bestimmt sind, zur Menge Γ gehört; denn nach der Bestimmungsweise des Schemas, gehört ja, für jedes m, der bei der m^{ten} Koordinate »abgeschnittene« Punkt zur Menge Γ_m, und nach dem Satze 4 genügt dies um schliessen zu können, dass der Punkt selbst zu Γ gehört.

Wir bilden nun, mit Hülfe des obigen Schemas, die lineare Substitution

$$(16)\begin{cases} x_1 = \beta_{1,1} y_1 \\ x_2 = \beta_{2,1} y_1 + \beta_{2,2} y_2 & (\beta_{n,n} \neq 0 \text{ für alle } n) \\ \cdots\cdots\cdots\cdots\cdots \\ x_n = \beta_{n,1} y_1 + \beta_{n,2} y_2 + \cdots + \beta_{n,n} y_n \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \end{cases}$$

und bezeichnen mit Γ^* die Punktmenge des unendlich-dimensionalen y-Raumes, welche die Bildmenge von Γ bei dieser Substitution bildet. Zu dieser Menge Γ^* gehören gewiss die sämtlichen Punkte der Form $(0, \cdots, 0, 1, 0, 0, \cdots)$ (d. h. sämtliche Punkte deren Koordinaten alle 0 sind, abgesehen von einer Koordinate, die gleich 1 ist); denn diese

Punkte entsprechen, nach (16), gerade den Punkten des x-Raumes, deren Koordinaten durch die Spalten des obigen Zahlenschemas gegeben sind. Wir werden zeigen, dass unser System (1) durch diese Substitution (16) in ein neues System

$$g_{n,1}y_1 + g_{n,2}y_2 + \cdots + g_{n,p_n}y_{p_n} \quad (n = 1, 2, \cdots)$$

mit lauter ganzen Koeffizienten übergeht. Hierzu haben wir nur zu benutzen, dass Γ^* (als Bildmenge von Γ) die Menge der simultanen Lösungen der neuen Nullkongruenzen

$$g_{n,1}y_1 + g_{n,2}y_2 + \cdots + g_{n,p_n}y_{p_n} \equiv 0 \pmod{1} \quad (n = 1, 2, \cdots)$$

darstellt, so dass diese letzten Kongruenzen gewiss die obigen Punkte $(0, 0, \cdots, 1, 0, 0, \cdots)$ als Lösungen besitzen. In der Tat folgt hieraus sofort, dass alle g ganze Zahlen sein müssen; denn daraus, dass $y_m = 1$, $y_n = 0$ (für $n \neq m$) ein Lösung der Kongruenzen ist, folgt ja, dass die sämtlichen Koeffizienten der Variablen y_m ganze Zahlen sein müssen.

In dem vorhergehenden Satze war nur von den simultanen Lösungen der unendlich vielen Nullkongruenzen (13) die Rede. Für den Beweis des Hauptsatzes in § 5 wird es aber nötig sein, das in diesem Satze 6 gefundene Kriterium der »Ganzartigkeit« eines Systems (1) etwas umzuformen, und zwar so, dass die Lösungen einer beliebig grossen endlichen Anzahl dieser Nullkongruenzen in den Mittelpunkt der Betrachtungen hineingezogen werden. Wir bezeichnen hierzu mit $\varDelta^{(N)}$ die Menge sämtlicher Lösungen $(x_1, x_2, \cdots)$ der N ersten der Nullkongruenzen [1], und mit $\varDelta_m^{(N)}$ die Projektion dieser Menge auf den m-dimensionalen Unter-

[1] Obwohl in diesen N Kongruenzen nur endlich viele x vorkommen, werden wir jedoch (vgl. Note 1, S. 4) unter einer Lösung der N Kongruenzen einen Punkt des unendlich-dimensionalen Raumes verstehen, so dass $\varDelta^{(N)}$ also eine Punktmenge dieses letzten Raumes bedeutet.

raum $x_1, \cdots, x_m$. Hierbei ist klar, dass bei jedem N die Relation $\varLambda^{(N)} > \varLambda^{(N+1)} > \varGamma$, und also auch die Relation $\varLambda_m^{(N)} > \varLambda_m^{(N+1)} > \varGamma_m$ besteht, und ferner, dass $\varLambda_m^{(N)}$ die Projektion von $\varLambda_{m+1}^{(N)}$ ist. Nach Voraussetzung kann jede der Variablen x aus den Linearformen des Systems (1) isoliert werden; wir bezeichnen mit $m^* = m^*(m)$ die kleinste Zahl, für welche jede der m ersten Variablen $x_1, \cdots, x_m$ aus den m^* ersten Linearformen isoliert werden kann.

Man sieht sofort, *dass bei jedem festen* $m = 1, 2, \cdots$ *und* $N > m^*$ *die Punktmenge* $\varLambda_m^{(N)}$ *ein echtes Gitter des m-dimensionalen Raumes bildet.* In der Tat:

1. Falls $(x'_1, \cdots, x'_m)$ und $(x''_1, \cdots, x''_m)$ zwei beliebige Punkte der Menge $\varLambda_m^{(N)}$ sind, wird auch der Differenzpunkt $(x'_1 - x''_1, \cdots, x'_m - x''_m)$ zu $\varLambda_m^{(N)}$ gehören. Denn es können ja die beiden Punkte durch Hinzufügung weiterer Koordinaten zu zwei Punkten $(x'_1, \cdots, x'_m, x'_{m+1}, \cdots)$ und $(x''_1, \cdots, x''_m, x''_{m+1}, \cdots)$ der Menge $\varLambda^{(N)}$ ergänzt werden, und aus der Definition von $\varLambda^{(N)}$ folgt sofort, dass der Differenzpunkt $(x'_1 - x''_1, \cdots, x'_m - x''_m, x'_{m+1} - x''_{m+1}, \cdots)$ ebenfalls zu $\varLambda^{(N)}$, also der »abgeschnittene« Punkt $(x'_1 - x''_1, \cdots, x'_m - x''_m)$ zu $\varLambda_m^{(N)}$ gehören wird.

2. Ferner ist klar, dass der Anfangspunkt $(0, 0, \cdots, 0)$ des m-dimensionalen Raumes kein Häufungspunkt der Menge $\varLambda_m^{(N)}$ sein kann, weil ja N so gross gewählt ist, dass jede der m Variablen $x_1, \cdots, x_m$ aus den N ersten Linearformen isoliert werden kann.

3. Und schliesslich ist klar, dass $\varLambda_m^{(N)}$ ein echtes Gitter bildet. In der Tat, falls H den Hauptnenner der (endlich vielen) rationalen Koeffizienten r der N ersten Kongruenzen bezeichnet, wird offenbar jeder Punkt $(x_1, x_2, \cdots)$ mit einer Koordinate gleich H und allen übrigen Koordinaten gleich 0 zur Menge $\varLambda^{(N)}$ gehören, so dass die Projektionsmenge $\varLambda_m^{(N)}$

gewiss auf jeder der m Koordinatenachsen einen vom Anfangspunkte verschiedenen Punkt enthalten wird (nämlich einen Punkt im Abstande H vom Anfangspunkte).

Es sei nunmehr m eine beliebig gewählte feste positive ganze Zahl; wir werden untersuchen, wie das Gitter $\Lambda_m^{(N)}$ $(N > m^*)$ sich ändert, wenn N ins Unendliche wächst. Aus der Relation $\Lambda_m^{(N)} > \Lambda_m^{(N+1)}$ ersieht man sofort, dass nur die beiden folgenden Möglichkeiten bestehen:

I. Entweder ändert sich das Gitter $\Lambda_m^{(N)}$ von einer gewissen Stelle an überhaupt nicht, d. h. es bleibt $\Lambda_m^{(N)}$ konstant für alle $N > N_0 = N_0(m)$. Wir bezeichnen in diesem Falle das konstante Endgitter $\Lambda_m^{(N)}$ $(N > N_0)$ mit Λ_m und nennen Λ_m *das* m^{te} *»Grenzgitter«*[1].

II. Oder es wächst das Volumen eines Grundparallelopipeds des Gitters $\Lambda_m^{(N)}$ mit $N \to \infty$ über alle Grenzen. Wir sagen in diesem Falle, dass (für das betrachtete m) *kein Grenzgitter* Λ_m *existiert.*

Satz 7. *Für die Ganzartigkeit des (reduzierten) Systems (1) ist notwendig und hinreichend, dass bei jedem festen* $m = 1, 2, \cdots$ *das Grenzgitter* Λ_m *existiert.*

Beweis. Nach dem Satze 6 handelt es sich darum zu zeigen, dass die beiden Bedingungen »es soll bei jedem m die Menge Γ_m ein echtes m-dimensionales Gitter sein« und »es soll bei jedem m ein Grenzgitter Λ_m existieren« inhaltlich gleichbedeutend sind.

1. Es ist unmittelbar ersichtlich, dass, falls die Mengen Γ_m alle echte Gitter sind, alle Grenzgitter Λ_m existieren müssen. In der Tat ist ja bei festem m und jedem N das

[1] Wir bemerken, dass falls die Grenzgitter Λ_m und Λ_{m+1} beide existieren, Λ_m die Projektion von Λ_{m+1} sein wird. In der Tat kann ein N so gross gewählt werden, dass $\Lambda_m = \Lambda_m^{(N)}$ und $\Lambda_{m+1} = \Lambda_{m+1}^{(N)}$ sind, und bei einem festen N wissen wir ja schon, dass $\Lambda_m^{(N)}$ die Projektion von $\Lambda_{m+1}^{(N)}$ ist.

Gitter Γ_m in dem Gitter $\Lambda_m^{(N)}$ enthalten, und es kann somit für $N \longrightarrow \infty$ das Volumen des Grundparallelopipeds von $\Lambda_m^{(N)}$ nicht über alle Grenzen wachsen.

2. Etwas tiefer liegt der Nachweis, dass umgekehrt aus der Existenz sämtlicher Grenzgitter Δ_m geschlossen werden kann, dass jedes Γ_m ein echtes Gitter ist. Wir führen diesen Nachweis dadurch, dass wir zeigen, dass Δ_m in Γ_m enthalten ist. Es sei also $(x_1, \cdots, x_m)$ ein beliebiger Punkt von Δ_m; wir haben zu beweisen, dass dieser Punkt ebenfalls zu Γ_m gehört, also dass er zu einem Punkte $(x_1, \cdots, x_m, x_{m+1}, \cdots)$ ergänzt werden kann, welcher zu Γ gehört, d. h. welcher eine simultane Lösung der sämtlichen Nullkongruenzen (13) darstellt. Hierzu müssen wir benutzen, dass nicht nur das Grenzgitter Δ_m, sondern auch die »höheren« Grenzgitter $\Delta_{m+1}, \Delta_{m+2}, \cdots$ alle existieren. Wir bilden (durch successive Wahl) die Folge der neuen Koordinaten $x_{m+1}, x_{m+2}, \cdots$ so, dass jeder »Abschnitt« $(x_1, \cdots, x_m, x_{m+1}, \cdots, x_{m+p})$ zur Menge Δ_{m+p} gehört (was möglich ist, weil, nach Note S. 35, Δ_n die Projektion von Δ_{n+1} ist), und behaupten, dass der durch diese Koordinaten ergänzte Punkt $(x_1, \cdots, x_m, x_{m+1}, \cdots)$ von der gewünschten Art ist, also eine simultane Lösung sämtlicher Nullkongruenzen darstellt, d. h. bei jedem festen N die N ersten der Kongruenzen befriedigt. In der Tat, es sei L der grösste Index eines x, welches in diesen N ersten Kongruenzen vorkommt; es genügt dann offenbar zu zeigen, dass der Punkt, welcher durch Projektion des Punktes $(x_1, x_2, \cdots, x_m, x_{m+1}, \cdots)$ auf den L-dimensionalen Unterraum entsteht, also der Punkt $(x_1, \cdots, x_L)$, in $\Lambda_L^{(N)}$ liegt; und dies ist gewiss der Fall, weil ja nach unserem Ergänzungsverfahren der Punkt $(x_1, \cdots, x_L)$ zu Δ_L gehört, und Δ_L in $\Lambda_L^{(N)}$ enthalten ist.

§ 5.
Beweis des Hauptsatzes.

Hauptsatz. *Für das Zusammenfallen der beiden zu einem System (1) gehörigen Punktmengen Π_1 und Π_2 ist notwendig und hinreichend, dass das System ganzartig ist* (also durch eine lineare Substitution in ein neues System mit lauter ganzen Koeffizienten übergeführt werden kann).

Beweis. Dass die Bedingung hinreichend ist, liegt auf der Hand. In der Tat wissen wir ja einerseits, dass der Übergang von einem System zu einem anderen mit Hilfe einer linearen Substitution sowohl die Menge Π_1 wie auch die Menge Π_2 ungeändert lässt, und andererseits (nach Satz 1), dass für ein System mit lauter ganzen Koeffizienten die Gleichung $\Pi_1 = \Pi_2$ besteht.

Die ganze Schwierigkeit liegt darin zu beweisen, dass die Bedingung der Ganzartigkeit für das Zusammenfallen der Mengen Π_1 und Π_2 auch notwendig ist. Hierbei können wir uns (nach § 3) ohne Beschränkung der Allgemeinheit auf die Betrachtung eines reduzierten Systems (1) beschränken.

Wir nehmen an, dass das System nicht ganzartig ist, und haben zu beweisen, dass Π_1 eine echte Teilmenge von Π_2 ist, also dass ein Punkt $(\theta_1, \theta_2, \cdots)$ derart existiert, dass bei jedem festen N die N ersten der Kongruenzen

$$(2) \quad r_{n,1}x_1 + r_{n,2}x_2 + \cdots + r_{n,q_n}x_{q_n} \equiv \theta_n \pmod{1} \quad (n = 1, 2, \cdots)$$

eine Lösung haben, aber keine simultane Lösung der sämtlichen Kongruenzen vorhanden ist. Das wesentlichste Hilfsmittel bei diesem Beweise ist der Satz 7, welcher ein Kriterium für die Ganzartigkeit (und also auch ein Kriterium für die Nicht-Ganzartigkeit) eines reduzierten Sy-

stems (1) gibt. Nach diesem Hilfsatz gibt es zu unserem (nicht ganzartigen) System (1) eine feste positive ganze Zahl m_0, so dass kein Grenzgitter $\varLambda_{m_0}$ existiert, d. h. *so dass das Volumen V_N eines Grundparallelopipeds des Gitters $\varLambda_{m_0}^{(N)}$ $(N \geq m_0^*)$ für $N \to \infty$ ins Unendliche wächst.*

Wir bestimmen zunächst eine unendliche Folge von wachsenden positiven ganzen Zahlen $N_1, N_2, \cdots, N_\nu, \cdots$ und zugehörigen positiven Zahlen $\varrho_1, \varrho_2, \cdots, \varrho_\nu, \cdots$ durch das folgende Verfahren[1].

1°. Es sei $N_1 > m_0^*$ so gewählt, dass das Grundvolumen V_{N_1} des Gitters $\varLambda_{m_0}^{(N_1)}$ grösser als das Kugelvolumen $K(1)$ ist, und somit in der Anfangskugel $\mathsf{K}(1)$ gewiss kein vollständiges Repräsentantensystem in bezug auf $\varLambda_{m_0}^{(N_1)}$ enthalten ist. Zu diesem N_1 bestimmen wir die positive Zahl ϱ_1 so gross, dass jede Kugel mit dem Radius ϱ_1 (wo auch ihr Zentrum liegt) ein vollständiges Repräsentantensystem in bezug auf $\varLambda_{m_0}^{(N_1)}$ enthält.

2°. Danach bestimmen wir $N_2 > N_1$ derart, dass das Grundvolumen V_{N_2} grösser als das Kugelvolumen $K(\varrho_1 + 2)$ ist, und daher die Anfangskugel $\mathsf{K}(\varrho_1 + 2)$ kein vollständiges Repräsentantensystem in bezug af $\varLambda_{m_0}^{(N_2)}$ enthält. Und zu diesem N_2 bestimmen wir die positive Zahl ϱ_2 derart,

[1] Hierbei werden wir, um uns bei den folgenden Überlegungen über die Verhältnisse im m_0-dimensionalen Raume $x_1, \cdots x_{m_0}$ übersichtlich ausdrücken zu können, fur einige immer wieder vorkommenden Begriffe abgekürzte Bezeichnungen einführen: Statt »Volumen eines Grundparallelopipeds des Gitters $\varLambda_{m_0}^{(N)}$« wollen wir »Grundvolumen des Gitters $\varLambda_{m_0}^{(N)}$« sagen, statt »m_0-dimensionale Kugel« einfach »Kugel«, statt »Volumen einer Kugel mit dem Radius ϱ« einfach »Kugelvolumen $K(\varrho)$« und statt »Kugel mit dem Anfangspunkt als Zentrum und dem Radius ϱ« einfach »Anfangskugel $\mathsf{K}(\varrho)$« oder nur »$\mathsf{K}(\varrho)$«. Ferner werden wir mit »einem vollständigen Repräsentantensystem in bezug auf $\varLambda_{m_0}^{(N)}$« eine solche Punktmenge bezeichnen, welche aus jedem System von Punkten $(x_1, \cdots, x_{m_0})$, welche in bezug auf das Gitter $\varLambda_{m_0}^{(N)}$ äquivalent liegen, genau einen Repräsentanten enthält.

dass jede Kugel mit dem Radius ϱ_2 ein vollständiges Repräsentantensystem in bezug auf $\Lambda_{m_0}^{(N_2)}$ enthält.

— — —

ν^0. Nachdem $N_{\nu-1}$ und $\varrho_{\nu-1}$ bestimmt sind, wählen wir $N_\nu > N_{\nu-1}$ derart, dass das Grundvolumen V_{N_ν} grösser als das Kugelvolumen $K(\varrho_{\nu-1}+\nu)$ ist, also die Anfangskugel $\mathbf{K}(\varrho_{\nu-1}+\nu)$ kein vollständiges Repräsentantensystem in bezug auf $\Lambda_{m_0}^{(N_\nu)}$ enthält, und danach wählen wir die positive Zahl ϱ_ν so, dass jede Kugel mit dem Radius ϱ_ν ein vollständiges Repräsentantensystem in bezug auf $\Lambda_{m_0}^{(N_\nu)}$ enthält.

— — —

Nach dieser Bestimmung von N_ν und ϱ_ν $(\nu = 1, 2, \cdots)$ schreiten wir nunmehr zu direktem Aufsuchen eines Punktes $(\theta_1, \theta_2, \cdots)$, welcher der Menge Π_2 aber nicht der Menge Π_1 angehört. Die Idee der (successiven) Bestimmung von $\theta_1, \theta_2, \cdots$ ist die, dass wir versuchen dafür zu sorgen, dass im m_0-dimensionalen Raume der Abstand des Anfangspunktes von der Projektion $(x_1, \cdots, x_{m_0})$ derjenigen Lösung $(x_1, x_2, \cdots)$ der N ersten Kongruenzen (2), für welche dieser Abstand am kleinsten ist, mit N über alle Grenzen wächst.

1^{ter} Schritt. Wir wählen zunächst einen beliebigen Punkt P' : $(x'_1, x'_2, \cdots)$ des unendlich-dimensionalen Raumes, welcher nur der Bedingung unterworfen sein soll, dass sein m_0^{ter} »Abschnitt« P'_{m_0} : $(x'_1, \cdots, x'_{m_0})$ ein derartiger Punkt des m_0-dimensionalen Raumes ist, dass kein mit ihm in bezug auf das Gitter $\Lambda_{m_0}^{(N_1)}$ äquivalenter Punkt in der Anfangskugel $\mathbf{K}(1)$ gelegen ist. (Ein solcher Punkt existiert nach 1^0). Wir setzen $(x_1, x_2, \cdots) = (x'_1, x'_2, \cdots)$ in die N_1 ersten Linearformen von (1) ein. Die dadurch entstehenden Zahlen sollen unsere Zahlen $\theta_1, \cdots, \theta_{N_1}$ sein. Wir bemerken, dass

die **sämtlichen** Lösungen $(x_1, x_2, \cdots)$ der N_1 ersten (mit den eben gewählten θ gebildeten) Kongruenzen (2) durch die Menge $P' + \Lambda^{(N_1)}$ gegeben sind[1], weil $\Lambda^{(N_1)}$ ja die Gesamtmenge der Lösungen der N_1 ersten **Nullkongruenzen** angibt. Aus der Bestimmung von P' folgt, dass in der Projektion dieser Menge $P' + \Lambda^{(N_1)}$ auf den m_0-dimensionalen Unterraum — d. h. in der Menge $P'_{m_0} + \Lambda^{(N_1)}_{m_0}$, welche aus allen mit P'_{m_0} in bezug auf $\Lambda^{(N_1)}_{m_0}$ äquivalenten Punkten besteht — kein Punkt enthalten ist, welcher in der Anfangskugel $\mathsf{K}(1)$ liegt.

2ter **Schritt.** Wir wählen danach einen Punkt $P'' : (x''_1, x''_2, \cdots)$ des unendlich-dimensionalen x-Raumes, **welcher der obigen Menge** $P' + \Lambda^{(N_1)}$ **angehört,** und ausserdem der Bedingung genügt, dass seine Projektion $P''_{m_0} : (x''_1, \cdots, x''_{m_0})$ auf den m_0-dimensionalen Unterraum in einer Kugel mit dem Radius ϱ_1 gelegen ist, deren Zentrum $C'' : (c''_1, \cdots, c''_{m_0})$ so gewählt ist, dass es keinen mit C'' in bezug auf $\Lambda^{(N_2)}_{m_0}$ äquivalenten Punkt in der Anfangskugel $\mathsf{K}(\varrho_1 + 2)$ gibt (was alles nach der Bestimmung von ϱ_1 und N_2 möglich ist). Wir setzen nun $(x_1, x_2, \cdots) = (x''_1, x''_2, \cdots)$ in die N_2 ersten Linearformen von (1) ein, und bezeichnen die herauskommenden Werte mit $\theta_1, \cdots, \theta_{N_2}$, wobei die N_1 ersten Zahlen $\theta_1, \cdots, \theta_{N_1}$ mit den obigen Zahlen $\theta_1, \cdots, \theta_{N_1}$ übereinstimmen, weil P'' zur Menge $P' + \Lambda^{(N_1)}$ gehört. Wir betrachten die Menge **sämtlicher** Lösungen $(x_1, x_2, \cdots)$ der N_2 ersten (mit den gewählten θ gebildeten) Kongruenzen (2), d. h. die Menge $P'' + \Lambda^{(N_2)}$, und behaupten,

[1] Unter $P + \Lambda$, wo P einen Punkt und Λ eine Punktmenge desselben Raumes bedeuten, verstehen wir die (mit Λ kongruente) Punktmenge, welche aus der Menge Λ entsteht, wenn zu jedem ihrer Punkte der Punkt P' »addiert« wird; hierbei bedeutet »Addition« zweier Punkte $(x_1, x_2, \cdots)$ und $(y_1, y_2, \cdots)$ einfach die Bildung des Punktes $(x_1 + y_1, x_2 + y_2, \cdots)$.

dass in der Projektion dieser Menge auf den m_0^{ten} Unterraum, d. h. in der Menge $P''_{m_0} + \varLambda^{(N_2)}_{m_0}$ aller mit P''_{m_0} in bezug auf $\varLambda^{(N_2)}_{m_0}$ aquivalenten Punkte, kein Punkt gelegen ist, welcher der Anfangskugel $\mathbf{K}(2)$ angehört. In der Tat gibt es in dieser Menge $P''_{m_0} + \varLambda^{(N_2)}_{m_0}$ einen Punkt, nämlich P''_{m_0} selbst, welcher von dem obigen Zentrum C'' um weniger als ϱ_1 abweicht, und weil $\mathbf{K}(\varrho_1 + 2)$ keinen mit C'' in bezug auf $\varLambda^{(N_2)}_{m_0}$ äquivalenten Punkt enthält, kann daher in $\mathbf{K}(2)$ kein mit P''_{m_0} äquivalenter Punkt (d. h. kein zu $P''_{m_0} + \varLambda^{(N_2)}_{m_0}$ gehöriger Punkt) gelegen sein.

— — —

ν^{ter} Schritt. Wir wählen einen Punkt $P^{(\nu)}: (x_1^{(\nu)}, x_2^{(\nu)}, \cdots)$ des unendlich-dimensionalen x-Raumes, welcher der Menge $P^{(\nu-1)} + \varLambda^{(N_{\nu-1})}$ angehört, und ausserdem der Bedingung genügt, dass sein m_0^{ter} Abschnitt $P^{(\nu)}_{m_0}: (x_1^{(\nu)}, \cdots, x_{m_0}^{(\nu)})$ in einer Kugel mit dem Radius $\varrho_{\nu-1}$ liegt, deren Zentrum $C^{(\nu)}: (c_1^{(\nu)}, \cdots, c_{m_0}^{(\nu)})$ so gewählt ist, dass kein mit $C^{(\nu)}$ in bezug auf $\varLambda^{(N_\nu)}_{m_0}$ äquivalenter Punkt der Anfangskugel $\mathbf{K}(\varrho_{\nu-1} + \nu)$ angehört. Dies ist alles möglich; denn nach der Bestimmung von N_ν gibt es gewiss einen Punkt $C^{(\nu)}$ der verlangten Art, und in die Kugel um $C^{(\nu)}$ mit dem Radius $\varrho_{\nu-1}$ fällt gewiss (nach der Bestimmung von $\varrho_{\nu-1}$) die Projektion eines Punktes der Menge $P^{(\nu-1)} + \varLambda^{(N_{\nu-1})}$, weil ja die Projektion $P^{(\nu-1)}_{m_0} + \varLambda^{(N_{\nu-1})}_{m_0}$ dieser letzten Menge ein ganzes System von untereinander äquivalenten Punkten in bezug auf $\varLambda^{(N_{\nu-1})}_{m_0}$ ausmacht. Wir setzen nun $(x_1, x_2, \cdots) = (x_1^{(\nu)}, x_2^{(\nu)}, \cdots)$ in die N_ν ersten Linearformen von (1) ein, und bezeichnen die herauskommenden Zahlen mit $\theta_1, \cdots, \theta_{N_\nu}$, wobei die $N_{\nu-1}$ ersten dieser Zahlen mit den bei dem $(\nu-1)^{\text{ten}}$ Schritte bestimmten Zahlen $\theta_1, \cdots, \theta_{N_{\nu-1}}$ übereinstimmen (weil $P^{(\nu)}$ der Menge $P^{(\nu-1)} + \varLambda^{(N_{\nu-1})}$ an-

gehört). Die sämtlichen Lösungen $(x_1, x_2, \cdots)$ der N_ν ersten (mit diesen θ gebildeten) Kongruenzen sind dann durch $P^{(\nu)} + \varLambda^{(N_\nu)}$ gegeben, also ihre Projektionen auf den $m_0$ten Unterraum durch $P^{(\nu)}_{m_0} + \varLambda^{(N_\nu)}_{m_0}$. In dieser letzten Menge gibt es keinen Punkt, welcher der Anfangskugel $\mathbf{K}(\nu)$ angehört, weil $P^{(\nu)}_{m_0}$ in einem Abstand $< \varrho_{\nu-1}$ vom $C^{(\nu)}$ gelegen ist, und in $\mathbf{K}(\varrho_{\nu-1} + \nu)$ kein Punkt liegt, welcher mit $C^{(\nu)}$ in bezug auf $\varLambda^{(N_\nu)}_{m_0}$ äquivalent ist.

— — —

Hiermit sind wir am Ende. In der Tat erfüllt der somit bestimmte Punkt $(\theta_1, \theta_2, \cdots)$ die angegebenen Bedingungen. Denn einerseits haben bei jedem N die N ersten der Kongruenzen (2) eine Lösung — weil bei jedem ν die N_ν ersten Kongruenzen eine Lösung, nämlich $P^{(\nu)}: (x_1^{(\nu)}, x_2^{(\nu)}, \cdots)$ besitzen, und N_ν mit ν über alle Grenzen wächst — und andererseits gibt es gewiss keine simultane Lösung der sämtlichen Kongruenzen (2), weil jede Lösung der N_ν ersten Kongruenzen eine Projektion auf den m_0-dimensionalen Unterraum besitzt, deren Abstand vom Anfangspunkte $\geqq \nu$ ist, also für $\nu \to \infty$ beliebig gross wird.

Forelagt paa Mødet den 21. Februar 1925.
Færdig fra Trykkeriet den 4. Maj 1925.

To nye simple Beviser for Kroneckers Sætning.

Af Harald Bohr og Børge Jessen.

I et Seminar afholdt for de Universitetsstuderende i dette Foraar diskuterede man bl. a. et analytisk Bevis for Kroneckers Sætning givet af H. Bohr*); ved denne Lejlighed efterlyste Prof. J. Nielsen et simpelt og instruktivt geometrisk Bevis og gav selv kort Tid efter et saadant baseret paa almindelige Betragtninger over Vektormoduler i et n-dimensionalt Rum**). Dette Bevis vakte paany Interessen for et endnu simplere analytisk Bevis. I det følgende skal nu meddeles to saadanne Beviser; det første (paa hvilket vi blev opmærksomme under Udarbejdelsen af et Arbejde om næstenperiodiske Funktioner) benytter Ideer af Bochner, Szidon og Fekete; det andet (maaske nok det simpleste) benytter tillige en Ide af Besicovitch, og minder forøvrig om et tidligere Bevis af Landau***).

Kroneckers Sætning kan formuleres saaledes Lad os antage, at $\lambda_1, \lambda_2, \cdots, \lambda_N$ er N lineært uafhængige reelle Tal (d. v. s. der bestaar ingen Relation af Formen $g_1\lambda_1 + g_2\lambda_2 + \cdots + g_N\lambda_N = 0$ i hele Tal $g_1, g_2, \cdots, g_N$, som ikke alle er 0) og lad endvidere $\varphi_1, \varphi_2, \cdots, \varphi_N$ betegne N vilkaarlige reelle Tal og ε et vilkaarligt positivt Tal. Da eksisterer der N hele Tal $h_1, h_2, \cdots, h_N$ og et reelt Tal t, saaledes at

$$|\lambda_\nu t - \varphi_\nu - h_\nu| < \varepsilon, \qquad (\nu = 1, 2, \cdots, N).$$

Anderledes udtrykt: Betragter man alle Punkter i det N-dimensionale Rum af Formen

$$(\lambda_1 t - h_1, \lambda_2 t - h_2, \cdots, \lambda_N t - h_N),$$

hvor t er et vilkaarligt reelt Tal og $h_1, h_2, \cdots, h_N$ er vilkaarlige hele Tal, saa ligger disse Punkter overalt tæt i det N-dimensionale Rum.

*) H. Bohr, Another proof of Kronecker's Theorem. Proc. of the London Math. Soc., Ser. 2, Vol. 21. (1922), S. 315–316.

**) J Nielsen, Om Strukturen af en Vektormodul med endelig Basis. Med Anvendelse paa Diofantiske Approksimationer. Matematisk Tidsskrift B, 1932, S. 29–42.

***) E. Landau, Über diophantische Approximationen. Scr. Univ. Bibl. Hierosolymitanarum, 1923, S. 1–4.

Ligesom det ovenfor nævnte Bevis af H. Bohr beror det første af de nye Beviser paa følgende ret indlysende Omskrivning af Sætningen (se Fig. 1, hvor $N = 3$): Danner man Funktionen

$$f(t) = 1 + e^{2\pi i(\lambda_1 t - \varphi_1)} + e^{2\pi i(\lambda_2 t - \varphi_2)} + \dots + e^{2\pi i(\lambda_N t - \varphi_N)},$$

saa er

$$\underset{-\infty<t<\infty}{\text{Øvre Grænse}} \; |f(t)| = 1 + N.$$

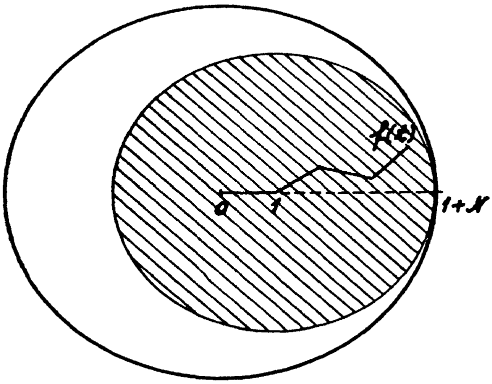


Fig. 1.

Sætter vi

$$\underset{-\infty<t<\infty}{\text{Øvre Grænse}} |f(t)| = \Gamma,$$

saa er, som man umiddelbart ser,

$$1 + N \geqq \Gamma.$$

Vi fører Beviset for Kroneckers Sætning, idet vi viser, at ogsaa omvendt

$$1 + N \leqq \Gamma.$$

I Beviset opererer vi uafbrudt med trigonometriske Polynomier, d. v. s. endelige Summer af Formen $F(t) = \Sigma a_n e^{i\mu_n t}$, hvor Eksponenterne μ_n er vilkaarlige indbyrdes forskellige reelle Tal, medens Koefficienterne a_n er vilkaarlige komplekse Tal. For ethvert saadant Polynomium eksisterer Middelværdien

$$M\{F(t)\} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} F(t)\,dt$$

og er lig med Polynomiets konstante Led, saafremt et saadant findes, d. v. s. saafremt ikke alle $\mu_n \neq 0$, i hvilket Tilfælde Middelværdien er lig med 0. Dette fremgaar umiddelbart af, at

$$M\{e^{i\mu t}\} = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} e^{i\mu t}\,dt = \begin{cases} 0 & \text{for } \mu \neq 0 \\ 1 & \text{for } \mu = 0. \end{cases}$$

Funktionen $f(t)$ er selv et trigonometrisk Polynomium med det konstante Led $M\{f(t)\} = 1$.

Beviset beror væsentlig paa den saakaldte Fejérske Kærne

$K_n(t)$. Saaledes betegner man, idet n er et helt Tal > 1, den fra Fourierrækkernes Teori velkendte Funktion

$$K_n(t) = \sum_{\nu=-n}^{n}\left(1 - \frac{|\nu|}{n}\right) e^{i\nu t} = \frac{1}{n}\left(\frac{\sin\frac{nt}{2}}{\sin\frac{t}{2}}\right)^2.$$

Overensstemmelsen mellem de to angivne Udtryk for $K_n(t)$ vises som velkendt ved simple trigonometriske Regninger, simplest vel gennem følgende Omskrivninger:

$$\sum_{\nu=-n}^{n}\left(1-\frac{|\nu|}{n}\right)e^{i\nu t} = \frac{1}{n}\sum_{\nu=-n}^{n}(n-|\nu|)\,e^{i\nu t} = \frac{1}{n}\,(D_0(t)+D_1(t)+\cdots+D_{n-1}(t)),$$

hvor $D_m(t)$ betegner den Dirichletske Kærne

$$D_m(t) = \sum_{\mu=-m}^{m} e^{i\mu t}.$$

Nu er som man let ser

$$D_m(t) = \sum_{\mu=-m}^{m} e^{i\mu t} = \frac{e^{i(m+\frac{1}{2})t} - e^{-i(m+\frac{1}{2})t}}{e^{i\frac{t}{2}} - e^{-i\frac{t}{2}}} = \frac{\sin(m+\frac{1}{2})t}{\sin\frac{t}{2}}$$

$$= \frac{\cos mt - \cos(m+1)t}{2\sin^2\frac{t}{2}};$$

altsaa faas

$$\frac{1}{n}\,(D_0(t)+D_1(t)+\cdots+D_{n-1}(t)) = \frac{1}{n}\,\frac{1-\cos nt}{2\sin^2\frac{t}{2}} = \frac{1}{n}\left(\frac{\sin\frac{nt}{2}}{\sin\frac{t}{2}}\right)^2.$$

De afgørende Egenskaber ved $K_n(t)$, som fremgaar af de to angivne Udtryk, er følgende to:

$$M\{K_n(t)\} = 1 \quad \text{og} \quad K_n(t) \geqq 0 \quad \text{for } -\infty < t < \infty.$$

Vi danner nu den „sammensatte Kærne"

$$\mathbf{K}_n(t) = K_n(2\pi(\lambda_1 t - \varphi_1))\cdot K_n(2\pi(\lambda_2 t - \varphi_2))\cdots K_n(2\pi(\lambda_N t - \varphi_N)).$$

Denne Kærne $\mathbf{K}_n(t)$ er aabenbart paany et trigonometrisk Polynomium, som bestemmes ved Udmultiplikation af det opskrevne Produkt. Da de herved optrædende, heltallige Kombinationer af Tallene $\lambda_1, \lambda_2, \cdots, \lambda_N$, som Følge af disse Tals for-

udsatte lineære Uafhængighed, alle maa blive forskellige, faas, idet man bemærker at

$$K_n(t) = 1 + \frac{n-1}{n}(e^{-it} + e^{it}) + \cdots,$$

for $\mathbf{K}_n(t)$ følgende Fremstilling:

$$\mathbf{K}_n(t) = 1 + \frac{n-1}{n}\left(e^{-2\pi i(\lambda_1 t - \varphi_1)} + e^{-2\pi i(\lambda_2 t - \varphi_2)} + \cdots + e^{-2\pi i(\lambda_N t - \varphi_N)}\right) + R(t),$$

hvor $R(t)$ betegner et trigonometrisk Polynomium $\Sigma a_m e^{i\mu_m t}$, hvis Eksponenter μ_m alle er forskellige fra Tallene $0, -2\pi\lambda_1, -2\pi\lambda_2, \ldots, -2\pi\lambda_N$. Af de to angivne Udtrykt for $\mathbf{K}_n(t)$ fremgaar, at denne Kærne ligesom den Fejérske Kærne har de to afgørende Egenskaber

$$M\{\mathbf{K}_n(t)\} = 1 \quad \text{og} \quad \mathbf{K}_n(t) \geqq 0 \quad \text{for} \quad -\infty < t < \infty.$$

Vi betragter nu Funktionen $f(t)\,\mathbf{K}_n(t)$; ogsaa denne Funktion er et trigonometrisk Polynomium, hvis konstante Led $M\{f(t)\,\mathbf{K}_n(t)\}$ umiddelbart beregnes til $1 + \frac{n-1}{n}N$. Af den saaledes fundne Formel

$$1 + \frac{n-1}{n}N = M\{f(t)\,\mathbf{K}_n(t)\}$$

faas nu umiddelbart ved Brug af de fremhævede Egenskaber ved $\mathbf{K}_n(t)$ Uligheden

$$1 + \frac{n-1}{n}N \leqq \Gamma \cdot M\{\mathbf{K}_n(t)\} = \Gamma.$$

Dette gælder for ethvert helt Tal $n > 1$; vi udfører nu Grænseovergangen $n \to \infty$ og faar derved Relationen

$$1 + N \leqq \Gamma,$$

hvormed Beviset er fuldført.

For at lette Forstaaelsen af den egentlige Grundtanke i det ovenfor givne Bevis tilføjer vi følgende Bemærkning: Det som skal vises er, at Funktionen $f(t)$ kommer i Nærheden af Værdien $1 + N$, som vides at være $\geqq$ Øvre Grænse Γ af Funktionens numeriske Værdi. Hertil har vi multipliceret $f(t)$ med en Kærne $\mathbf{K}_n(t)$, som er reel og $\geqq 0$ og som netop fremhæver de even-

tuelle Værdier af t, for hvilke $f(t)$ overhovedet kan komme i Nærheden af $1+N$, d. v. s. for hvilke Tallene $\lambda_1 t - \varphi_1$, $\lambda_2 t - \varphi_2$, $\cdots$, $\lambda_N t - \varphi_N$ samtidig er i Nærheden af hele Tal. Denne Egenskab ved den sammensatte Kærne $\mathbf{K}_n(t)$ fremgaar umiddelbart af den tilsvarende Egenskab ved den Fejérske Kærne $K_n(t)$, hvilken Kærne, som det ses af Udtrykket

$$K_n(t) = \frac{1}{n}\left(\frac{\sin\frac{nt}{2}}{\sin\frac{t}{2}}\right)^2,$$

netop fremhæver de Værdier af t, som er heltallige Multipla af 2π (se Fig. 2, hvor $n = 8$).

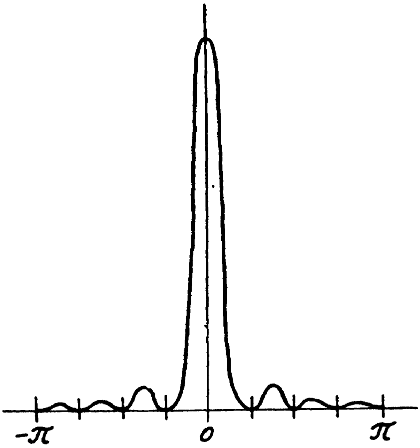


Fig. 2.

Ved Beregning af Middelværdien $M\{f(t)\mathbf{K}_n(t)\}$ viser det sig nu, at denne Middelværdi bliver saa stor, som det kun er muligt, naar $f(t)$ virkelig kommer i Nærheden af $1+N$.

Den sidste Bemærkning danner Udgangspunktet for det andet Bevis for Kroneckers Sætning, som vi skal meddele. Vi gaar her ud fra Sætningen i dens oprindelige Formulering og vælger først svarende til det givne positive Tal ε (som vi kan antage $<\frac{1}{2}$) et Tal n saa stort, at

$$K_n(t) \leqq k < \frac{1}{N},$$

saa snart t afviger mindst $2\pi\varepsilon$ fra ethvert helt Multiplum af 2π. Herved er $k < \frac{1}{N}$ et vilkaarligt, men fast valgt positivt Tal. Et saadant Valg af n er sikkert muligt; thi for enhver Værdi af t, som afviger mindst $2\pi\varepsilon$ fra ethvert helt Multiplum af 2π, har man

$$K_n(t) = \frac{1}{n}\left(\frac{\sin\frac{nt}{2}}{\sin\frac{t}{2}}\right)^2 \leqq \frac{1}{n}\frac{1}{(\sin\pi\varepsilon)^2};$$

den angivne Betingelse er altsaa for den valgte Værdi af k opfyldt, saasnart

$$n > \frac{1}{k}\,\frac{1}{(\sin \pi\varepsilon)^2}.$$

Herefter føres Beviset for Kroneckers Sætning indirekte saaledes: Antaget at Sætningen var forkert, at det altsaa for enhver Værdi af t var umuligt at vælge de N hele Tal $h_1, h_2, \cdots, h_N$, saaledes at

$$|\,\lambda_\nu t - \varphi_\nu - h_\nu\,| < \varepsilon \quad (\nu = 1, 2, \cdots, N),$$

saa maatte for enhver Værdi af t mindst et af Tallene $\lambda_1 t - \varphi_1$, $\lambda_2 t - \varphi_2, \cdots, \lambda_N t - \varphi_N$ afvige mindst ε fra ethvert helt Tal, og altsaa mindst et af Tallene $2\pi(\lambda_1 t - \varphi_1)$, $2\pi(\lambda_2 t - \varphi_2), \cdots$, $2\pi(\lambda_N t - \varphi_N)$ afvige mindst $2\pi\varepsilon$ fra ethvert helt Multiplum af 2π. Følgelig maatte for enhver Værdi af t mindst en af Faktorerne $K_n(2\pi(\lambda_1 t - \varphi_1))$, $K_n(2\pi(\lambda_2 t - \varphi_2)), \cdots, K_n(2\pi(\lambda_N t - \varphi_N))$ i Produktet $\mathbf{K}_n(t)$ være $\leqq k$; da nu disse Faktorer alle er $\geqq 0$ for $-\infty < t < \infty$, saa gjaldt altsaa for enhver Værdi af t Uligheden

$$\mathbf{K}_n(t) \leqq k \sum_{\nu=1}^{N} \frac{\mathbf{K}_n(t)}{K_n(2\pi(\lambda_\nu t - \varphi_\nu))}.$$

Nu er hver af de N Summander paa højre Side af Ulighedstegnet en Kærne af samme Type som $\mathbf{K}_n(t)$, blot med $N-1$ Faktorer i Stedet for N Faktorer. Hver af disse Kærner har altsaa ligesom $\mathbf{K}_n(t)$ Middelværdien 1. Tager vi altsaa i den fundne Ulighed paa begge Sider Middelværdien efter t, saa faar vi

$$1 \leqq kN;$$

men dette er en Modstrid, idet vi netop har valgt $k < \frac{1}{N}$. Hermed er Beviset fuldført.

Det sidste Bevis er paa en Maade mere direkte end det første, for saa vidt som det ikke benytter Funktionen $f(t)$. Dets Ide er den, at vise, at saafremt Kroneckers Sætning var forkert, d. v. s. saafremt Tallene $\lambda_1 t - \varphi_1$, $\lambda_2 t - \varphi_2, \cdots, \lambda_N t - \varphi_N$ aldrig samtidig kom i Nærheden af hele Tal, saa kunde Kærnen $\mathbf{K}_n(t)$, naar n er valgt tilstrækkelig stor, ikke blive stor nok til at faa Middelværdien 1.

ONE MORE PROOF OF KRONECKER'S THEOREM

H. BOHR and B. JESSEN*.

Kronecker's theorem may be stated as follows. Suppose that $\lambda_1, \ldots, \lambda_N$ are N linearly independent real numbers, $\phi_1, \ldots, \phi_N$ are N arbitrary real numbers, and ϵ is positive. Then there exist N integers $h_1, \ldots, h_N$ and a real number t such that

$$|\lambda_n t - \phi_n - h_n| < \epsilon \quad (n = 1, \ldots, N),$$

i.e. such that the N complex numbers

$$e^{2\pi i(\lambda_n t - \phi_n)} \quad (n = 1, \ldots, N)$$

all differ by less than ϵ_1 from $e^0 = 1$. In other words, the upper limit Γ of the numerical value of the function

$$F(t) = 1 + e^{2\pi i(\lambda_1 t - \phi_1)} + \ldots + e^{2\pi i(\lambda_N t - \phi_N)}$$

for real values of t is equal to $1+N$. Evidently it is enough to prove that $1+N \leqslant \Gamma$.

A simple proof of this latter assertion was given some years ago by Bohr†; in this note we shall give a further and perhaps still simpler proof.

We consider Fejér's kernel

$$K_n(t) = \sum_{\nu=-n}^{n} \frac{n-|\nu|}{n} e^{i\nu t} = \frac{1}{n} \left(\frac{\sin \frac{1}{2}nt}{\sin \frac{1}{2}t}\right)^2,$$

and form the composite kernel

$$\mathbf{K}_n(t) = K_n\{2\pi(\lambda_1 t - \phi_1)\} \ldots K_n\{2\pi(\lambda_N t - \phi_N)\}.$$

Multiplying out, and remembering that the λ's are linearly independent, we obtain

$$\mathbf{K}_n(t) = 1 + \frac{n-1}{n}\left(e^{-2\pi i(\lambda_1 t - \phi_1)} + \ldots + e^{-2\pi i(\lambda_N t - \phi_N)}\right) + R(t),$$

where $R(t)$ is a trigonometrical polynomial whose exponents, divided by 2π, are all different from the numbers $0, -\lambda_1, \ldots, -\lambda_N$. Hence

$$F(t)\mathbf{K}_n(t) = 1 + \frac{n-1}{n} N + S(t),$$

* Received 5 October, 1932; read 10 November, 1932.

† H. Bohr, "Another proof of Kronecker's theorem", *Proc. London Math. Soc.* (2), 21 (1922), 315–316.

where $S(t)$ is a polynomial whose exponents are all different from zero; and so

$$1+\frac{n-1}{n}N = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} F(t)\,\mathbf{K}_n(t)\,dt.$$

Now $\mathbf{K}_n(t) \geqslant 0$ for all t and

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\mathbf{K}_n(t)\,dt = 1.$$

It therefore follows that

$$1+\frac{n-1}{n}N \leqslant \Gamma,$$

and finally
$$1+N = \lim_{n\to\infty}\left(1+\frac{n-1}{n}N\right) \leqslant \Gamma.$$

At the suggestion of Prof. Hardy we add the following remark.

Kronecker's theorem is often stated in a slightly different form, namely as follows. Suppose that the numbers $1, \lambda_1, \ldots, \lambda_N$ are linearly independent, $\phi_1, \ldots, \phi_N$ are arbitrary numbers, and ϵ is positive. Then there exist N integers $h_1, \ldots, h_N$ and an integer t such that

$$|\lambda_n t - \phi_n - h_n| < \epsilon \quad (n = 1, \ldots, N).$$

This theorem and the theorem above are easily deduced from each other. It may, however, be of interest to observe that the proof given above is also directly applicable in this case.

In fact, denoting now by Γ the upper limit of the numerical value of $F(t)$ for all integral values of t, we have again to prove that $1+N \leqslant \Gamma$. We define $\mathbf{K}_n(t)$, $R(t)$, and $S(t)$ as before; and we now conclude that the exponents of $R(t)$, divided by 2π, are all incongruent to the numbers $0, -\lambda_1, \ldots, -\lambda_N \pmod 1$. Hence the exponents of $S(t)$, divided by 2π, are not integers. Substituting $\lim_{p\to\infty}\frac{1}{2p+1}\sum_{t=-p}^{p}\ldots$ for $\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\ldots\,dt$, and using the equations

$$\lim_{p\to\infty}\frac{1}{2p+1}\sum_{t=-p}^{p} e^{2\pi i\lambda t} = \begin{cases} 1 \text{ for } \lambda \text{ an integer,} \\ 0 \text{ for } \lambda \text{ not an integer,}\end{cases}$$

instead of

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} e^{2\pi i\lambda t}\,dt = \begin{cases} 1 & \text{for} \quad \lambda = 0, \\ 0 & \text{for} \quad \lambda \neq 0.\end{cases}$$

the proof runs exactly as before.

ZUM KRONECKERSCHEN SATZ.

Von **Harald Bohr** und **Börge Jessen** (Kopenhagen).

Adunanza del 13 novembre 1932.

1. Der KRONECKERsche Satz über diophantische Approximationen kann zum Beispiel folgendermassen formuliert werden:

Es seien die N reellen Zahlen $\lambda_1, \ldots, \lambda_N$ *von einander linear unabhängig, d. h. es bestehe keine Relation* $\nu_1\lambda_1 + \cdots + \nu_N\lambda_N = 0$ *in ganzen, nicht sämtlich verschwindenden Zahlen* $\nu_1, \ldots, \nu_N$; *ferner seien* $\varphi_1, \ldots, \varphi_N$ *beliebige reelle Grössen und* $\varepsilon\left(\leqq \frac{1}{2}\right)$ *eine gegebene positive Zahl. Dann gibt es ein reelles t, so dass die N Ungleichungen*

$$(1) \qquad |\lambda_n t - \varphi_n - g_n| < \varepsilon \qquad (n = 1, \ldots, N)$$

bei passender Wahl von ganzen Zahlen $g_1, \ldots, g_N$ *alle erfüllt sind.*

Für diesen Satz gibt es bekanntlich eine Reihe verschiedenartiger Beweise. In dieser Note soll noch ein Beweis gebracht werden, der uns besonders einfach erscheint. Wie mehrere der früheren Beweise ist auch dieser Beweis von analytischem Charakter. Es mag vielleicht gestattet sein, bevor wir zu der Darstellung des neuen Beweises übergehen, in aller Kürze an einige ebenfalls sehr einfache dieser Beweise zu erinnern; dies geschieht nicht nur zur Orientierung des Lesers, sondern ist auch deshalb angebracht, weil die Idee des neuen Beweises mit früher verwendeten Beweismethoden eng zusammenhängt.

Gemeinsam für alle die zu erörternden Beweise ist die Heranziehung der Exponentialfunktion $e^{2\pi i t}$ als « Invariante mod. Eins », welche auf WEYL zurückgeht [1]). Die lineare Unabhängigkeit der Zahlen λ_n wird überall dazu ausgenutzt um zu sichern,

[1]) H. WEYL, *Über die Gleichverteilung von Zahlen mod. Eins* [Mathematische Annalen, Bd. LXXVII (1916), S 313-352].

dass eine Exponentialfunktion der Form

$$e^{2\pi i(\nu_1\lambda_1+\cdots+\nu_N\lambda_N)t}$$

sich nur dann auf die Konstante $e^0 = 1$ reduziert, wenn die ganzen Zahlen $\nu_1, \ldots, \nu_N$ alle gleich Null sind (oder was auf dasselbe hinauskommt, dass zwei Exponentialfunktionen dieser Form, etwa $e^{2\pi i(\nu_1'\lambda_1+\cdots+\nu_N'\lambda_N)t}$ und $e^{2\pi i(\nu_1''\lambda_1+\cdots+\nu_N''\lambda_N)t}$ nur dann dieselbe Funktion sind, wenn sie formal übereinstimmen, d. h. wenn $\nu_1' = \nu_1'', \ldots, \nu_N' = \nu_N''$). Ferner wird in allen Beweisen von der elementaren Tatsache Gebrauch gemacht, dass bei einem reellen μ

$$M\{e^{i\mu t}\} = \begin{cases} 0 & \text{für } \mu \neq 0 \\ 1 & \text{für } \mu = 0, \end{cases}$$

wobei $M\{f(t)\}$ wie überall im folgenden den Mittelwert

$$M\{f(t)\} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(t)\,dt$$

bezeichnet (speziell existiert also für jedes Exponentialpolynom $f(t) = \sum a_m e^{i\mu_m t}$ der Mittelwert $M\{f(t)\}$ und ist gleich dem konstanten Glied des Polynoms).

2. Ein kurzer und in seinem Ansatz sehr primitiver Beweis rührt von LANDAU her [2]). LANDAU betrachtet diejenige stetige, periodische Funktion $F(t)$ der Periode 1, welche im Periodenintervall $\left(-\frac{1}{2}, \frac{1}{2}\right)$ gleich Null ist, ausser für $-\varepsilon < t < \varepsilon$, wo sie durch $F(t) = \varepsilon - |t|$ definiert wird, und entwickelt sie in ihre (absolut und gleichmässig konvergente) FOURIERreihe

$$F(t) = \sum_{\nu=-\infty}^{\infty} A_\nu e^{2\pi i\nu t}.$$

Sodann bildet er das Produkt

$$\Phi(t) = \prod_{n=1}^{N} F(\lambda_n t - \varphi_n)$$

und schliesst nun wie folgt: Gäbe es kein t, für welches bei passender Wahl der Zahlen $g_1, \ldots, g_N$ die obigen Ungleichungen (1) alle erfüllt wären, so wäre die Funktion $\Phi(t)$ identisch Null. Andererseits aber ergibt sich durch Multiplikation der N FOURIERreihen der einzelnen Faktoren $F(\lambda_n t - \varphi_n)$ für die Funktion $\Phi(t)$ eine (wiederum absolut und gleichmässig konvergente) Reihenentwicklung der Form

$$\Phi(t) = \sum_{m=1}^{\infty} a_m e^{i\mu_m t},$$

[2]) E. LANDAU, *Über Diophantische Approximationen* [Scripta universitatis atque bibliothecae Hierosolymitanarum. I, 1 (1923), S. 1-4].

aus der man unmittelbar ersieht, dass der Mittelwert $M\{\Phi(t)\}$ existiert und gleich dem konstanten Glied dieser Entwicklung ist [übrigens ist die erwähnte Reihe $\sum a_m e^{i\mu_m t}$ einfach die FOURIERreihe der fastperiodischen Funktion $\Phi(t)$]. Dieses konstante Glied ist aber von Null verschieden, weil es — wegen der linearen Unabhängigkeit der Zahlen λ_n — gleich der N-ten Potenz des konstanten Gliedes $A_0 = \varepsilon^2$ der FOURIERreihe von $F(t)$ ist. Aus dem Nichtverschwinden des Mittelwertes $M\{\Phi(t)\}$ folgt aber a fortiori, dass $\Phi(t)$ nicht identisch verschwindet.

3. Ein anderer einfacher, obwohl nicht so primitiver Beweis wurde von BOHR gegeben [3]). Die Behauptung des KRONECKERschen Satzen wird hier zunächst auf die folgende (von ε freie) Form gebracht, dass die obere Grenze Γ des absoluten Betrages des Exponentialpolynoms

$$G(t) = 1 + \sum_{n=1}^{N} e^{2\pi i(\lambda_n t - \varphi_n)},$$

welche offenbar höchstens gleich $1 + N$ ist, tatsächlich den grösstmöglichen Wert $1 + N$ erreicht. Zu diesem Zwecke wird die Funktion der N «freien» Veränderlichen $x_1, \ldots, x_N$

$$G^*(x_1, \ldots, x_N) = 1 + \sum_{n=1}^{N} e^{2\pi i x_n}$$

herangezogen, welche in jeder Veränderlichen periodisch mit der Periode 1 ist, und für welche die obere Grenze des absoluten Betrages offenbar gleich $1 + N$ ist. Diese obere Grenze wird nun in der Form

$$\lim_{p \to \infty} \sqrt[2p]{M\{|G^*(x_1, \ldots, x_N)|^{2p}\}}$$

dargestellt, wo der Mittelwert hier in bezug auf die Variablen $x_1, \ldots, x_N$ zu nehmen ist. Vergleicht man nun die Entwicklungen von $|G(t)|^{2p}$ und $|G^*(x_1, \ldots, x_N)|^{2p}$ (die beide in der üblichen Weise mit Hilfe von $|A|^2 = A\bar{A}$ gewonnen werden), so ergibt sich unmittelbar, da wegen der linearen Unabhängigkeit der Zahlen λ_n die Rechnungen parallel verlaufen, dass

$$M\{|G^*(x_1, \ldots, x_N)|^{2p}\} = M\{|G(t)|^{2p}\},$$

so dass auch

$$\lim_{p \to \infty} \sqrt[2p]{M\{|G(t)|^{2p}\}} = 1 + N$$

ist. Diese Relation zeigt aber, dass die obere Grenze Γ des absoluten Betrages von $G(t)$ nicht kleiner als $1 + N$ sein kann.

---

3) H. BOHR, *Another Proof of* KRONECKER'*s Theorem* [Proceedings of the London Mathematical Society, Series III, vol. XXI (1923), S. 315-316].

4. Neuerdings wurde von den Verfassern ein weiterer sehr kurzer Beweis mitgeteilt, welcher mit noch elementareren Hilfsmitteln als die beiden vorangehenden operiert [4]). Hierbei wurde wieder die obige Formulierung des Satzes

$$\Gamma = \underset{-\infty<t<\infty}{\text{obere Grenze}} |G(t)| = 1 + N$$

zugrundegelegt. Der Beweis basierte nun auf die Heranziehung des Fejérschen Kernes

$$\mathfrak{K}_p(t) = \sum_{\nu=-p}^{p} \left(1 - \frac{|\nu|}{p}\right) e^{2\pi i \nu t}$$

$$= \frac{1}{p} \left(\frac{\sin \pi p t}{\sin \pi t}\right)^2$$

und die Bildung des « zusammengesetzten Kernes »

$$K_p(t) = \prod_{n=1}^{N} \mathfrak{K}_p(\lambda_n t - \varphi_n).$$

Dieser Kern $K_p(t)$ ist, wie der Fejérsche Kern, ein nichtnegatives Exponentialpolynom, dessen konstante Glied, wegen der linearen Unabhängigkeit der Zahlen λ_n, gleich 1 ist.

Es wurden nun die beiden Exponentialpolynome $G(t)$ und $K_p(t)$ mit einander multipliziert; dann ergibt sich ein Exponentialpolynom, dessen konstante Glied, wiederum wegen der linearen Unabhängigkeit der Zahlen λ_n, gleich $1 + N\left(1 - \frac{1}{p}\right)$ ist. Aus den so gewonnenen Relationen

$$K_p(t) \geqq 0, \qquad M\{K_p(t)\} = 1$$

und

$$M\{G(t)K_p(t)\} = 1 + N\left(1 - \frac{1}{p}\right)$$

folgt nun sofort, dass $\Gamma \geqq 1 + N\left(1 - \frac{1}{p}\right)$ und also (da p beliebig ist), dass Γ mindestens gleich $1 + N$ sein muss.

5. Nunmehr gehen wir zu dem neuen Beweis über. Der Grundgedanke dieses Beweises ist derselbe wie der des oben skizzierten Beweises von Landau. Indem wir aber wie in dem vorangehenden Beweis den Fejérschen Kern $\mathfrak{K}_p(t)$, an Stelle der Landauschen Funktion $F(t)$, heranziehen, welcher ebenfalls, obwohl nicht in so primitiver Weise, die ganzzahligen Werte von t hervorhebt, vermeiden wir jede Verwendung von Sätzen aus der Fourierreihentheorie; in der Tat wird in diesem Beweis

---

[4]) H. Bohr and B. Jessen, *One more Proof of* Kronecker's *Theorem* [Journal of the London Mathematical Society, vol. VII (1932), S. 274-275].

wie in den beiden vorangehenden nur mit endlichen Summen und nicht mit unendlichen Reihen gerechnet.

Wir gehen hier wieder von der anfangs gegebenen Fassung des KRONECKERschen Satzes aus, und bestimmen, nachdem zunächst eine positive Zahl $k < \frac{1}{N}$ festgelegt ist, zu dem gegebenen ε ein p so gross, dass

$$\mathfrak{K}_p(t) \leqq k,$$

sobald t von jeder ganzen Zahl um mindestens ε abweicht (d. h. also überall, wo die LANDAUsche Funktion $F(t) = 0$ ist).

Wir bilden nunmehr wieder den zusammengesetzten Kern

$$K_p(t) = \prod_{n=1}^{N} \mathfrak{K}_p(\lambda_n t - \varphi_n) \tag{2}$$

sowie die entsprechenden, aus je $N - 1$ Faktoren bestehenden Kerne

$$K_p^{(n)}(t) = \frac{K_p(t)}{\mathfrak{K}_p(\lambda_n t - \varphi_n)} \qquad (n = 1, \ldots, N).$$

Diese Kerne sind alle nichtnegativ und haben das konstante Glied 1 und also auch den Mittelwert 1.

Gäbe es nun kein t, für welches die N Ungleichungen (1) gleichzeitig erfüllt wären, so wäre also in dem Produkt (2) für jeden Wert von t zumindest einer der Faktoren $\mathfrak{K}_p(\lambda_n t - \varphi_n)$ höchstens gleich k, und es bestünde somit für jeden Wert von t die Ungleichung

$$K_p(t) \leqq k \sum_{n=1}^{N} K_p^{(n)}(t).$$

Somit wäre auch

$$M\{K_p(t)\} \leqq k \sum_{n=1}^{N} M\{K_p^{(n)}(t)\},$$

d. h. $1 \leqq kN$, was der Wahl von k wiederspricht.

6. Der besprochene KRONECKERsche Satz ist bekanntlich der wichtigste und prägnanteste Fall des folgenden allgemeinen KRONECKERschen Satzes über lineare diophantische Approximationen:

Ein System von Ungleichungen der Form

$$|\lambda_{n1} t_1 + \cdots + \lambda_{nM} t_M - \varphi_n - g_n| < \varepsilon \tag{3} \qquad (n = 1, \ldots, N)$$

ist dann und nur dann bei jedem positiven ε in reellen Zahlen $t_1, \ldots, t_M$ und ganzen Zahlen $g_1, \ldots, g_N$ lösbar, wenn bei jedem System von N ganzen Zahlen $\nu_1, \ldots, \nu_N$,

für welches der Ausdruck

$$(4) \qquad \nu_1(\lambda_{11} t_1 + \cdots + \lambda_{1M} t_M) + \cdots + \nu_N(\lambda_{N1} t_1 + \cdots + \lambda_{NM} t_M)$$

identisch in den M Variablen $t_1, \ldots, t_M$ verschwindet, die entsprechende Zahl

$$\nu_1 \varphi_1 + \cdots + \nu_N \varphi_N$$

ganzzahlig ausfällt.

Die Notwendigkeit dieser Bedingung leuchtet ein, wenn man bemerkt, dass sonst für hinreichend kleine ε die Ungleichungen (3) einen offenkundigen Wiederspruch aufweisen würden. Es handelt sich also nur darum, zu zeigen, dass die Bedingung auch hinreichend ist.

Ohne auf Einzelheiten einzugehen, bemerken wir nur, dass jede der oben zum Beweis des spezielleren Satzes verwendeten Methoden nach geringfügiger Modifikation im Stande ist auch diesen allgemeineren Satz zu liefern. Für den in 3. besprochenen Beweis ist die Übertragung auf den allgemeineren Fall schon früher in einer Note von BOHR explizite durchgeführt [5]). Auch in dem allgemeinen Fall dürfte aber die in 5. dargestellte Beweismethode die einfachste sein. Der Beweis verläuft hier wie folgt.

Wiederum von dem FEJÉRschen Kern $\mathfrak{K}_p(t)$ ausgehend wird nun an Stelle der obigen Kerne $K_p(t)$ und $K_p^{(n)}(t)$ die entsprechenden Kerne

$$K_p(t_1, \ldots, t_M) = \prod_{n=1}^{N} \mathfrak{K}_p(\lambda_{n1} t_1 + \cdots + \lambda_{nM} t_M - \varphi_n)$$

und

$$K_p^{(n)}(t_1, \ldots, t_M) = \frac{K_p(t_1, \ldots, t_M)}{\mathfrak{K}_p(\lambda_{n1} t_1 + \cdots + \lambda_{nM} t_M - \varphi_n)} \qquad (n = 1, \ldots, N)$$

gebildet. Diese Kerne sind nichtnegative Exponentialpolynome in den M Veränderlichen $t_1, \ldots, t_M$. Ihre konstanten Glieder — also ihre Mittelwerte in bezug auf die Variablen $t_1, \ldots, t_M$ — brauchen aber nicht mehr gleich 1 zu sein, sondern können unter Umständen grösser als 1 ausfallen. Das konstante Glied A_0 von $K_p(t_1, \ldots, t_M)$ wird nämlich durch den Ausdruck

$$A_0 = 1 + \sum\nolimits_* \left(1 - \frac{|\nu_1|}{p}\right) \cdots \left(1 - \frac{|\nu_N|}{p}\right) e^{-2\pi i(\nu_1\varphi_1 + \cdots + \nu_N \varphi_N)}$$

gegeben, wo die Summe $\sum_*$ über diejenigen von $0, \ldots, 0$ verschiedenen Kombinationen $\nu_1, \ldots, \nu_N$ mit $|\nu_n| \leqq p$ zu erstrecken ist, für welche die Linearform (4) identisch verschwindet; für jede solche Kombination $\nu_1, \ldots, \nu_N$ ist aber der Faktor

[5]) H. BOHR, *Neuer Beweis eines allgemeinen* KRONECKER*schen Approximationssatzes* [D. K. D. Vidensk. Selsk. Math.-fys. Meddelelser VI, 8 (1924), S. 1-8].

$e^{-2\pi i(\nu_1\varphi_1+\cdots+\nu_N\varphi_N)}$ nach Annahme gleich 1, und es ist somit

$$A_0 = 1 + \sum\nolimits_* \left(1 - \frac{|\nu_1|}{p}\right) \cdots \left(1 - \frac{|\nu_N|}{p}\right) \quad (\geqq 1).$$

Entsprechenderweise ergibt sich für das konstante Glied $A_0^{(n)}$ von $K_p^{(n)}(t_1, \ldots, t_M)$ ein Ausdruck der Form

$$A_0^{(n)} = 1 + \sum\nolimits_*^{(n)} \left(1 - \frac{|\nu_1|}{p}\right) \cdots \left(1 - \frac{|\nu_N|}{p}\right),$$

wo jetzt die Summe $\sum_*^{(n)}$ nur über diejenigen der obigen Kombinationen $\nu_1, \ldots, \nu_N$ zu erstrecken ist, für welche $\nu_n = 0$ ist. Entscheidend für das folgende ist nur, dass für jedes n dieses Glied $A_0^{(n)}$ jedenfalls nicht grösser als das Glied A_0 ist.

Wären nun die Ungleichungen (3) nicht lösbar, so wäre also für jedes Wertesystem $t_1, \ldots, t_M$ zumindest eine der N Faktoren $\mathfrak{K}_p(\lambda_{n1}t_1 + \cdots + \lambda_{nM}t_M - \varphi_n)$ in $K_p(t_1, \ldots, t_M)$ höchstens gleich k, und es gälte somit für alle Werte von $t_1, \ldots, t_M$ die Ungleichung

$$K_p(t_1, \ldots, t_M) \leqq k \sum_{n=1}^{N} K_p^{(n)}(t_1, \ldots, t_M).$$

Hieraus ergibt sich aber durch Mittelwertbildung in bezug auf die Variablen $t_1, \ldots, t_M$, dass

$$A_0 \leqq k \sum_{n=1}^{N} A_0^{(n)},$$

und also a fortiori dass $A_0 \leqq kNA_0$, womit wir wieder zu dem obigen Wiederspruch $1 \leqq kN$ gelangt sind.

Kopenhagen, October 1932.

HARALD BOHR und BÖRGE JESSEN.

Endnu engang Kroneckers Sætning.

Af Harald Bohr.

I forrige Aargang af Matematisk Tidsskrift var der flere Gange Tale om Kroneckers Approksimationssætning om diofantiske Uligheder, idet dels Prof. Nielsen (S. 29 – 42) dels Dr. Jessen og jeg (S. 53 – 58) diskuterede denne Sætning og gav nye Beviser for den, henholdsvis af geometrisk og af analytisk Art. Naar jeg alligevel nu igen tillader mig at komme tilbage til Sætningen, er det, fordi jeg senere har fundet et nyt analytisk Bevis for den, som jeg tror næppe kan være simplere — selvom jeg er klar over, hvor farlige den Slags „Forhaabninger“ i Reglen er — og som jeg derfor tænkte mig, det maaske kunde være naturligt at fremstille her i Tidsskriftet, hvor de nævnte andre Beviser er fremkommet.

Nedenfor beviser jeg først den „lille“ Kroneckerske Sætning, d. v. s. den berømte Approksimationssætning om lineære diofantiske Uligheder med een Variabel og lineært uafhængige Koefficienter. Dernæst viser jeg, hvorledes den anvendte Bevismetode — det samme gælder forøvrigt, som Dr. Jessen og jeg har fremhævet, om forskellige af de tidligere anvendte Metoder – umiddelbart kan almindeliggøres saaledes, at den ogsaa kan benyttes til Bevis for den saakaldte „store“ Kroneckerske Sætning, som omhandler vilkaarlige lineære diofantiske Uligheder med vilkaarlig mange Ubekendte. Min Fremstilling har jeg med Forsæt gjort lidt bred, for at den let skal kunne læses, ogsaa af Læsere, der ikke er fortrolige med Emnet eller kender de ovennævnte tidligere Afhandlinger. Det eneste Hjælpemiddel, der benyttes i Beviset (ligesom forøvrigt i Dr. Jessens og mit tidligere Bevis), er den klassiske elementære Relation

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} e^{i\alpha t}\,dt = \begin{cases} 0 & \text{for } \alpha \text{ reel} \neq 0, \\ 1 & \text{for } \alpha = 0. \end{cases} \tag{0}$$

Den „lille“ Kroneckerske Sætning.

Lad $\lambda_1, \lambda_2, \cdots, \lambda_N$ være N givne (reelle) Tal. Skal de N diofantiske Uligheder*)

*) Ved $|a| < b \pmod 1$, hvor a og $b > 0$ er reelle Tal, forstaas, at der findes et helt Tal g saaledes, at $|a-g| < b$.

$$\begin{aligned} &|\lambda_1 t-\varphi_1| < \varepsilon \quad (\text{mod } 1) \\ &|\lambda_2 t-\varphi_2| < \varepsilon \quad (\text{mod } 1) \\ &\ldots\ldots\ldots\ldots \\ &|\lambda_N t-\varphi_N| < \varepsilon \quad (\text{mod } 1) \end{aligned} \tag{1}$$

for vilkaarligt valgte Tal $\varphi_1, \varphi_2, \cdots, \varphi_N$ og et vilkaarligt lille $\varepsilon > 0$ have en Løsning $t = t_0$, kræves øjensynlig, at Tallene $\lambda_1, \lambda_2, \cdots, \lambda_N$ er lineært uafhængige, d. v. s. at Størrelsen

$$g_1 \lambda_1 + g_2 \lambda_2 + \cdots + g_N \lambda_N$$

hvor $g_1, g_2, \cdots, g_N$ er hele Tal, kun bliver 0 i det Tilfælde, hvor alle g^{erne} er 0; thi fandtes der en Relation af Formen

$$G_1 \lambda_1 + G_2 \lambda_2 + \cdots + G_N \lambda_N = 0,$$

hvori ikke alle G^{er} var 0, og multiplicerede vi de N Uligheder (1) med henholdsvis $G_1, G_2, \cdots, G_N$ og lagde dem derefter sammen, vilde vi jo faa

$$|G_1 \varphi_1 + G_2 \varphi_2 + \cdots + G_N \varphi_N| < \varepsilon(|G_1| + |G_2| + \cdots + |G_N|) \quad (\text{mod } 1)$$

altsaa, da ε kan vælges vilkaarlig lille,

$$G_1 \varphi_1 + G_2 \varphi_2 + \cdots + G_N \varphi_N \equiv 0 \qquad (\text{mod } 1)$$

i aabenbar Strid med, at φ^{erne} skulde kunne vælges fuldkommen vilkaarligt.

Den (lille) Kroneckerske Sætning udsiger nu, at den fundne nødvendige Betingelse, nemlig *at Tallene* $\lambda_1, \lambda_2, \cdots, \lambda_N$ *skal være lineært uafhængige,* ogsaa er tilstrækkelig for, *at Ulighederne (1) kan løses for vilkaarlig valgte* φ^{er} *og vilkaarlig valgt* ε.

Idet vi indfører Eksponentialfunktionen $e^{2\pi i x}$ („Invarianten mod 1") kan vi ogsaa udtrykke Kroneckers Sætning saaledes: Er $\lambda_1, \lambda_2, \cdots, \lambda_N$ lineært uafhængige, og $\varphi_1, \varphi_2, \cdots, \varphi_N$ vilkaarlige Tal, kan t vælges saaledes, at de N komplekse Tal

$$e^{2\pi i(\lambda_1 t-\varphi_1)}, \; e^{2\pi i(\lambda_2 t-\varphi_2)}, \ldots, \; e^{2\pi i(\lambda_N t-\varphi_N)}$$

alle afviger vilkaarlig lidt fra $e^0 = 1$, altsaa at de N tilsvarende „Vektorer" (af Længden 1) alle, med vilkaarlig foreskreven Nøjagtighed, er rettet i Retning af den positive reelle Akse.

Sætningen siger med andre Ord, at øvre Grænse af den numeriske Værdi af Funktionen

$$f(t) = 1 + e^{2\pi i(\lambda_1 t - \varphi_1)} + e^{2\pi i(\lambda_2 t - \varphi_2)} + \dots + e^{2\pi i(\lambda_N t - \varphi_N)}$$
$$(-\infty < t < \infty)$$

er lig med $N+1$, d. v. s. lig med den Værdi, som Funktionen

$$F(x_1, x_2, \cdots x_N) = 1 + x_1 + x_2 + \cdots + x_N$$

i de N uafhængige komplekse Variable $x_1, x_2, \cdots, x_N$ antager for $x_1 = x_2 = \cdots = x_N = 1$.

Bevis: Vi betragter (for et vilkaarligt positivt helt Tal p) de p^{te} Potenser

$$\{f(t)\}^p = \sum a_\nu e^{i\beta_\nu t} \tag{2}$$

og

$$\{F(x_1, x_2, \cdots x_N)\}^p = \sum a_{n_1 n_2 \ldots n_N} x_1^{n_1} x_2^{n_2} \cdots x_N^{n_N}. \tag{3}$$

Da λ^{erne} er lineært uafhængige, kan to forskellige Led i Polynomialudviklingen af $\{f(t)\}^p$ aldrig trækkes sammen, d. v. s. de kan ikke indeholde den samme Eksponentialfaktor $e^{i\beta t}$, og Koefficienterne a_ν i (2) maa derfor have de samme numeriske Værdier som de „tilsvarende" positive Koefficienter $a_{n_1 n_2 \ldots n_N}$ i (3). Følgelig er

$$\sum |a_\nu| = \sum a_{n_1 n_2 \ldots n_N} = \{F(1, 1, \cdots, 1)\}^p = (N+1)^p.$$

Vi drejer nu Beviset indirekte og antager altsaa, at der findes en Konstant $k < N+1$, saaledes at

$$|f(t)| \leqq k \qquad \text{for } -\infty < t < \infty.$$

I Følge (0) vilde da enhver af Koefficienterne a_ν i (2) tilfredsstille Uligheden

$$|a_\nu| = \left| \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \{f(t)\}^p e^{-i\beta_\nu t} dt \right| \leqq k^p.$$

Nu er imidlertid Antallet af Led i Polynomialudviklingen (2) eller (3) øjensynlig $\leqq (p+1)^N$ [forøvrigt, som velkendt, lig med $\binom{N+p}{p}$], og vi vilde følgelig faa

$$\sum |a_\nu| \leqq (p+1)^N \cdot k^p.$$

Denne Ulighed strider imidlertid for tilstrækkelig store p mod den ovenfor udledte Ligning $\sum |a_\nu| = (N+1)^p$, idet jo

$$\frac{(p+1)^N \cdot k^p}{(N+1)^p} = (p+1)^N \left(\frac{k}{N+1}\right)^p \to 0 \qquad \text{for } p \to \infty .$$

Den „store" Kroneckerske Sætning.

Her betragtes N lineære diofantiske Uligheder i M Ubekendte $t_1, t_2, \cdots, t_M$

$$\begin{array}{lll} |\lambda_{11} t_1 + \lambda_{12} t_2 + \cdots + \lambda_{1M} t_M - \varphi_1 | < \varepsilon & \pmod 1 & \\ |\lambda_{21} t_1 + \lambda_{22} t_2 + \cdots + \lambda_{2M} t_M - \varphi_2 | < \varepsilon & \pmod 1 & \quad (\mathrm{I}) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & & \\ |\lambda_{N1} t_1 + \lambda_{N2} t_2 + \cdots + \lambda_{NM} t_M - \varphi_N | < \varepsilon & \pmod 1, & \end{array}$$

og der spørges om, hvorledes λ^{erne} og φ^{erne} skal være beskafne, for at disse Uligheder (I) for ethvert Valg af $\varepsilon > 0$ skal have en Løsning $t_1, t_2, \cdots, t_M$.

En nødvendig Betingelse herfor er øjensynlig, *at det for ethvert System af N hele Tal $g_1, g_2, \cdots, g_N$, for hvilket Udtrykket*

$$g_1(\lambda_{11} t_1 + \cdots + \lambda_{1M} t_M) + g_2(\lambda_{21} t_1 + \cdots + \lambda_{2M} t_M) + \cdots + g_N(\lambda_{N1} t_1 + \cdots + \lambda_{NM} t_M)$$

forsvinder identisk i t^{erne}, gælder, at Tallet

$$g_1 \varphi_1 + g_2 \varphi_2 + \cdots + g_N \varphi_N$$

er et helt Tal.

Dette indses, ganske som ovenfor, ved blot at multiplicere Ligningerne (I) med henholdsvis $g_1, g_2, \cdots, g_N$ og lægge dem sammen, hvorved faas, at

$$|g_1 \varphi_1 + g_2 \varphi_2 + \cdots + g_N \varphi_N| < \varepsilon(|g_1| + |g_2| + \cdots + |g_N|) \pmod 1,$$

altsaa — da det skal gælde for ethvert ε — at

$$g_1 \varphi_1 + g_2 \varphi_2 + \cdots + g_N \varphi_N = 0 \pmod 1.$$

Kroneckers „store" Sætning udsiger nu, *at den angivne nødvendige Betingelse ogsaa er en tilstrækkelig Betingelse for, at de N diofantiske Uligheder (I) kan løses for et vilkaarligt lille ε.* Med andre Ord: De nævnte N diofantiske Uligheder kan altid løses, medmindre de ei i „aabenbar" Modstrid med hinanden.

Bevis: Til Afkortning sætter vi

$$\lambda_{n1} t_1 + \lambda_{n2} t_2 + \cdots + \lambda_{nM} t_M = L_n \qquad (n = 1, 2, \cdots, N).$$

Det drejer sig om at vise, at der findes saadanne Værdier af $t_1, t_2, \cdots, t_M$, at de N Vektorer af Længden 1

$$e^{2\pi i (L_1 - \varphi_1)}, \; e^{2\pi i (L_2 - \varphi_2)}, \ldots, \; e^{2\pi i (L_n - \varphi_N)}$$

alle med vilkaarlig foreskreven Nøjagtighed er rettet efter den positive reelle Akse, altsaa at øvre Grænse af den numeriske Værdi af Funktionen

$$f(t_1, t_2, \cdots, t_M) = 1 + e^{2\pi i (L_1 - \varphi_1)} + \ldots + e^{2\pi i (L_N - \varphi_N)}$$

er lig med $N + 1$, d. v. s. lig med den Værdi, som Funktionen

$$F(x_1, x_2, \cdots, x_N) = 1 + x_1 + \cdots + x_N$$

i de N uafhængige komplekse Variable $x_1, x_2, \cdots, x_N$ antager for $x_1 = x_2 = \cdots = x_N = 1$.

Ligesom før betragter vi de p^{te} Potenser

$$\left\{f(t_1, \cdots, t_M)\right\}^p = \sum a_\nu e^{i(\beta'_\nu t_1 + \ldots + \beta_\nu^{(M)} t_M)} \tag{II}$$

og

$$\left\{F(x_1, \cdots, x_N)\right\}^p = \sum a_{n_1 n_2 \ldots n_N} x_1^{n_1} x_2^{n_2} \cdots x_N^{n_N}. \tag{III}$$

Her kan det imidlertid ved Polynomialudviklingen af venstre Side af (II) meget vel ske, at to (eller flere) Led kommer til at indeholde den samme Eksponentialfaktor $e^{i(\beta' t_1 + \ldots + \beta^{(M)} t_M)}$ og derfor skal sammendrages til eet Led ved Dannelsen af Resultatet paa højre Side af (II). Dette vil dog kun da ske for to Led med Eksponentialfaktorerne

$$e^{2\pi i (n'_1 L_1 + \ldots + n'_N L_N)} \quad \text{og} \quad e^{2\pi i (n''_1 L_1 + \ldots + n''_N L_N)},$$

naar de hele Tal

$$g_1 = n_1' - n_1'', \; g_2 = n_2' - n_2'', \cdots, \; g_N = n_N' - n_N''$$

er saaledes beskafne, at Udtrykket

$$g_1 L_1 + \cdots + g_N L_N = g_1 (\lambda_{11} t_1 + \cdots + \lambda_{1M} t_M) + \cdots + g_N (\lambda_{N1} t_1 + \cdots + \lambda_{NM} t_M)$$

forsvinder identisk i t^{erne}. Men i dette Tilfælde vil efter For-

udsætning Tallet $g_1\varphi_1+\cdots+g_N\varphi_N$ være et helt Tal, d. v. s. de to Størrelser

$$n_1'\varphi_1+\cdots+n_N'\varphi_N \quad \text{og} \quad n_1''\varphi_1+\cdots+n_N''\varphi_N$$

afviger indbyrdes med et helt Tal, og det (komplekse) „Fortegn" for de to Koefficienter til de to omhandlede Led vil derfor være det samme. Følgelig gælder det ogsaa her – ligesom i Beviset for den lille Kroneckerske Sætning – at

$$\sum|a_\nu| = \sum a_{n_1 n_2 \ldots n_N} = (N+1)^p.$$

Nu løber Beviset til Ende ganske som før. Var Sætningen forkert, altsaa

$$|f(t_1, t_2, \cdots, t_M)| \leqq k < N+1$$

for alle $t_1, t_2, \cdots, t_M$, maatte i Følge (0) alle Koefficienter a_ν tilfredsstille Uligheden

$$= \left|\lim_{\substack{T_1\to\infty\\ \cdots\cdots\\ T_M\to\infty}} \frac{1}{2^M T_1\cdots T_M}\int_{-T_1}^{T_1}\cdots\int_{-T_M}^{T_M}\{f(t_1,\cdots t_M)\}^p e^{-i(\beta'_\nu t_1+\cdots+\beta_\nu^{(M)}t_M)}dt_1\cdots dt_M\right| \leqq k^p$$

og da Antallet af Led i (III) og følgelig yderligere i (II) er $\leqq (p+1)^N$, maatte

$$\sum|a_\nu| \leqq (p+1)^N\cdot k^p$$

i Modstrid med Ligningen $\sum|a_\nu| = (N+1)^p$.

Anwendung einer Landauschen Beweismethode auf den Kroneckerschen Approximationssatz.

(Auszug aus einem Briefe an Professor Landau.)

Von

Harald Bohr in Kopenhagen.

In Ihrem äußerst geistreichen neuen Beweis (Göttinger Nachr. 1933) des alten Satzes von mir, daß die Riemannsche Zetafunktion $\zeta(\sigma+it)$ in der Halbebene $\sigma > 1$ jeden Wert $c \neq 0$ annimmt, ist es Ihnen unter anderem gelungen, die Verwendung des Kroneckerschen Approximationssatzes zu vermeiden — und trotzdem Funktionen von mehreren freien Veränderlichen heranzuziehen. Beim Studium Ihres Beweises habe ich nun bemerkt, daß eine der diesem Beweis zugrunde liegenden Ideen unmittelbar zu einem neuen Beweis des Kroneckerschen Satzes verwendet werden kann. Obwohl es einfachere Beweise dieses letzteren Satzes gibt, scheint der so entstehende neue Beweis, den ich in den folgenden Zeilen darstellen werde, mir trotzdem von einem gewissen Interesse zu sein, vor allem, weil er die erwähnte schöne Beweisidee von Ihnen besonders klar hervortreten läßt.

Außer ganz geläufigen Hilfsmitteln werden bei diesem neuen Beweis des Kroneckerschen Satzes nur die beiden folgenden Bemerkungen benutzt (die übrigens triviale Spezialfälle von allgemeinen Sätzen über fastperiodische Funktionen sind): Falls eine Funktion $\psi(t)$ $(-\infty < t < \infty)$ in eine (endliche oder unendliche) Reihe der Form $\Sigma \alpha_\nu e^{i\beta_\nu t}$ mit reellen, unter einander verschiedenen β_ν und konvergenter Majorantenreihe $\Sigma|\alpha_\nu|$ entwickelbar ist, so gilt:

1. Die Entwicklung ist eindeutig bestimmt.
2. Jeder Koeffizient α_ν ist numerisch $\leqq \underset{-\infty<t<\infty}{\text{Obere Grenze}} |\psi(t)|$.

Beides folgt sofort aus der, unter der gemachten Annahme der Konvergenz von $\Sigma|\alpha_\nu|$ trivialen Koeffizientendarstellungsformel

$$\alpha_\nu = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \psi(t)\, e^{-i\beta_\nu t}\, dt.$$

Ich spreche den Kroneckerschen Satz in der folgenden analytischen Fassung aus:

Es seien $\lambda_1, \ldots, \lambda_N$ linear unabhängige und $\varphi_1, \ldots, \varphi_N$ beliebige reelle Größen. Ferner seien $N+1$ positive Größen $a_0, \ldots, a_N$ mit der Summe 1 festgewählt $\left(\text{z. B. } a_0 = \ldots = a_N = \frac{1}{N+1}\right)$. Dann ist die obere Grenze des absoluten Betrages der Summe

$$f(t) = a_0 + a_1 e^{2\pi i(\lambda_1 t - \varphi_1)} + \ldots + a_N e^{2\pi i(\lambda_N t - \varphi_N)} \qquad (-\infty < t < \infty)$$

gleich 1.

Der Beweis ist indirekt zu führen. Es sei also die Existenz einer positiven Konstanten $k < 1$ derart angenommen, daß die Ungleichung $|f(t)| \leqq k$ für alle t besteht.

Es wird nun die Funktion

$$g(t) = \frac{1}{1-f(t)} = 1 + f(t) + \ldots + (f(t))^n + \ldots \qquad (-\infty < t < \infty)$$

betrachtet. Hierbei sind — wegen der linearen Unabhängigkeit der Zahlen λ_ν — in der Polynomialentwicklung

$$(f(t))^n = \sum_{l_j \geqq 0,\, l_1 + \ldots + l_N \leqq n} g^{(n)}_{l_1, \ldots, l_N} e^{2\pi i t(l_1\lambda_1 + \ldots + l_N\lambda_N)}$$

keine zwei Glieder zusammenzufassen; daher ist nach der obigen Bemerkung 2. jeder der Koeffizienten $g^{(n)}$ numerisch $\leqq k^n$, und es gilt also — da die Anzahl der Glieder $\binom{N+n}{n}$ beträgt — die Abschätzung

$$\sum |g^{(n)}_{l_1, \ldots, l_N}| \leqq \binom{N+n}{n} k^n.$$

Somit konvergiert die unendliche Majorantenreihe

$$\sum_{n=0}^{\infty} \sum |g^{(n)}_{l_1, \ldots, l_N}|,$$

nämlich mit einer Summe $\leqq$

$$\sum_{n=0}^{\infty} \binom{N+n}{n} k^n = \frac{1}{(1-k)^{N+1}}.$$

Folglich gilt für $g(t)$ eine absolut konvergente Entwicklung der Form

$$g(t) = \sum_{l_j \geqq 0} g_{l_1, \ldots, l_N} e^{2\pi i t(l_1\lambda_1 + \ldots + l_N\lambda_N)}.$$

Nun ist aber $(1 - f(t)) \cdot g(t) = 1$ (für alle t), also

$$(*) \quad \left(1 - a_0 - \sum_{\nu=1}^{N} a_\nu e^{2\pi i(t\lambda_\nu - \varphi_\nu)}\right) \cdot \sum g_{l_1, \ldots, l_N} e^{2\pi i t(l_1\lambda_1 + \ldots + l_N\lambda_N)} = 1,$$

und nach der obigen Bemerkung 1. kann daher die durch gliedweises Ausmultiplizieren der beiden Faktoren auf der linken Seite und nachfolgende Zusammenfassung von Gliedern mit demselben Exponentialfaktor $e^{i\beta_\nu t}$ entstehende, wiederum absolut konvergente Reihe nur aus dem

einzigen konstanten Gliede 1 bestehen, d. h. die Relation(*) gilt auch im Sinne „formaler“ Reihenmultiplikation.

Nunmehr ersetzen wir in der Relation(*) überall die Größen $e^{2\pi i t \lambda_\nu}$ $(\nu = 1, \ldots, N)$ durch freie Veränderliche x_ν. Hierbei geht (*) — da das Ausrechnen der linken Seite nach dieser Ersetzung dem Ausrechnen vor der Ersetzung (wiederum wegen der linearen Unabhängigkeit der λ_ν) völlig parallel verläuft — in die jedenfalls „formal“ gültige Reihenrelation

$$\left(1 - a_0 - \sum_{\nu=1}^{N} a_\nu x_\nu e^{-2\pi i \varphi_\nu}\right) \cdot \sum g_{l_1, \ldots, l_N} x_1^{l_1} \ldots x_N^{l_N} = 1$$

über, welche aber jedenfalls für $|x_1| = \ldots = |x_N| = 1$ (wo die unendliche Reihe links absolut konvergiert) als „reale“ Funktionenrelation gültig ist, etwa

$$\left(1 - a_0 - \sum_{\nu=1}^{N} a_\nu x_\nu e^{-2\pi i \varphi_\nu}\right) \cdot G(x_1, \ldots, x_N) = 1.$$

Dies letzte Resultat ist aber offenbar unsinnig, da der erste Faktor links für $x_\nu = e^{2\pi i \varphi_\nu} (\nu = 1, \ldots, N)$ verschwindet. Hiermit ist der Widerspruch erreicht und der Beweis vollendet.

(Eingegangen am 16. Juli 1933.)

AGAIN THE KRONECKER THEOREM.

HARALD BOHR*.

There are perhaps rather too many different analytical proofs of the famous approximation theorem of Kronecker, a fact for which the present author is partly responsible. Nevertheless, I shall allow myself to give one more which seems to me especially simple and elementary.

As in an earlier proof of mine†, I state the theorem in the following form:

Let $\lambda_1, \ldots, \lambda_N$ *be linearly independent and* $\phi_1, \ldots, \phi_N$ *arbitrary given (real) numbers. Then the upper limit of the absolute value of the function*

$$f(t) = 1 + e^{2\pi i(\lambda_1 t - \phi_1)} + \ldots + e^{2\pi i(\lambda_N t - \phi_N)} \quad (-\infty < t < \infty)$$

is equal to $N+1$, *i.e. is equal to the value of the function of* N *independent complex variables*

$$F(x_1, \ldots, x_N) = 1 + x_1 + \ldots + x_N$$

at the point $(x_1, \ldots, x_N) = (1, \ldots, 1)$.

As in the earlier proof, I start by comparing (for an arbitrary positive integer p) the p-th powers

$$\{f(t)\}^p = \Sigma\, a_\nu e^{i\beta_\nu t} \tag{1}$$

and

$$\{F(x_1, \ldots, x_N)\}^p = \Sigma\, a_{n_1, \ldots, n_N} x_1^{n_1} \ldots x_N^{n_N}. \tag{2}$$

Since the λ_n are linearly independent, no two terms in $\{f(t)\}^p$ can have the same exponential factor, and so the coefficients a_ν in (1) have the same absolute values as the "corresponding" positive coefficients $a_{n_1, \ldots, n_N}$ in (2). Hence

$$\Sigma |a_\nu| = \Sigma\, a_{n_1, \ldots, n_N} = \{F(1, \ldots, 1)\}^p = (N+1)^p. \tag{3}$$

Now I turn the proof indirectly, assuming the existence of a constant $k < N+1$ such that $|f(t)| \leqslant k$ for $-\infty < t < \infty$. Then each coefficient a_ν in (1) would satisfy the inequality

$$|a_\nu| = \left| \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \{f(t)\}^p e^{-i\beta_\nu t} dt \right| \leqslant k^p,$$

* Received 28 September, 1933; read 16 November, 1933.

† H. Bohr, *Proc. London Math. Soc.* (2), 21 (1922), 315–316. Other proofs of Kronecker's theorem have been given recently in the *Journal* by H. Bohr and B. Jessen [7 (1932), 274–275] and by T. Estermann [8 (1933), 18–20.]

and, since the number of terms in the polynomial development (1) or (2) is less than $(p+1)^N$, we should get

$$\Sigma |a_\nu| < (p+1)^N k^p.$$

But, for sufficiently large p, this is an obvious contradiction of (3), since

$$\frac{(p+1)^N k^p}{(N+1)^p} = (p+1)^N \left(\frac{k}{N+1}\right)^p \to 0$$

as $p \to \infty$.

DET KGL. DANSKE VIDENSKABERNES SELSKAB
MATEMATISK-FYSISKE MEDDELELSER, BIND XXIV, NR. 12

INFINITE SYSTEMS OF LINEAR CONGRUENCES WITH INFINITELY MANY VARIABLES

BY

HARALD BOHR AND ERLING FØLNER

KØBENHAVN
I KOMMISSION HOS EJNAR MUNKSGAARD
1948

 Dan. Mat. Fys. Medd. 24, no. 12 (1948).

TABLE OF CONTENTS

Printed in Denmark
Bianco Lunos Bogtrykkeri

§ 1. Introduction.

In the present paper we shall investigate a general problem concerning an arbitrary enumerable system of linear congruences with an enumerable number of variables

$$
(1) \qquad \begin{array}{l}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n_1}x_{n_1} \equiv \theta_1 \pmod 1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n_2}x_{n_2} \equiv \theta_2 \pmod 1 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{array}
$$

where every congruence only contains a finite number of variables and the a's and the θ's are arbitrary (real) numbers.

By the consideration of certain classifications of the almost periodic functions one of the authors[1] met with a problem concerning a system of congruences of the above form but in the special case where all the a's were *rational* numbers. The problem was to give a convenient necessary and sufficient condition on the system of linear forms

$$
(2) \qquad \begin{array}{l}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n_1}x_{n_1} \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n_2}x_{n_2} \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{array}
$$

in order that it possesses the following property: For every choice of the numbers $\theta_1, \theta_2, \cdots$ for which any finite subsystem of the system of congruences (1) has a solution[2]—or, what amounts to the same, for which for any N the system of the N first of

1) H. Bohr: Unendlich viele lineare Kongruenzen mit unendlich vielen Unbekannten. Kgl. Danske Videnskabernes Selskab. Math.-fys. Meddelelser, Bind VII, 1925. In the following this paper is cited by (I). We do not, however, assume the reader to be acquainted with (I).

2) It will be convenient to interpret, not only a solution of the whole system (1), but also a solution of a finite subsystem of (1) as a point $(x_1, x_2, \cdots)$ in the infinite-dimensional space, although for a subsystem only a finite number of the variables really enters in the congruences in question (and the rest of the variables therefore can be chosen quite arbitrarily).

the congruences (1) has a solution—there shall exist a solution of the whole system (1).

If instead of the congruences (1) we consider the corresponding system of equations (now without limitation to rational coefficients) there exists no analogous problem. In fact, it follows from a general investigation of Toeplitz on such systems of equations that for an arbitrary given system the existence of a solution of any finite subsystem always will involve the existence of a solution of the whole system of equations. A direct proof of this special theorem can be found in the paper (I).

That the analogous theorem really is not true for congruences (not even if we restrict ourselves to rational coefficients) can be seen from the following simple example where, moreover, only a single variable x_1 explicitly enters (all the other variables $x_2, x_3, \cdots$ having the coefficients 0).

Example 1. We consider the system of congruences

$$\begin{aligned} \frac{1}{3}x_1 &\equiv \theta_1 \pmod 1 \\ \frac{1}{9}x_1 &\equiv \theta_2 \pmod 1 \\ &\cdots\cdots\cdots \\ \frac{1}{3^n}x_1 &\equiv \theta_n \pmod 1 \\ &\cdots\cdots\cdots \end{aligned}$$

for $\theta_1 = \theta_2 = \cdots = \frac{1}{2}$. The solutions of the n^{th} congruence are all points $(x_1, x_2, \cdots)$ where $x_2, x_3, \cdots$ are arbitrary numbers and x_1 is a number from the "lattice" $x_1 \equiv \frac{3^n}{2} \pmod{3^n}$. These solutions are also solutions of the $(n-1)^{\text{th}}$ congruence, for if $x_1 \equiv \frac{3^n}{2} \pmod{3^n}$ then also $x_1 \equiv \frac{3^n}{2} \pmod{3^{n-1}}$, i. e. $x_1 \equiv \frac{3^{n-1}}{2} \pmod{3^{n-1}}$, since $\frac{3^n}{2} = \frac{3^{n-1}}{2} + 3^{n-1}$. Hence for every N the N first congruences have solutions, viz. all the solutions $x_1 \equiv \frac{3^N}{2} \pmod{3^N}$ of the N^{th} congruence. But nevertheless, there is no solution of the whole system of congruences, for if $(x_1, x_2, \cdots)$ is a solution of the N^{th} congruence then $|x_1| \geqq \frac{3^N}{2}$ which $\to \infty$ for $N \to \infty$.

For a given system of linear forms (2) we shall denote by π_1 the set of points $(\theta_1, \theta_2, \cdots)$ for which the corresponding infinite

system (1) has a solution, and by π_2 the set of points $(\theta_1, \theta_2, \cdots)$ for which any finite subsystem of (1) has a solution. It is plain that $\pi_1 \subseteq \pi_2$ and that both sets contain the point $(0, 0, \cdots)$.

The previous, in (I) treated, problem was to indicate a necessary and sufficient condition that a given system of linear forms (2) with rational coefficients have $\pi_1 = \pi_2$. Before stating the result we shall have to mention the notion of a *substitution* in an enumerable number of variables. A substitution is a linear transformation of the form

$$
\begin{aligned}
y_1 &= \alpha_{11}x_1 + \alpha_{12}x_2 + \cdots + \alpha_{1p_1}x_{p_1} \\
y_2 &= \alpha_{21}x_1 + \alpha_{22}x_2 + \cdots + \alpha_{2p_2}x_{p_2} \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned} \tag{3}
$$

which establishes a one-to-one mapping of the whole infinite-dimensional space on the *whole* infinite-dimensional space. As shown in (I) (cp. also § 4 of the present paper) a necessary and sufficient condition that the transformation (3) be a substitution is that no linear dependance exists amongst (any finite number of) the linear forms on the right-hand side of (3) and that each of the variables x_m can be "isolated", i. e. written as a linear combination of a finite number of the linear forms. In particular, any substitution has an "inverse substitution"

$$
\begin{aligned}
x_1 &= \beta_{11}y_1 + \beta_{12}y_2 + \cdots + \beta_{1q_1}y_{q_1} \\
x_2 &= \beta_{21}y_1 + \beta_{22}y_2 + \cdots + \beta_{2q_2}y_{q_2} \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}
$$

If a substitution is applied to a linear form we get a new linear form. The importance of substitutions in our problem is plain because a substitution applied to a system of linear forms will not change any of the sets π_1 and π_2 simply because two linear forms which correspond by the substitution will take the same value for corresponding values of the variables.

The solution of the former problem can now be stated as follows. *A necessary and sufficient condition that a system of linear forms with rational coefficients have $\pi_1 = \pi_2$ is that the system by a substitution can be transfered into an integral system, i. e. a system with mere integral coefficients.*

We remark, for orientation, that the *sufficiency* of the condition is easy to prove. In fact, on account of the invariance of the sets π_1 and π_2 by a substitution (applied to the linear forms) we need only show that every integral system (2) has $\pi_1 = \pi_2$. Denoting by $(\theta_1, \theta_2, \cdots)$ an arbitrary point from π_2 we shall show that it also lies in π_1. Let $P_N = (\xi_1^{(N)}, \xi_2^{(N)}, \cdots)$ be a solution of the N first congruences (1), $N = 1, 2, \cdots$. Since all a's are integral we can assume all ξ's reduced modulo 1 to lie in the interval $0 \leqq \xi < 1$. Hence we can choose a subsequence P_{N_p}, $p = 1, 2, \cdots$, of the sequence P_N, such that every coordinate sequence $\xi_i^{(N_p)}$ (i fixed) converges towards a number ξ_i for $p \to \infty$. The "limit-point" $(\xi_1, \xi_2, \cdots)$ will then be a solution of all the congruences (1), for if N_0 is an arbitrary positive integral number then $(\xi_1, \xi_2, \cdots)$ from continuity reasons will satisfy the N_0^{th} congruence because this congruence only contains a finite number of variables and the point $(\xi_1^{(N_p)}, \xi_2^{(N_p)}, \cdots)$ for every $p \geqq N_0$ is a solution of the congruence.—The real problem in (I) was to show the necessity of the condition, i. e. that amongst the rational systems there are no other systems than those mentioned above which have $\pi_1 = \pi_2$.

In the present paper we shall treat the corresponding problem for congruences with *arbitrary* coefficients. Also in this general case the systems with $\pi_1 = \pi_2$ can be characterized as systems which by substitutions can be transfered into systems of a certain simple type, denoted by S, which obviously has $\pi_1 = \pi_2$ and whose algebraic structure can be accounted for.

By *a system of linear forms of the type S* we shall understand a system where certain of the variables (finite or infinite in number) have mere integral coefficients while each of the other variables (finite or infinite in number) necessarily becomes 0 if for a sufficiently large N (i. e. for $N \geqq N_0$ where N_0 depends on the variable) one solves the N first "zero-congruences" corresponding to the linear forms, i. e. the congruences (1) with $\theta_1 = \theta_2 = \cdots = 0$.

Our purpose is to prove the following

Main Theorem[1]. *A necessary and sufficient condition that a*

1) Incidentally, our proof of the main theorem in reality yields a stronger form of this theorem than the one indicated here. For the formulation of the theorem in the stronger form we refer to § 5.

system of linear forms have $\pi_1 = \pi_2$ is that the system by a substitution can be transfered into a system of the type S.

Also in this case it is easy to prove that the condition is *sufficient*. We only have to show that every system of the type S has $\pi_1 = \pi_2$. Denoting by $(\theta_1, \theta_2, \cdots)$ an arbitrary point from π_2 we shall show that it also belongs to π_1. Let $P_N = (\xi_1^{(N)}, \xi_2^{(N)}, \cdots)$ be a solution of the N first congruences (1). We may assume those coordinates which in all congruences have integral coefficients reduced modulo 1 to lie in the interval $0 \leqq \xi < 1$. Everyone of the remaining coordinates $\xi_i^{(N)}$ will possess a constant value ξ_i for $N \geqq N_0$ where $N_0 = N_0(i)$ is determined such that every solution $(x_1, x_2, \cdots)$ of the N_0 first zero-congruences will have $x_i = 0$; for as the two points $(\xi_1^{(N_0)}, \xi_2^{(N_0)}, \cdots)$ and $(\xi_1^{(N)}, \xi_2^{(N)}, \cdots)$ are both solutions of the N_0 first congruences (1) their difference $(\xi_1^{(N)} - \xi_1^{(N_0)}, \xi_2^{(N)} - \xi_2^{(N_0)}, \cdots)$ will be a solution of the N_0 first zero-congruences and hence $\xi_i^{(N)} - \xi_i^{(N_0)} = 0$, i. e. $\xi_i^{(N)} = \xi_i^{(N_0)} = \xi_i$ for $N \geqq N_0$. We now extract a subsequence from our sequence of points $P_N = (\xi_1^{(N)}, \xi_2^{(N)}, \cdots)$ such that any coordinate sequence $\xi_i^{(N)}$ (i fixed) which does not end in being a constant will converge towards a number ξ_i; this can be done since they are all lying in the interval $0 \leqq \xi < 1$. The limit point $(\xi_1, \xi_2, \cdots)$ will obviously (for continuity reasons) be a solution of all the congruences (1) and hence the point $(\theta_1, \theta_2, \cdots)$ will lie in π_1.

That the main theorem above contains the main theorem in (I) can be seen in the following way. Since every integral system is also a system of the type S the "trivial" part of the main theorem in (I) (concerning the sufficiency of the condition) is contained in the trivial part of the general main theorem. To show that the non-trivial part of the general main theorem involves the non-trivial part of the main theorem in (I) requires a little consideration. We are to show that any rational system (2) with $\pi_1 = \pi_2$ can be transfered into an integral system. The general main theorem only states that it can be transfered into a system of the type S. By using, however, that the system is rational we can easily prove that the resulting system of the type S always must be integral. Otherwise, in fact, there would exist in this system a variable y_m which for N sufficiently large necessarily becomes 0 by solution of the N first zero-congruences. The

solutions of the N first zero-congruences in the original system would therefore satisfy an equation $a_{m1}x_1 + \cdots + a_{mp_m}x_{p_m} = 0$ whose left-hand side is that linear form which in the substitution used is put equal to y_m. Denoting, however, by G a common denominator of all the coefficients in the N first linear forms in the original system, obviously all points $(h_1G, h_2G, \cdots)$ where $h_1, h_2, \cdots$ are arbitrary integers will be solutions of the corresponding zero-congruences, and these points cannot possibly all satisfy the equation $a_{m1}x_1 + \cdots + a_{mp_m}x_{p_m} = 0$ (whose coefficients are not all 0). Hence our assumption has led to a contradiction.

That the proof of the general main theorem cannot follow quite the same line as the proof in the rational case given in (I) is due to the fact that certain finite-dimensional sets which enter in the investigation (see § 2), and which in (I) without real limitation could be supposed to be lattices, in the present case are modules of a more general kind. If, however, closures are taken of the sets in question these closures will get properties analogous to the sets in (I). But in order to obtain the substitution which transfers a given system of linear forms with $\pi_1 = \pi_2$ into a system of the special type S we should still as in (I) have to consider the mentioned sets themselves and not their closures. Now, however, from the properties of the closures it would be possible to get at analogous properties for the sets themselves which would allow the seeking out of the substitution wanted. This would be a similar, though more complicated line to that followed in (I) and until recently our intension had been to use this arrangement. Then, however, B. Jessen asked us whether in the infinite-dimensional space in question a structural theorem existed for closed modules analogous to that holding for such modules in a finite-dimensional space. That this is really the case we could answer affirmatively by help of our main theorem. Later on we found a more direct proof of this structural theorem for closed modules in the infinite-dimensional space by using the dual connection between our space and another infinite-dimensional space, a connection which in case of the finite-dimensional space was introduced by M. Riesz. Now, conversely, it turned out that a more perspicious proof of the main theorem could be obtained by first establishing the structural theorem for closed

modules and then applying it to our problem. In fact, by applying this structural theorem to the closed module Γ formed by the set of all solutions of the zero-congruences corresponding to the given system of linear forms we could directly obtain the desired substitution, i. e. the substitution which takes our system (1) into a system of the type S and thus avoiding all difficulties arising from the consideration of the above mentioned non-closed modules.

In the present paper we have prefered to give the proof in this latter arrangement.

§ 2. Some important sets.

Already by the definition of a system of linear forms of the type S we had to consider the corresponding zero-congruences. In our treatment of the arbitrary system of congruences (1) the corresponding system of zero-congruences

$$(4)\qquad \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n_1}x_{n_1} \equiv 0 \pmod 1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n_2}x_{n_2} \equiv 0 \pmod 1 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{array}$$

will play an important role. In connection with the zero-congruences (4) we introduce the following notations.

$$(5)\left\{\begin{array}{ll} \Gamma & : \text{The set of solutions of the zero-congruences (4).} \\ \Gamma_m & : \text{The projection of } \Gamma \text{ on the } x_1 \cdots x_m\text{-space.} \\ H_m & : \text{The closure of } \Gamma_m. \\ \Lambda^{(N)} & : \text{The set of solutions of the } N \text{ first zero-congruences in (4).} \\ \Lambda_m^{(N)} & : \text{The projection of } \Lambda^{(N)} \text{ on the } x_1 \cdots x_m\text{-space.} \\ H_m^{(N)} & : \text{The closure of } \Lambda_m^{(N)}. \end{array}\right.$$

Here Γ and $\Lambda^{(N)}$ are point sets in the infinite-dimensional space while the four other sets (with lower index m) are point sets in the m-dimensional $x_1 \cdots x_m$-space. Γ_m and $\Lambda_m^{(N)}$ are obviously (vector-) modules and hence H_m and $H_m^{(N)}$ are closed modules. Further, for $m_1 < m$, the module Γ_{m_1} is the projection of Γ_m on the $x_1 \cdots x_{m_1}$-space, and similarly $\Lambda_{m_1}^{(N)}$ is the projection of $\Lambda_m^{(N)}$.

As well-known the closed modules in the $x_1 \cdots x_m$-space have an especially simple structure. Let H be an arbitrary closed module in the m-dimensional space. Then it is possible to find a system of *linearly independent* vectors $F_1, \cdots, F_p, V_1, \cdots, V_q$ $(p + q \leqq m)$ such that H consists of all vectors (points) of the form

$$P = \xi_1 F_1 + \xi_2 F_2 + \cdots + \xi_p F_p + h_1 V_1 + \cdots + h_q V_q$$

where the ξ's are arbitrary numbers and the h's are arbitrary integers. Conversely, each such point set is a closed module. We shall say that the vectors $F_1, \cdots, F_p$ and $V_1, \cdots, V_q$ (together) generate H with respectively arbitrary and integral coefficients.

If H does not contain any vector space (with exception of the space 0 consisting only of the origin) there can be no F-vectors and H is a *lattice*. The parallelotope determined by the vectors $V_1, \cdots, V_q$ is then called a *fundamental parallelotope* of the lattice.

The general closed module H can be called a *lattice cylinder* erected on the lattice generated by the vectors $V_1, \cdots, V_q$ (integral coefficients) with the space determined by the vectors $F_1, \cdots, F_p$ as space of generatrix directions. Concerning the freedom by which one can choose a generating system of linearly independent vectors for a closed module in the m-dimensional space we state the following well-known

Theorem. *If H is a closed module and T an arbitrary (vector-) space both lying in the m-dimensional space we can determine a system of linearly independent vectors which generates H (with arbitrary, respectively integral coefficients) by determining first in an arbitrary manner such a generating system of the closed submodule $H \cap T$[1], and next supplementing these vectors with certain other vectors (if necessary).*

Let us consider the sets (5) for a numerically given system of zero-congruences.

Example 2. Let the system of zero-congruences be

$$\begin{aligned} x_1 - x_2 &\equiv 0 \pmod 1 \\ \sqrt{2}\, x_2 &\equiv 0 \pmod 1 \\ \frac{1}{2}(x_1 - x_2) &\equiv 0 \pmod 1 \\ \frac{1}{2}\sqrt{2}\, x_2 &\equiv 0 \pmod 1 \end{aligned}$$

1) $H \cap T$ denotes the common part of H and T.

$$\frac{1}{4}(x_1 - x_2) \equiv 0 \pmod 1$$
$$\frac{1}{4}\sqrt{2}x_2 \equiv 0 \pmod 1$$
$$\cdots\cdots\cdots\cdots\cdots$$

Only the two variables x_1 and x_2 occur in these congruences. Hence for $m \geqq 2$ the set $\varLambda_m^{(N)}$ consists of all points $(x_1, \cdots, x_m)$ whose projections on the x_1x_2-plane lie in $\varLambda_2^{(N)}$, just as $\varLambda^{(N)}$ consists of all points $(x_1, x_2, \cdots)$ whose projections on the x_1x_2-plane lie in $\varLambda_2^{(N)}$. The set $\varLambda_2^{(1)}$ is the closed module in the x_1x_2-plane determined by $x_1 - x_2 \equiv 0 \pmod 1$ (it may for instance be generated by $F_1 = (1, 1)$ and $V_1 = (1, 0)$). The sets $\varLambda_2^{(2)} \supset \varLambda_2^{(3)} \supset \cdots$ form a strictly decreasing sequence of lattices in the x_1x_2-plane; for instance $\varLambda_2^{(2)}$ is the lattice generated by the vectors $V_1 = (1, 0)$ and $V_2 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, and more generally $\varLambda_2^{(2n)}$ is the lattice generated by the vectors $V_1 = (2^{n-1}, 0)$ and $V_2 = \left(\frac{2^{n-1}}{\sqrt{2}}, \frac{2^{n-1}}{\sqrt{2}}\right)$. As to the projections on the x_1-axis we see that $\varLambda_1^{(1)}$ is the whole x_1-axis while $\varLambda_1^{(2)} \supset \varLambda_1^{(3)} \supset \cdots$ is a strictly decreasing sequence of non-closed modules which are all lying everywhere dense on the x_1-axis. All these modules can be generated by a finite number of vectors, though of course not by linearly independent vectors; for instance $\varLambda_1^{(2)}$ is generated by the vectors $V_1 = 1$ and $V_2 = \frac{1}{\sqrt{2}}$ and more generally $\varLambda_1^{(2n)}$ is generated by the vectors $V_1 = 2^{n-1}$ and $V_2 = \frac{2^{n-1}}{\sqrt{2}}$.[1] Since the sets $\varLambda_1^{(n)}$ are everywhere dense on the x_1-axis it follows that their closures $H_1^{(n)}$ are all equal to the whole x_1-axis. Finally we see that $\Gamma = \{(0, 0, x_3, x_4, \cdots)\}$ where $x_3, x_4, \cdots$ are arbitrary numbers so that the sets Γ_1 and Γ_2 consist only of the origin.

In the rational case the knowledge of Γ is sufficient to decide whether $\pi_1 = \pi_2$ or not. In fact, by help of the main theorem in the rational case we can easily show that a necessary and sufficient condition that $\pi_1 = \pi_2$ is that Γ by a substitution can be transfered into a set which contains the "unit lattice" in the infinite-dimensional space, i. e. the set $\{(h_1, h_2, \cdots)\}$ where the h's are arbitrary integers. This can be seen in the following way.

[1] It can easily be seen that for any m and N the set $\varLambda_m^{(N)}$ also in the case of an *arbitrary* system of linear forms may be generated by a *finite* number of (generally non-independent) vectors with arbitrary, respectively integral coefficients. In fact if $M > m$ denotes a positive integer so large that no variable with larger index than M really occurs (i. e. has a coefficient different from 0) in any of the N first linear forms we see that $\varLambda_M^{(N)}$ is a closed module in the $x_1 \cdots x_M$-space and that $\varLambda_m^{(N)}$ is its projection on the $x_1 \cdots x_m$-space. The projection of a system of (linearly independent) generators of the closed module $\varLambda_M^{(N)}$ will therefore be a system of (in general linearly dependent) generators of $\varLambda_m^{(N)}$.

(i). If Γ by a substitution can be transfered into a set which contains the unit lattice, then the linear forms by the substitution must be transfered into linear forms whose corresponding zero-congruences amongst their solutions have all points $(h_1, h_2, \cdots)$. If this is used for the points $(1, 0, 0, \cdots)$, $(0, 1, 0, 0, \cdots)$, $\cdots$ it follows that the coefficients of x_1, the coefficients of x_2, $\cdots$ are all integral. Hence, on account of the main theorem, $\pi_1 = \pi_2$.

(ii). If $\pi_1 = \pi_2$, the linear forms can, on account of the main theorem, be transfered into an integral system. The corresponding system of zero-congruences of this integral system is obviously satisfied by all points from the unit lattice. Hence, by the substitution, Γ is transfered into a set which contains the unit lattice.

In the general case where the coefficients are arbitrary numbers the knowledge of Γ is not sufficient to decide whether $\pi_1 = \pi_2$. In fact we can easily indicate two systems of linear forms which have the same Γ but such that $\pi_1 \neq \pi_2$ for the one system and $\pi_1 = \pi_2$ for the other. This we do in the following example.

Example 3. We consider the two systems of linear forms

$$\begin{array}{ccc} \frac{1}{3}x_1 & & x_1 \\ \frac{1}{9}x_1 & & \sqrt{2}x_1 \\ \frac{1}{27}x_1 & & 0x_1 \\ \vdots & , & \vdots \\ \frac{1}{3^n}x_1 & & 0x_1 \\ \vdots & & \vdots \end{array}$$

where the first system is the same as that used in example 1, § 1. In both systems only the one variable x_1 really occurs. It is clear that the two systems have the same Γ, namely the set $\{(0, x_2, x_3, \cdots)\}$ where $x_2, x_3, \ldots$ are arbitrary numbers. The first system, however, has $\pi_1 \neq \pi_2$—in fact we proved in example 1 that the point $\left(\frac{1}{2}, \frac{1}{2}, \cdots\right)$ was lying in π_2 but not in π_1—while the second system obviously has $\pi_1 = \pi_2$ since in reality it only contains a finite number (namely 2) of linear forms.

While, thus, a consideration of Γ alone cannot decide whether $\pi_1 = \pi_2$ we shall see in the following paragraph that the knowledge of the sets $H_m^{(N)}$ is sufficient for that purpose.

§ 3. The sets $H_m^{(N)}$, H_m and the condition $\pi_1 = \pi_2$.

In this paragraph we shall indicate as a statement on the sets $H_m^{(N)}$ a necessary and sufficient condition for the validity of $\pi_1 = \pi_2$. Moreover, in the case $\pi_1 = \pi_2$ we shall find a connection between the sets $H_m^{(N)}$ and H_m.

Theorem. *A necessary and sufficient condition that $\pi_1 = \pi_2$ is that for every $m = 1, 2, \cdots$ the sequence of m-dimensional sets*

$$H_m^{(1)} \supseteqq H_m^{(2)} \supseteqq H_m^{(3)} \supseteqq \cdots$$

is constant from a certain step (depending on m).

Additional Theorem. *If $H_m^{(1)} \supseteqq H_m^{(2)} \supseteqq H_m^{(3)} \supseteqq \cdots$ for every m is constant from a certain step (and hence $\pi_1 = \pi_2$) this constant set is just the set H_m.*

We remark that if for a given m the sequence

$$H_m^{(1)} \supseteqq H_m^{(2)} \supseteqq H_m^{(3)} \supseteqq \cdots$$

is constant ($= \Phi_m$) from a certain step N_0 then for every $m_1 < m$ the sequence

$$H_{m_1}^{(1)} \supseteqq H_{m_1}^{(2)} \supseteqq H_{m_1}^{(3)} \supseteqq \cdots$$

will also—at the latest from the same step—be constant (= the closure of the projection of Φ_m on the $x_1 \cdots x_{m_1}$-space); for two sets (viz. Φ_m and $\Lambda_m^{(N)}$ for $N \geqq N_0$) in the $x_1 \cdots x_m$-space with identical closures (viz. Φ_m) are projected into two sets in the $x_1 \cdots x_{m_1}$-space with identical closures, because the condition that two sets have identical closures is that every point in each of the sets can be approximated by points in the other and this property obviously is preserved by projection.

We divide the theorem above, together with its addition, in a theorem A for the sufficiency and the addition and a theorem B for the necessity.

Theorem A. *If for every m the sequence*

$$H_m^{(1)} \supseteqq H_m^{(2)} \supseteqq H_m^{(3)} \supseteqq \cdots$$

is constant from a certain step, then $\pi_1 = \pi_2$ and the constant set is equal to H_m.

Proof. We first show that $\pi_1 = \pi_2$. Denoting by $(\theta_1, \theta_2, \cdots)$ an arbitrary point from π_2 we are to show that it also lies in π_1, i. e. that there exists a solution $Y = (y_1, y_2, \cdots)$ of all the congruences (1). Let

$$H_m^{(1)} \supseteqq H_m^{(2)} \supseteqq H_m^{(3)} \supseteqq \cdots \supseteqq H_m^{(N)} \supseteqq \cdots$$

be constant for $N \geqq N_m$ where the integral sequence N_m moreover is chosen to be strictly increasing (and hence $\rightarrow \infty$).

We take our starting-point in an arbitrary positive integer M[1] and in an arbitrary chosen solution $Y^{(M)} = (y_1^{(M)}, y_2^{(M)}, \cdots)$ of the N_M first congruences (1). Next we choose a solution $X^{(M+1)} = (x_1^{(M+1)}, x_2^{(M+1)}, \cdots)$ of the N_{M+1} first congruences. This solution can be altered by an arbitrary point $Z^{(M+1)}$ from $\Lambda^{(N_{M+1})}$, i. e. for any point $Z^{(M+1)}$ from $\Lambda^{(N_{M+1})}$ (and no other points) the point $X^{(M+1)} + Z^{(M+1)}$ is again a solution of the N_{M+1} first congruences; this is true since $\Lambda^{(N_{M+1})}$ is the set of solutions of the N_{M+1} first zero-congruences (4). *Hence we can alter the solution* $X^{(M+1)} = (x_1^{(M+1)}, x_2^{(M+1)}, \cdots)$ *such that the projected point* $(x_1^{(M+1)}, \cdots, x_M^{(M+1)})$ *is altered by an arbitrary point from* $\Lambda_M^{(N_{M+1})}$ when only the other coordinates of $X^{(M+1)}$ are altered in a suitable manner. Our wish is now that the altered point $X^{(M+1)} + Z^{(M+1)}$ shall lie "near to" $Y^{(M)}$. Since $N_{M+1} > N_M$ the point $X^{(M+1)}$ is as $Y^{(M)}$ a solution of the N_M first congruences and hence their difference $Y^{(M)} - X^{(M+1)}$ is lying in $\Lambda^{(N_M)}$. The difference of the projected points $(y_1^{(M)}, \cdots, y_M^{(M)}) - (x_1^{(M+1)}, \cdots, x_M^{(M+1)})$ will therefore lie in $\Lambda_M^{(N_M)}$ and hence a fortiori in $H_M^{(N_M)}$ *and hence also in* $H_M^{(N_{M+1})}$. Since, as mentioned above, the solution $X^{(M+1)}$ of the N_{M+1} first congruences can be altered to another solution $Y^{(M+1)} = (y_1^{(M+1)}, y_2^{(M+1)}, \cdots)$ of these congruences such that the difference $(y_1^{(M+1)}, \cdots, y_M^{(M+1)}) - (x_1^{(M+1)}, \cdots, x_M^{(M+1)})$ becomes an arbitrarily chosen point of $\Lambda_M^{(N_{M+1})}$ and since the previous difference $(y_1^{(M)}, \cdots, y_M^{(M)}) - (x_1^{(M+1)}, \cdots, x_M^{(M+1)})$ is lying in the closure $H_M^{(N_{M+1})}$ of the set $\Lambda_M^{(N_{M+1})}$ it is clear that to every $\varepsilon_M > 0$ we can choose our solution $Y^{(M+1)}$ such that the first of the two M-dimensional point-differences ε_M-approximates the latter, i. e. such that

[1] For the proof of $\pi_1 = \pi_2$ we could choose $M = 1$. When M is chosen arbitrarily it is in view of the proof of the additional theorem.

$$\left| (y_1^{(M+1)} - x_1^{(M+1)}) - (y_1^{(M)} - x_1^{(M+1)}) \right| = \left| y_1^{(M+1)} - y_1^{(M)} \right| < \varepsilon_M$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\left| (y_M^{(M+1)} - x_M^{(M+1)}) - (y_M^{(M)} - x_M^{(M+1)}) \right| = \left| y_M^{(M+1)} - y_M^{(M)} \right| < \varepsilon_M.$$

Next, let $X^{(M+2)} = (x_1^{(M+2)}, x_2^{(M+2)}, \cdots)$ be a solution of the N_{M+2} first congruences (1). This solution can be altered by an arbitrary point from $\Lambda^{(N_{M+2})}$ and hence $X^{(M+2)}$ can be altered such that the projected point $(x_1^{(M+2)}, \cdots, x_{M+1}^{(M+2)})$ is altered by an arbitrary point from $\Lambda_{M+1}^{(N_{M+2})}$ when only the other coordinates of $X^{(M+2)}$ are altered in a suitable manner. Our wish is that the altered point shall lie "near to" $Y^{(M+1)}$. Since $N_{M+2} > N_{M+1}$ the point $X^{(M+2)}$ is as $Y^{(M+1)}$ a solution of the N_{M+1} first congruences. The difference $(y_1^{(M+1)}, \cdots, y_{M+1}^{(M+1)}) - (x_1^{(M+2)}, \cdots, x_{M+1}^{(M+2)})$ is therefore lying in $\Lambda_{M+1}^{(N_{M+1})}$ and hence a fortiori in $H_{M+1}^{(N_{M+1})}$ *and hence also in* $H_{M+1}^{(N_{M+2})}$. Since, as mentioned above, the solution $X^{(M+2)}$ of the N_{M+2} first congruences can be altered to another solution $Y^{(M+2)} = (y_1^{(M+2)}, y_2^{(M+2)}, \cdots)$ of these congruences such that the difference $(y_1^{(M+2)}, \cdots, y_{M+1}^{(M+2)}) - (x_1^{(M+2)}, \cdots, x_{M+1}^{(M+2)})$ becomes an arbitrarily chosen point of $\Lambda_{M+1}^{(N_{M+2})}$ and since the previous difference $(y_1^{(M+1)}, \cdots, y_{M+1}^{(M+1)}) - (x_1^{(M+2)}, \cdots, x_{M+1}^{(M+2)})$ is lying in the closure $H_{M+1}^{(N_{M+2})}$ of the set $\Lambda_{M+1}^{(N_{M+2})}$ it is clear that to every $\varepsilon_{M+1} > 0$ we can choose the solution $Y^{(M+2)}$ such that the first of the two $(M+1)$-dimensional point-differences ε_{M+1}-approximates the latter, i. e. such that

$$\left| (y_1^{(M+2)} - x_1^{(M+2)}) - (y_1^{(M+1)} - x_1^{(M+2)}) \right| = \left| y_1^{(M+2)} - y_1^{(M+1)} \right| < \varepsilon_{M+1}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\left| (y_{M+1}^{(M+2)} - x_{M+1}^{(M+2)}) - (y_{M+1}^{(M+1)} - x_{M+1}^{(M+2)}) \right| = \left| y_{M+1}^{(M+2)} - y_{M+1}^{(M+1)} \right| < \varepsilon_{M+1}.$$

- -

In general, i. e. for an arbitrary $n \geqq M+1$, let the point $X^{(n)} = (x_1^{(n)}, x_2^{(n)}, \cdots)$ be a solution of the N_n first congruences (1). This solution can be altered by an arbitrary point from $\Lambda^{(N_n)}$ and hence $X^{(n)}$ can be altered such that the projected point $(x_1^{(n)}, \cdots, x_{n-1}^{(n)})$ is altered by an arbitrary point from $\Lambda_{n-1}^{(N_n)}$ when only the other coordinates of $X^{(N)}$ are altered in a suitable manner. Our wish is that the altered point shall lie "near to" $Y^{(n-1)}$. Since $N_n > N_{n-1}$ the point $X^{(n)}$ is as $Y^{(n-1)}$ a solution

of the N_{n-1} first congruences. The difference $(y_1^{(n-1)}, \cdots, y_{n-1}^{(n-1)}) - (x_1^{(n)}, \cdots, x_{n-1}^{(n)})$ is therefore lying in $\Lambda_{n-1}^{(N_{n-1})}$ and hence a fortiori in $H_{n-1}^{(N_{n-1})}$ *and hence also in* $H_{n-1}^{(N_n)}$. Since, as mentioned above, the solution $X^{(n)}$ of the N_n first congruences can be altered to another solution $Y^{(n)} = (y_1^{(n)}, y_2^{(n)}, \cdots)$ of these congruences such that the difference $(y_1^{(n)}, \cdots, y_{n-1}^{(n)}) - (x_1^{(n)}, \cdots, x_{n-1}^{(n)})$ becomes an arbitrarily chosen point of $\Lambda_{n-1}^{(N_n)}$ and since the previous difference $(y_1^{(n-1)}, \cdots, y_{n-1}^{(n-1)}) - (x_1^{(n)}, \cdots, x_{n-1}^{(n)})$ is lying in the closure $H_{n-1}^{(N_n)}$ of the set $\Lambda_{n-1}^{(N_n)}$ it is clear that to every $\varepsilon_{n-1} > 0$ we can choose the point $Y^{(n)}$ such that the first of the two $(n-1)$-dimensional point-differences ε_{n-1}-approximates the other, i. e. such that

$$\left|(y_1^{(n)} - x_1^{(n)}) - (y_1^{(n-1)} - x_1^{(n)})\right| = \left|y_1^{(n)} - y_1^{(n-1)}\right| < \varepsilon_{n-1}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\left|(y_{n-1}^{(n)} - x_{n-1}^{(n)}) - (y_{n-1}^{(n-1)} - x_{n-1}^{(n)})\right| = \left|y_{n-1}^{(n)} - y_{n-1}^{(n-1)}\right| < \varepsilon_{n-1}.$$

- -

Choosing our ε's such that $\sum_M^\infty \varepsilon_r$ is convergent we consider the sequence

$$\begin{aligned} Y^{(M)} &= (y_1^{(M)}, y_2^{(M)}, \cdots) \\ Y^{(M+1)} &= (y_1^{(M+1)}, y_2^{(M+1)}, \cdots) \\ Y^{(M+2)} &= (y_1^{(M+2)}, y_2^{(M+2)}, \cdots) \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

The M first coordinate sequences $y_\nu^{(M)}, y_\nu^{(M+1)}, y_\nu^{(M+2)}, \cdots$ $(\nu = 1, 2, \cdots, M)$ satisfy

$$\left|y_\nu^{(p)} - y_\nu^{(q)}\right| \leqq \sum_q^\infty \varepsilon_r \quad \text{for} \quad p > q \geqq M$$

while each of the following coordinate sequences $y_n^{(M)}, y_n^{(M+1)}, y_n^{(M+2)}, \cdots$ $(n \geqq M+1)$ satisfy

$$\left|y_n^{(p)} - y_n^{(q)}\right| \leqq \sum_q^\infty \varepsilon_r \quad \text{for} \quad p > q \geqq n.$$

Hence, in particular, all the coordinate sequences converge towards respective numbers $y_1, y_2, \cdots$. The limit point

$$Y = (y_1, y_2, \cdots)$$

will then be a solution of all the congruences (1). In fact, to see that Y is a solution of the N^{th} congruence we observe that $Y^{(n)} = (y_1^{(n)}, y_2^{(n)}, \cdots)$ from a certain step is a solution of this congruence. Since only a finite number of variables really occurs in the congruence the statement follows from continuity reasons. Thus $(\theta_1, \theta_2, \cdots)$ is lying in π_1 and hence $\pi_1 = \pi_2$. Out of regard to the following we observe that the M first coordinates $y_1, \cdots, y_M$ of Y satisfy the inequalities

$$
(6) \qquad \begin{aligned}
&|y_1 - y_1^{(M)}| \leqq \sum_M^\infty \varepsilon_r \\
&\cdots\cdots\cdots\cdots\cdots \\
&|y_M - y_M^{(M)}| \leqq \sum_M^\infty \varepsilon_r .
\end{aligned}
$$

Now, to conclude the proof of theorem A, we have to show that the constant final set $H_M^{(N_M)}$ in the sequence

$$H_M^{(1)} \supseteqq H_M^{(2)} \supseteqq H_M^{(3)} \supseteqq \cdots$$

for every $M = 1, 2, \cdots$ is equal to H_M. Since $\Gamma_M \subseteqq \Lambda_M^{(N_M)}$, it is plain that $H_M \subseteqq H_M^{(N_M)}$. In order to show that, conversely, $H_M^{(N_M)} \subseteqq H_M$ for an arbitrarily given M we use the proof above in the case $\theta_1 = \theta_2 = \cdots = 0$ with our present M as the M in the proof. The previous point $Y^{(M)} = (y_1^{(M)}, y_2^{(M)}, \cdots)$ is then an arbitrary point from $\Lambda^{(N_M)}$ and the projected point $(y_1^{(M)}, \cdots, y_M^{(M)})$ is therefore *an arbitrary point from* $\Lambda_M^{(N_M)}$. We are to show that $(y_1^{(M)}, \cdots, y_M^{(M)})$ can be approximated by points from Γ_M. But this is an immediate consequence of the fact that (in the present case $\theta_1 = \theta_2 = \cdots = 0$) the point Y constructed in the proof above is lying in Γ and that its M first coordinates satisfy the inequalities (6) where $\sum_M^\infty \varepsilon_r$ can be chosen arbitrarily small.

Theorem B. *If* $\pi_1 = \pi_2$ *the sequence*

$$H_m^{(1)} \supseteqq H_m^{(2)} \supseteqq H_m^{(3)} \supseteqq \cdots$$

will for every m *be constant from a certain step.*

Proof. Indirectly, we assume that there exists an m_0 for which $H_{m_0}^{(1)} \supseteqq H_{m_0}^{(2)} \supseteqq H_{m_0}^{(3)} \supseteqq \cdots$ is not constant from a certain step and

are to show that $\pi_1 \neq \pi_2$, i. e. that there exists a $(\theta_1, \theta_2, \cdots)$ which belongs to π_2 but not to π_1. We first consider the geometric appearance of the sequence of modules $H^{(n)}_{m_0}$ $(n = 1, 2, \cdots)$. This sequence is an essentially decreasing[1] sequence of lattice cylinders (see § 2). It is therefore plain that from a certain step $n \geqq N_0$ the least space (vector space) which contains $H^{(n)}_{m_0}$, and the space of generatrix directions of the cylinder $H^{(n)}_{m_0}$, will be constant spaces R_p and R_{p_1} of dimensions (say) p and p_1. Furthermore from this step the lattice base G_n of $H^{(n)}_{m_0}$ can be chosen in such a way that the least space which contains G_n is *a fixed space* R_q (dimension q with $p = p_1 + q$). The lattices G_n form from this step an essentially decreasing sequence in their common least space R_q. Therefore the q-dimensional content of the fundamental parallelotope of G_n ("fundamental content G_n") is an essentially increasing sequence which $\rightarrow \infty$ (since the fundamental content is at least doubled by the transition from one lattice to the next every time the lattices are different).

By $K(\varrho)$ we denote the open sphere in R_q with radius ϱ and center O as also the q-dimensional content of this sphere. By $C(\varrho)$ we denote the corresponding sphere cylinder in R_p with the sphere $K(\varrho)$ as base and the space of generatrix directions R_{p_1}. We also consider spheres in R_q whose centers are not lying in O and the corresponding sphere cylinders in R_q. In the following we denote for abbreviation sphere cylinders with base-sphere in R_q and space of generatrix directions R_{p_1} as "sphere cylinders" without further specification. By the sphere cylinder around the point P in R_p with radius ϱ we understand the sphere cylinder corresponding to the sphere with radius ϱ and center in the projection of P on R_q in the direction of R_{p_1}.

We first determine a sequence of strictly increasing positive numbers $N_1, N_2, \cdots, N_\nu, \cdots$ and the corresponding positive numbers $\varrho_1, \varrho_2, \cdots, \varrho_\nu, \cdots$ by the following procedure.

1°. Let $N_1 \geqq N_0$ be chosen such that the fundamental content G_{N_1} is larger than the sphere content $K(1)$. Then the sphere $K(1)$ cannot contain a complete system of representatives in R_q modulo G_{N_1} and hence the sphere cylinder $C(1)$ cannot contain a complete

1) An essentially decreasing sequence of sets is here and in the following a sequence where every element is contained in the preceding and which is not constant from a certain step. The expression, an essentially increasing sequence of numbers, used below, has an analogous meaning.

system of representatives in R_p modulo $H_{m_0}^{(N_1)}$. To this N_1 we choose the positive number ϱ_1 so large that every sphere in R_q with radius ϱ_1 contains a complete system of representatives in R_q modulo G_{N_1} and hence also a complete system of representatives in R_p modulo $H_{m_0}^{(N_1)}$. In particular, everyone of our sphere cylinders in R_p with radius ϱ_1 will contain a complete system of representatives in R_p modulo $H_{m_0}^{(N_1)}$.

2°. Next we determine $N_2 > N_1$ such that the fundamental content G_{N_2} is larger than $K(\varrho_1 + 2)$. Then the sphere cylinder $C(\varrho_1 + 2)$ cannot contain a complete system of representatives in R_p modulo $H_{m_0}^{(N_2)}$. To this N_2 we determine the positive number ϱ_2 so large that everyone of our sphere cylinders in R_p with radius ϱ_2 contains a complete system of representatives in R_p modulo $H_{m_0}^{(N_2)}$.

- -

ν°. After having determined $N_{\nu-1}$ and $\varrho_{\nu-1}$ we determine $N_\nu > N_{\nu-1}$ such that the fundamental content G_{N_ν} is larger than $K(\varrho_{\nu-1} + \nu)$. Then the sphere cylinder $C(\varrho_{\nu-1} + \nu)$ cannot contain a complete system of representatives in R_p modulo $H_{m_0}^{(N_\nu)}$. To this N_ν we determine the positive number ϱ_ν so large that everyone of our sphere cylinders in R_p with radius ϱ_ν contains a complete system of representatives in R_p modulo $H_{m_0}^{(N_\nu)}$.

- -

After having determined N_ν and ϱ_ν $(\nu = 1, 2, \cdots)$ we now pass to the direct searching of a point $(\theta_1, \theta_2, \cdots)$ which belongs to the set π_2 but not to the set π_1. The idea in this (successive) determination modulo 1 of the numbers $\theta_1, \theta_2, \cdots$, the kernel of which can be found in example 1, § 1, is that we try to see that the set of projections $(x_1, \cdots, x_{m_0})$ on the $x_1 \cdots x_{m_0}$-space of all solutions $(x_1, x_2, \cdots)$ of the N first congruences (1) will lie farther and farther away from O for increasing values of N. More precisely, we will see that the set of projections for $N = N_\nu$ will lie in R_p and outside $C(\nu)$.

1st step. We first choose an arbitrary point $P^{(1)} = (x_1^{(1)}, x_2^{(1)}, \cdots)$ in the infinite-dimensional space which only satisfies the condition that the projected point $P_{m_0}^{(1)} = (x_1^{(1)}, \cdots, x_{m_0}^{(1)})$ is lying in R_p and has no equivalent point modulo $H_{m_0}^{(N_1)}$ lying in $C(1)$.

Such a point exists on account of 1° since $C(1)$ does not contain a complete system of representatives in R_p modulo $H_{m_0}^{(N_1)}$. We substitute $(x_1, x_2, \cdots) = (x_1^{(1)}, x_2^{(1)}, \cdots)$ in the N_1 first linear forms (2). The numbers thus determined (but only considered modulo 1) shall be our numbers $\theta_1, \theta_2, \cdots, \theta_{N_1}$. We observe that the total set of solutions of the N_1 first congruences (1) (with the θ's just chosen) is the set $P^{(1)} + \Lambda^{(N_1)}$ because $\Lambda^{(N_1)}$ is the set of solutions of the N_1 first zero-congruences. From the choice of $P^{(1)}$ it follows that the projection of this set $P^{(1)} + \Lambda^{(N_1)}$ on the $x_1 \cdots x_{m_0}$-space—i. e. the set $P_{m_0}^{(1)} + \Lambda_{m_0}^{(N_1)}$ which consists of all points equivalent to $P_{m_0}^{(1)}$ modulo $\Lambda_{m_0}^{(N_1)}$—is lying in R_p and outside $C(1)$.

2nd step. Next we choose (which is possible from the choice of N_2) a point $D_{m_0}^{(2)} = (d_1^{(2)}, \cdots, d_{m_0}^{(2)})$ in R_p which has no equivalent point modulo $H_{m_0}^{(N_2)}$ in $C(\varrho_1 + 2)$. The sphere cylinder in R_p around $D_{m_0}^{(2)}$ with radius ϱ_1 contains (on account of the choice of ϱ_1) a point equivalent to $P_{m_0}^{(1)}$ modulo $H_{m_0}^{(N_1)}$. Since $\Lambda_{m_0}^{(N_1)}$ is lying everywhere dense in $H_{m_0}^{(N_1)}$ this cylinder also contains a point $P_{m_0}^{(2)} = (x_1^{(2)}, \cdots, x_{m_0}^{(2)})$ equivalent to $P_{m_0}^{(1)}$ modulo $\Lambda_{m_0}^{(N_1)}$. We can therefore choose a point $P^{(2)} = (x_1^{(2)}, x_2^{(2)}, \cdots)$ whose projection on the $x_1 \cdots x_{m_0}$-space is $P_{m_0}^{(2)}$ and which is equivalent to $P^{(1)}$ modulo $\Lambda^{(N_1)}$. In particular $P^{(2)}$ is a solution of the N_1 first congruences (1). We now substitute $(x_1, x_2, \cdots) = (x_1^{(2)}, x_2^{(2)}, \cdots)$ in the N_2 first linear forms (2) and denote the numbers thus determined (modulo 1) by $\theta_1, \cdots, \theta_{N_2}$. The N_1 first of these numbers coincide with the numbers $\theta_1, \cdots, \theta_{N_1}$ determined by the first step, since $P^{(2)}$ satisfies the N_1 first congruences (formed with these θ's). We now consider the set of solutions $(x_1, x_2, \cdots)$ of the N_2 first (with the above θ's formed) congruences (1), i. e. the set $P^{(2)} + \Lambda^{(N_2)}$. Then the projection of this set on the $x_1 \cdots x_{m_0}$-space—i. e. the set $P_{m_0}^{(2)} + \Lambda_{m_0}^{(N_2)}$ which consists of all points equivalent to $P_{m_0}^{(2)}$ modulo $\Lambda_{m_0}^{(N_2)}$—is lying in R_p and outside $C(2)$; that the set is lying in R_p is plain, and the second statement follows from the fact that $P_{m_0}^{(2)}$ is lying in a sphere cylinder around $D_{m_0}^{(2)}$ with radius ϱ_1 where $D_{m_0}^{(2)}$ has no equivalent point in $C(\varrho_1 + 2)$ modulo $H_{m_0}^{(N_2)}$ and hence a fortiori no equivalent point modulo $\Lambda_{m_0}^{(N_2)}$.

- -

ν^{th} *step.* We choose (which is possible from the choice of N_ν) a point $D^{(\nu)}_{m_0} = (d^{(\nu)}_1, \cdots, d^{(\nu)}_{m_0})$ in R_p which has no equivalent point modulo $H^{(N_\nu)}_{m_0}$ in $C(\varrho_{\nu-1} + \nu)$. The sphere cylinder in R_p around $D^{(\nu)}_{m_0}$ with radius $\varrho_{\nu-1}$ contains (on account of the choice of $\varrho_{\nu-1}$) a point equivalent to $P^{(\nu-1)}_{m_0}$ modulo $H^{(N_\nu-1)}_{m_0}$ and hence also a point $P^{(\nu)}_{m_0} = (x^{(\nu)}_1, \cdots, x^{(\nu)}_{m_0})$ equivalent to $P^{(\nu-1)}_{m_0}$ modulo $\Lambda^{(N_\nu-1)}_{m_0}$. We can therefore choose a point $P^{(\nu)} = (x^{(\nu)}_1, x^{(\nu)}_2, \cdots)$ whose projection on the $x_1 \cdots x_{m_0}$-space is $P^{(\nu)}_{m_0}$ and which is equivalent to $P^{(\nu-1)}$ modulo $\Lambda^{(N_\nu-1)}$. In particular $P^{(\nu)}$ is a solution of the $N_{\nu-1}$ first congruences (1). We now substitute $(x_1, x_2, \cdots) = (x^{(\nu)}_1, x^{(\nu)}_2, \cdots)$ in the N_ν first linear forms (2) and denote the numbers thus determined (modulo 1) by $\theta_1, \cdots, \theta_{N_\nu}$. The $N_{\nu-1}$ first of these numbers coincide with the numbers $\theta_1, \cdots, \theta_{N_{\nu-1}}$ determined by the $(\nu - 1)^{\text{th}}$ step. We consider the set of solutions $(x_1, x_2, \cdots)$ of the N_ν first (with the above θ's formed) congruences (1). Then the projection of this set on the $x_1 \cdots x_{m_0}$-space—i. e. the set $P^{(\nu)}_{m_0} + \Lambda^{(N_\nu)}_{m_0}$ which consists of all points equivalent to $P^{(\nu)}_{m_0}$ modulo $\Lambda^{(N_\nu)}_{m_0}$—lies in R_p and outside $C(\nu)$; that the set is lying in R_p is plain, and the second statement follows from the fact that $P^{(\nu)}_{m_0}$ is lying in a sphere cylinder around $D^{(\nu)}_{m_0}$ with radius $\varrho_{\nu-1}$ where $D^{(\nu)}_{m_0}$ has no equivalent point in $C(\varrho_{\nu-1} + \nu)$ modulo $H^{(N_\nu)}_{m_0}$ and hence a fortiori no equivalent point modulo $\Lambda^{(N_\nu)}_{m_0}$.

- -

In this manner we have got a point $(\theta_1, \theta_2, \cdots)$ with the desired properties. In fact, the point is belonging to π_2 since for every ν the N_ν first (with these θ's formed) congruences (1) have a solution $P^{(\nu)} = (x^{(\nu)}_1, x^{(\nu)}_2, \cdots)$, and here $N_\nu \to \infty$ for $\nu \to \infty$. On the other hand the point $(\theta_1, \theta_2, \cdots)$ does not belong to π_1, i. e. there is no solution of the whole system of congruences (1); for every solution of the N_ν first congruences has a projection on the $x_1 \cdots x_{m_0}$-space which lies in R_p and outside $C(\nu)$.

Remark. The theorems of this paragraph connect the condition $\pi_1 = \pi_2$ with the closures $H^{(N)}_m$ and H_m of the modules $\Lambda^{(N)}_m$ and Γ_m. We shall mention that analogous theorems hold for the sets $\Lambda^{(N)}_m$ and Γ_m themselves, viz.

Theorem. *A necessary and sufficient condition that* $\pi_1 = \pi_2$ *is that for every* $m = 1, 2, \cdots$ *the sequence*

$$\Lambda_m^{(1)} \supseteqq \Lambda_m^{(2)} \supseteqq \Lambda_m^{(3)} \supseteqq \cdots$$

is constant from a certain step (depending on m).

Additional Theorem. *If* $\Lambda_m^{(1)} \supseteqq \Lambda_m^{(2)} \supseteqq \Lambda_m^{(3)} \supseteqq \cdots$ *for every* m *is constant from a certain step this constant set is just the set* Γ_m.

If these theorems, as their analogues for the closures, are divided in a theorem A for the sufficiency and the addition and a theorem B for the necessity, the theorem A is even simpler to prove than the previous theorem A. Theorem B, however, lies deeper than its analogue. We can obtain the new theorem B from the old one by the following

Theorem. *For an arbitrary system of linear forms* (1) *(with* $\pi_1 = \pi_2$ *or* $\pi_1 \neq \pi_2$*) there exists to every positive integer* m *an integer* $M \geqq m$ *and a positive integer* N *such that the sequence* $\Lambda_m^{(N)} \supseteqq \Lambda_m^{(N+1)} \supseteqq \Lambda_m^{(N+2)} \supseteqq \cdots$ *is the projection on the* $x_1 \cdots x_m$*-space of the sequence* $H_M^{(N)} \supseteqq H_M^{(N+1)} \supseteqq H_M^{(N+2)} \supseteqq \cdots$.

We omit, however, the proofs of these theorems which are unnecessary for the proof of our main theorem in its present framing (cp. p. 8-9).

§ 4. The structure of closed modules in the infinite-dimensional space.

In this paragraph we shall study the closed modules in our infinite-dimensional space—which from now on is denoted by R^∞—where the underlying convergence notion, occasionally used in the previous paragraphs, is that of convergence in each of the coordinates. As we shall see the closed modules in the space R^∞ possess quite a similar structure as that of the closed modules in the usual m-dimensional space R_m (see § 2).

In order to prove the structure theorem in R^∞ we shall use the analogous structure theorem in R_m, $m = 1, 2, \cdots$. The transition from the finite-dimensional case is, however, not a trivial one. We shal have to put in an intermediate space R_∞ between the finite-dimensional spaces R_m and the space R^∞. The space R_∞ is as R^∞ an infinite-dimensional space, but while a point $X = (x_1, x_2, \cdots)$ in R^∞ may have quite arbitrary coor-

dinates, a point $A = (a_1, a_2, \circ\circ\circ)$[1] in R_∞ always has coordinates which from a certain step (depending on the point) are 0, i. e. $a_n = 0$ for $n \geqq N = N(A)$.

Between the spaces R_∞ and R^∞ there exists, when a convergence notion in R_∞ is suitably chosen, a *duality*. Once established this duality permits us to get at the structure theorem for closed modules in R^∞ from an analogous structure theorem for closed modules in R_∞. Now, as mentioned, the space R_∞ is lying nearer to the finite-dimensional spaces R_m than does R^∞, in fact it can be exhausted by the $a_1 a_2 \cdots a_m$-space for $m \to \infty$. This is the reason why, as we shall see, the structure theorem in R_∞ can easily be obtained from the finite-dimensional case.

The duality, mentioned above, between R_∞ and R^∞ is analogous to a duality considered by M. Riesz between two m-dimensional spaces $R_m = \{(a_1, \cdots, a_m)\}$ and $R_m = \{(x_1, \cdots, x_m)\}$.

If M is an arbitrary module in R_m Riesz considers the point set in (the other space) R_m consisting of all points $A = (a_1, \cdots, a_m)$ from this latter R_m for which

$$A \cdot X = a_1 x_1 + a_2 x_2 + \cdots + a_m x_m \equiv 0 \pmod 1$$

for every point $X = (x_1, x_2, \cdots, x_m)$ from M. This point set is a closed module in R_m and is called the *dual* module of M. We denote it by M'. If we repeat the operation of passing to the dual module we get a closed module $M'' = (M')'$ in (the original space) R_m. The relation between M and M'' appears from the following important theorem.

Riesz's Theorem. *If M is an arbitrary module in R_m the dual module M'' of its dual module M' is the closure $\overline{M}$ of M, i. e.*

$$M'' = \overline{M}.$$

For a closed module H in R_m we get in particular $H'' = H$.

We now pass to the establishment of the duality between R_∞ and R^∞, or rather that side of the duality which will be needed in the following. A full account of the duality can be found in another paper[2] where the topic of this paragraph is discussed in more detail.

[1] For points in R_∞ we use the notation $(a_1, a_2, \circ\circ\circ)$ in order to make apparent that their coordinates are all zero from a certain step.

[2] H. Bohr and E. Følner: On a structure theorem for closed modules in an infinite-dimensional space, to appear elsewhere.

Let T be an arbitrary linear transformation in R_∞ and let the fundamental points $(1, 0, 0, \circ\circ\circ)$, $(0, 1, 0, \circ\circ\circ)$, $\cdots$ by the transformation be taken into the points

$$T\{(1, 0, 0, \circ\circ\circ)\} = S_1 = (t_{11}, t_{21}, \circ\circ\circ)$$
$$T\{(0, 1, 0, \circ\circ\circ)\} = S_2 = (t_{12}, t_{22}, \circ\circ\circ)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

from R_∞. The arbitrary point $A = (a_1, a_2, \circ\circ\circ)$ from R_∞ will then be carried into the point

$$B = T(A) = a_1 S_1 + a_2 S_2 + \circ\circ\circ.$$

Introducing the matrix $T = \{t_{rs}\}$ the linear transformation may be written $B = TA$. In the following we denote a linear transformation in R_∞ and the corresponding (uniquely determined) matrix by the same letter T.

Conversely, each such matrix equation

$$\begin{Bmatrix} b_1 \\ b_2 \\ \circ \\ \circ \\ \circ \end{Bmatrix} = \begin{Bmatrix} t_{11} t_{12} & \cdot & \cdot & \cdot \\ t_{21} t_{22} & \cdot & \cdot & \cdot \\ \circ\ \circ & & & \\ \circ\ \circ & & & \\ \circ\ \circ & & & \end{Bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ \circ \\ \circ \\ \circ \end{Bmatrix}$$

where the column vectors are arbitrary points from R_∞ is a linear transformation in R_∞.

We now define the scalar product between two points $A = (a_1, a_2, \circ\circ\circ)$ and $X = (x_1, x_2, \cdots)$ from R_∞ and R^∞, respectively. We put

$$A \cdot X = X \cdot A = a_1 x_1 + a_2 x_2 + \circ\circ\circ.$$

In matrix notation the scalar product is expressed by A^*X or X^*A[1)] when we agree on considering the points as column vectors (for convenience we usually write them horizontally).

For a given linear transformation T in R_∞ and two variable points X and Y from R^∞ we now set up the condition

(7) $$A \cdot X = T(A) \cdot Y \text{ for every } A \text{ from } R_\infty.$$

1) The star denotes the operation of transposing a matrix.

We shall show that this condition on X and Y is equivalent to a linear transformation in R^∞ (expressed by linear expressions as (3), § 1) of Y into X (and thus, in particular, that to any given Y there exists one and only one X satisfying (7)).

In matrix notation the condition runs as follows

$$A^*X = (TA)^*Y \quad \text{or} \quad A^*X = A^*T^*Y.$$

Putting successively $A^* = (1, 0, 0, \circ\circ\circ), (0, 1, 0, \circ\circ\circ), \cdots$ in this relation we get

$$X = T^*Y \tag{8}$$

and conversely the former condition follows from (8) by left-multiplying it with A^*.

Putting (8) into (7) and changing Y to X we get the relation

$$A \cdot T^*(X) = T(A) \cdot X \text{ for every } A \text{ from } R_\infty \text{ and every } X \text{ from } R^\infty. \tag{9}$$

We now define a *substitution in* R_∞ as a linear, one-to-one transformation of R_∞ onto R_∞.

If T is a substitution the condition (7) is equivalent to the condition

$$A \cdot Y = T^{-1}(A) \cdot X \text{ for every } A \text{ from } R_\infty, \tag{10}$$

in fact we have only substituted $T^{-1}(A)$ for A and interchanged the two sides of the equation (7). Here T^{-1} denotes the inverse substitution of T. Since (7) is equivalent to (8) we see that (10) is equivalent to

$$Y = (T^{-1})^*X. \tag{11}$$

Hence also the relations (8) and (11) are equivalent which shows that T^* is a one-to-one transformation of R^∞ onto R^∞ and therefore what we have called a substitution in R^∞ (see § 1). Putting $T' = (T^*)^{-1}$ and replacing X by $T'(X)$ in (9) we obtain the following

Theorem 1. *If T is a substitution in R_∞ then T^* is a substitution in R^∞ and there exists a uniquely determined substitution T' in R^∞ such that*

$$(12)\qquad A \cdot X = T(A) \cdot T'(X) \textit{ for every } A \textit{ from } R_\infty \textit{ and every } X \textit{ from } R^\infty,$$

viz. the substitution $T' = (T^*)^{-1} = (T^{-1})^*$.

We call T' the *dual substitution* of T.

In order to speak of *closed* modules in R_∞ and R^∞ we must know the underlying convergence notion of the two spaces. We have already mentioned that in R^∞ our convergence notion is that of convergence in every coordinate. In order to define a suitable convergence notion in R_∞ we first observe that our convergence notion in R^∞ may also be stated as follows:

A sequence $X^{(n)}$ converges towards X if and only if

$$A \cdot X^{(n)} \to A \cdot X \textit{ for every } A \textit{ from } R_\infty.$$

In fact, since a point A from R_∞ only contains a finite number of non-zero coordinates the former condition involves the latter, and conversely, the former condition is obtained from the latter by putting successively $A = (1, 0, 0, \circ\circ\circ), (0, 1, 0, \circ\circ\circ), \cdots$.

In the new form the notion of convergence in R^∞ has a dual notion of convergence in R_∞:

A sequence $A^{(n)}$ of points from R_∞ is said to converge towards a point A from R_∞ if and only if

$$X \cdot A^{(n)} \to X \cdot A \textit{ for every } X \textit{ from } R^\infty.$$

This is going to be our convergence notion in R_∞.[1]

Remark. Our substitutions in R^∞ are obviously *bicontinuous.* In order to show that our substitutions in R_∞ are also bicontinuous we remark that on account of (9) every linear transformation T in R_∞ is continuous; in fact, when $A^{(n)} \to A$ we get from (9)

$$X \cdot T(A^{(n)}) = T^*(X) \cdot A^{(n)} \to T^*(X) \cdot A = X \cdot T(A)$$

for every X from R^∞ which shows that $T(A^{(n)}) \to T(A)$. It can

1) In the following we shall only use the definition of convergence in R_∞ in the above form; we may, however, mention that this definition, as easily seen, is equivalent to the following (more direct) one: Convergence of a sequence in R_∞ means convergence in every coordinate and moreover the existence of a p only depending on the sequence, such that all points of the sequence have 0 on the coordinate places with higher number than p

easily be shown that our substitutions in R^∞ or R_∞ are just the linear, one-to-one, bicontinuous transformations of the space onto itself (in the case of R_∞ nothing is left to prove).

For an arbitrary closed module H in R_∞ we consider the point set H' in R^∞ which consists of all points X for which

$$A \cdot X \equiv 0 \pmod{1} \text{ for every } A \text{ from } H.$$

Obviously the set H' is a module. Furthermore H' is closed, for if $X^{(n)} \to X$ in R^∞ and all $X^{(n)}$ are lying in H', then for every A from H we have $0 \equiv A \cdot X^{(n)} \to A \cdot X$ so that $A \cdot X \equiv 0$. We call the closed module H' the *dual module* of the closed module H. The following simple theorem indicates the connection between the two notions, dual module and dual substitution.

Theorem 2. *If we subject a closed module H in R_∞ to a substitution T and subject the dual module H' in R^∞ to the dual substitution T' then the resulting module $T'(H')$ in the latter case is the dual module of the resulting module $T(H)$ in the former case, i. e.*

$$T'(H') = (T(H))'.$$

This is an immediate consequence of the relation (12) when we only observe that $T(A)$ runs through $T(H)$ and $T'(X)$ runs through $T'(H')$ when A runs through H and X through H'.

We have defined above the dual module of a closed module from R_∞. Analogously, we define the dual module H' of a closed module H from R^∞ as the point set (eo ipso closed module) consisting of the points A from R_∞ for which

$$X \cdot A \equiv 0 \pmod{1} \text{ for every } X \text{ from } H.$$

Then we have the following important

Theorem 3. *For an arbitrary closed module H in R^∞ the dual module H'' of its dual module H' is the module itself, i. e.*

$$H'' = H.$$

Obviously $H'' \supseteq H$. Thus we only have to prove that $H'' \subseteq H$. Let then $Y = (y_1, y_2, \cdots)$ be an arbitrary point from H''. In order to show that Y is lying in H, let m be an arbitrary positive

integer. We consider the points $(a_1, a_2, \cdots, a_m, 0, 0, \circ\circ\circ) = (a_1, a_2, \cdots, a_m)$ from the common part L of H' and the $a_1 a_2 \cdots a_m$-space. Then for every point in L we have

$$(13) \qquad (y_1, y_2, \cdots, y_m) \cdot (a_1, a_2, \cdots, a_m) \equiv 0 \pmod 1.$$

Next, let M denote the projection of H on the $x_1 x_2 \cdots x_m$-space (i. e. the set of points $(x_1, x_2, \cdots, x_m)$ arising from the points $(x_1, x_2, \cdots)$ of H by cancelling all coordinates with indices $> m$). M is again a module, but not necessarily a closed module. Plainly, $L = M'$ and thus on account of (13) the point $(y_1, y_2, \cdots, y_m)$ belongs to M''. Now, according to Riesz's theorem

$$M'' = \overline{M}$$

and hence $(y_1, y_2, \cdots, y_m)$ can be approximated by points $(x_1, x_2, \cdots, x_m)$ from M. Since m is arbitrary it follows that $Y = (y_1, y_2, \cdots)$ can be approximated by points $(x_1, x_2, \cdots)$ from H, i. e. Y must lie in $\overline{H} = H$, q. e. d.

We shall now prove the following structure theorem for closed modules in R_∞.

Structure Theorem R_∞. *A closed module H in the infinite-dimensional space R_∞ is a point set E which by a substitution can be transfered into a point set of a special form, in the following denoted by S_∞, namely a point set $\{(a_1, a_2, \circ\circ\circ)\}$ of the following structure: The indices $1, 2, \cdots, n, \cdots$ can be divided into three fixed classes $\{n_r\}$, $\{n_s\}$, $\{n_t\}$ depending only on the point set, such that the coordinates a_{n_r} independently run through all numbers, and the coordinates a_{n_s} independently run through all integers, while all the remaining coordinates a_{n_t} are constantly zero. Only, of course, the simultaneous variation of the a_{n_r} and the a_{n_s} in the set is limited by the obvious demand that $(a_1, a_2, \circ\circ\circ)$ always shall lie in R_∞, i. e. have 0 from a certain coordinate place (depending on the point). Conversely, each such point set E is a closed module.*

The latter part of the theorem follows immediately from the remark on p. 26.

In order to prove the first (and real) part of the theorem, let H_m denote the common part of H and the $x_1 \cdots x_m$-space. Then, obviously, H_m is in the usual sense a closed module in

the $a_1 \cdots a_m$-space. Furthermore, H_m is the common part of H_{m+1} and the $a_1 \cdots a_m$-space. Hence it follows from the theorem on p. 10, for $m = 1, 2, \cdots$, that we can generate successively the closed modules $H_1, H_2, \cdots$ by linearly independent vectors with arbitrary and integral coefficients in such a way that the generating vectors of H_{m+1} are the generating vectors of H_m with the same types of coefficients, in connection with other vectors (if necessary). In this way we get a sequence of linearly independent vectors $G_1, G_2, \cdots$ which provided with suitable types of coefficients (integral or arbitrary) will generate H (generation of course in the sense that for each vector of H only a finite number of generators is used). With arbitrary coefficients the vectors spann a subspace $R(H)$ of R_∞. Let R_1 denote the common part of $R(H)$ and the x_1-axis. If the space R_1 is not the whole x_1-axis, but only the 0-vector we place a non-zero vector on the x_1-axis. Then this vector together with $R(H)$ will spann a space $R^{(1)}$ which contains the x_1-axis. If $R(H)$ itself contains the x_1-axis we put $R^{(1)} = R(H)$. Next, let R_2 denote the common part of $R^{(1)}$ and the x_1x_2-plane. If the space R_2 is not the whole x_1x_2-plane, but only the x_1-axis we place a vector in the x_1x_2-plane outside the x_1-axis. Then this vector together with $R^{(1)}$ will spann a space $R^{(2)}$ which contains the x_1x_2-plane. If $R^{(1)}$ itself contains the x_1x_2-plane we put $R^{(2)} = R^{(1)}$. In this way we continue. If the vectors thus found in some way or other are put into a sequence with the vectors $G_1, G_2, \cdots$ we get a sequence of linearly independent vectors $U_1, U_2, \cdots$ which provided with suitable types of coefficients (zero, integral or arbitrary) will generate H and with mere arbitrary coefficients the whole space R_∞. The linear independence of $U_1, U_2, \cdots$ secures that each point in R_∞ has only one representation by this generation. Hence

$$B = a_1 U_1 + a_2 U_2 + \circ\circ\circ$$

is a substitution in R_∞ of $A = (a_1, a_2, \circ\circ\circ)$ into B. It takes the fundamental vectors $(1, 0, 0, \circ\circ\circ)$, $(0, 1, 0, \circ\circ\circ)$, $\cdots$ into the vectors $U_1, U_2, \cdots$. Therefore the inverse substitution, which takes $U_1, U_2, \cdots$ into the fundamental vectors, will take the closed module H into a set $\{(a_1, a_2, \circ\circ\circ)\}$ determined by $a_i = 0$ for certain i, a_i arbitrary integral for certain i, and

a_i arbitrary for the remaining i. This proves structure theorem R_∞.

By help of structure theorem R_∞ and the duality between R_∞ and R^∞ we shall now obtain the main result of this paragraph, viz.

Structure Theorem R^∞. *A closed module in the infinite-dimensional space R^∞ is a point set E which by a substitution can be transfered into a point set of a special form, denoted by S^∞, namely a point set $\{(x_1, x_2, \cdots)\}$ of the following structure: The indices $1, 2, \cdots, n, \cdots$ can be divided into three fixed classes $\{n_r\}$, $\{n_s\}$, $\{n_t\}$ which depend only on the point set, such that the coordinates x_{n_r} independently run through all numbers, and the coordinates x_{n_s} independently run through all integers, while all the remaining coordinates x_{n_t} are constantly zero. Conversely, each such point set E is a closed module.*

Again, the latter part of the theorem follows immediately from the remark on p. 26.

In order to obtain a proof of the first (and real) part of the theorem by help of the corresponding theorem in R_∞ let us first show that the dual module of a closed module of the special form S_∞ is a closed module of the special form S^∞. More precisely we shall prove

Theorem 4. *For a closed module H in R_∞ of the special form S_∞, explicitly $\{(a_1, a_2, \circ\circ\circ)\}$ with the coordinates a_{n_r} arbitrary, the coordinates a_{n_s} integral, and the coordinates a_{n_t} zero, the dual module H' in R^∞ is of the special form S^∞, and more precisely the dual module is $\{(x_1, x_2, \cdots)\}$ where the x_{n_r} are zero, the x_{n_s} integral, and the x_{n_t} arbitrary.*

We first observe that obviously all points X of the form mentioned are lying in H'. Conversely, we have to show that all points in H' have the form mentioned. Since the points X in H' have to fulfill

$$(\circ\circ\circ\ \xi_{n_r}\ \circ\circ\circ)\cdot X \equiv 0 \pmod 1 \text{ for all values } \xi_{n_r}$$

it follows that the n_r^{th} coordinate of X must be zero, and since

$$\underset{1\,2\cdots n_s\cdots}{(\circ\circ\circ\ 1\ \circ\circ\circ)}\cdot X \equiv 0 \pmod 1$$

it follows that the n_s^{th} coordinate of X must be integral. This proves theorem 4.

We have now got all means necessary to prove structure theorem R^∞. Let first H be an arbitrary closed module in R_∞. Then on account of structure theorem R_∞ there exists a substitution T in R_∞ such that $T(H)$ has the special form S_∞. The dual module H' of H is a closed module in R^∞. We shall show that H' by a substitution can be taken into a closed module of the special form S^∞. In fact, the dual substitution T' of T has this property, for it follows from theorem 2 that $T'(H') = (T(H))'$ and from theorem 4 that $(T(H))'$, as the dual module of a closed module of the special form S_∞, is itself a closed module of the special form S^∞. Hence we see that every closed module in R^∞ which is the dual module of a closed module in R_∞ by a substitution can be taken into a set of the form S^∞. In order to complete the proof of structure theorem R^∞ we therefore only have to show that *every* closed module H in R^∞ can be written in the form K' where K is a closed module in R_∞. This, however, is a consequence of theorem 3 which tells that $H = H''$ so that for K we may use H'.

§ 5. Proof of the main theorem.

Already in § 1 we have formulated the main theorem and proved the simple "half" of it, namely that a sufficient condition that a system of linear forms (2) have $\pi_1 = \pi_2$ is that the system by a substitution can be taken into a system of the type S. We shall now show that this condition is also necessary, i. e. that every system of linear forms which has $\pi_1 = \pi_2$ by a substitution can be taken into a system of the type S.

For a system of congruences (1) the set Γ of solutions of the corresponding zero congruences is obviously always (i. e. whether $\pi_1 = \pi_2$ or not) a closed module in R^∞. Hence the structure theorem R^∞ from § 4 states that there exists a substitution T which takes Γ into a point set of the form S^∞, corresponding (say) to the classes $\{n_r\}$, $\{n_s\}$, $\{n_t\}$. By this substitution T the system of linear forms will be taken into a system where the coefficient columns corresponding to the variables x_{n_r} are zero

columns while the coefficient columns corresponding to the variables x_{n_s} are integral columns. This is seen by putting $(\circ\circ\circ\ \xi_{n_r}\ \circ\circ\circ)$ with arbitrary ξ_{n_r}, respectively $(\underset{1}{\circ}\underset{2}{\circ}\underset{\cdot}{\circ}\ \underset{n_s}{1}\ \circ\circ\circ)$ into the zero-congruences. Conversely, a coefficient column of zero's corresponds to a variable x_{n_r} and an integral coefficient column which is not a zero column to a variable x_{n_s}.

Now, we shall show that if $\pi_1 = \pi_2$ for the original system, and hence also for the transformed system, the latter of these systems will be of the type S.

Obviously it makes no real difference if all the coefficient columns corresponding to the variables x_{n_r} are removed together with their respective variables. For since all these columns consist of zero's this removal will neither change the property of having or not having $\pi_1 = \pi_2$, nor the property of being or not being a system of the type S.

We shall use theorem A and B from § 3 on the system after the removal. Since $\pi_1 = \pi_2$ the modules $H_m^{(N)}$ of this system will for each m be constant from a certain step $N \geqq N_0 = N_0(m)$ and equal to the modul H_m. Since Γ_m is a module of the form $\{(x_1, x_2, \cdots, x_m)\}$ where the indices $1, 2, \cdots, m$ can be divided into two classes $\{n_s\}$ and $\{n_t\}$ such that the coordinates x_{n_s} are integral and the coordinates x_{n_t} are zero, Γ_m is in particular a closed modul so that $H_m = \Gamma_m = \{(x_1, x_2, \cdots, x_m)\} = \{$(integral, zero)$\}$. Hence from the step N_0 also $H_m^{(N)} = \{$(integral, zero)$\}$. Finally, using that $\Lambda_m^{(N)} \subseteqq H_m^{(N)}$ we find the following property of our new system: Each of the variables x_{n_t} becomes zero if one solve the N first zero-congruences for sufficiently large N (depending on the variable). Hence the system is of the type S. *This proves the main theorem.* Furthermore we see that each of the variables x_{n_s} becomes integral if one solve the N first zero-congruences for sufficiently large N (depending on the variable). The same of course is also true for the system before the removal of the variables x_{n_r} with mere zero coefficients.— This proves the following

Stronger form of the main theorem. *A necessary (and sufficient) condition that a system of linear forms have $\pi_1 = \pi_2$ is that the linear forms by a substitution can be transfered into a system which is of the type S and moreover possesses the property*

that each of the variables belonging to the integral columns necessarily becomes integral if one solve the N first zero-congruences, corresponding to the linear forms, for sufficiently large N (depending on the variable).

Remark. A (necessary and) sufficient condition that a system of linear forms of the type S have the additional property mentioned in the theorem above is that the variables mentioned necessarily become integral if one solve the system of *all* the zero-congruences corresponding to the linear forms.

In fact, to prove this, we may use theorem A and B from § 3 in a similar way as above.

§ 6. A remark on the algebraic structure of a system of the special type S.

The notion of a system of linear forms of the type S was defined in § 1 as a system of linear forms where certain variables had mere integral coefficients while each of the other variables necessarily became 0 by solution of a suitable finite selection of the zero-congruences corresponding to the linear forms.

The question, therefore, naturally arises how a finite system of zero-congruences (in a finite number of variables) can force one of the variables to be zero. In this final paragraph we treat this problem by giving *a necessary and sufficient condition that a system of linear zero-congruences in* $x_1, \cdots, x_n$

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \equiv 0 \pmod 1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \equiv 0 \pmod 1 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \equiv 0 \pmod 1 \end{array}$$

will involve $x_1 = 0$.

Let in the corresponding matrix

$$\left\{ \begin{array}{l} a_{11}a_{12} \cdots a_{1n} \\ a_{21}a_{22} \cdots a_{2n} \\ \cdots\cdots\cdots\cdots \\ a_{m1}a_{m2} \cdots a_{mn} \end{array} \right\}$$

the system of row vectors $R_1, R_2, \cdots, R_m$ have the maximal rank ϱ. Then we can find ϱ linearly independent vectors amongst these row vectors. Let it be, for instance, $R_1, R_2, \cdots, R_\varrho$. Then numbers α exist such that

$$\begin{aligned} R_{\varrho+1} &= \alpha_{11}R_1 + \alpha_{12}R_2 + \cdots + \alpha_{1\varrho}R_\varrho \\ R_{\varrho+2} &= \alpha_{21}R_1 + \alpha_{22}R_2 + \cdots + \alpha_{2\varrho}R_\varrho \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ R_m &= \alpha_{m-\varrho,1}R_1 + \alpha_{m-\varrho,2}R_2 + \cdots + \alpha_{m-\varrho,\varrho}R_\varrho . \end{aligned}$$

The column vectors in the abridged matrix

$$\begin{Bmatrix} a_{11} \cdots a_{1n} \\ \cdots\cdots\cdots \\ a_{\varrho 1} \cdots a_{\varrho n} \end{Bmatrix}$$

are denoted by $S_1, \cdots, S_n$. They have the maximal rank ϱ. The column vectors in the matrix

$$\begin{Bmatrix} \alpha_{11} \cdots\cdots\cdots\cdots \alpha_{1\varrho} \\ \cdots\cdots\cdots\cdots\cdots \\ \alpha_{m-\varrho,1} \cdots \alpha_{m-\varrho,\varrho} \end{Bmatrix}$$

are denoted by $\mathfrak{S}_1, \cdots, \mathfrak{S}_\varrho$.

Instead of the congruences we can equally well consider the equations

$$\begin{aligned} a_{11}x_1 &+ \cdots + a_{1n}x_n = h_1 \\ a_{21}x_1 &+ \cdots + a_{2n}x_n = h_2 \\ &\cdots\cdots\cdots\cdots\cdots \\ a_{\varrho 1}x_1 &+ \cdots + a_{\varrho n}x_n = h_\varrho \\ a_{\varrho+1,1}x_1 &+ \cdots + a_{\varrho+1,n}x_n = h_{\varrho+1} \\ &\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 &+ \cdots + a_{mn}x_n = h_m \end{aligned} \tag{14}$$

where the h's are new integral variables. This system of equations can be solved for a given choice of $h_1, \cdots, h_\varrho$ if and only if

$$h_1\mathfrak{S}_1 + h_2\mathfrak{S}_2 + \cdots + h_\varrho\mathfrak{S}_\varrho \equiv 0 \pmod 1^{1)};$$

1) Here, by $A \equiv 0 \pmod 1$ we mean that A is an integral vector.

for the ϱ first equations can always be solved and they involve the validity of the others if the condition above is satisfied, while otherwise at least one equation is not satisfied. In particular, the condition is satisfied if $h_1 = h_2 = \cdots = h_\varrho = 0$.

If the vectors $S_2, \cdots, S_n$ have the maximal rank ϱ we can choose x_1 arbitrarily by the solution of the ϱ first equations with $h_1 = \cdots = h_\varrho = 0$. If our congruences have no solutions with $x_1 \neq 0$ it follows that $S_2, \cdots, S_n$ must have the maximal rank $\varrho - 1$. Let this necessary condition be satisfied. The integral solutions $(h_1, h_2, \cdots, h_\varrho)$ of

$$h_1\mathfrak{S}_1 + h_2\mathfrak{S}_2 + \cdots + h_\varrho\mathfrak{S}_\varrho \equiv 0 \pmod 1$$

form a lattice. Then obviously a necessary and sufficient condition that every solution of the equations (14) has $x_1 = 0$ is that the lattice $\{(h_1, h_2, \cdots, h_\varrho)\}$ is contained in the space spanned by $S_2, \cdots, S_n$. Hence we have the result:

A necessary and sufficient condition that the congruences involve $x_1 = 0$ is that $S_2, \cdots, S_n$ have the maximal rank $\varrho - 1$ and that the lattice $\{(h_1, h_2, \cdots, h_\varrho)\}$ of integral solutions $(h_1, h_2, \cdots, h_\varrho)$ of

$$h_1\mathfrak{S}_1 + h_2\mathfrak{S}_2 + \cdots + h_\varrho\mathfrak{S}_\varrho \equiv 0 \pmod 1$$

is contained in the space spanned by $S_2, \cdots, S_n$.

Indleveret til Selskabet d. 31. Oktober 1947.
Færdig fra Trykkeriet d. 12. Juli 1948.

COURANT
Anniversary Volume
1948

ON A STRUCTURE THEOREM FOR CLOSED MODULES IN AN INFINITE-DIMENSIONAL SPACE

Harald Bohr and Erling Følner

1. In a paper, to be published in Det Kgl. Danske Videnskabernes Selskabs Skrifter, we have studied a problem concerning systems of infinitely[1)] many linear congruences with infinitely many variables. The variables x_1, x_2, ... could here naturally be considered as coordinates of a variable point $(x_1, x_2, \ldots)$ of an infinite-dimensional space in which we were led to consider the convergence notion defined by convergence in each of the coordinates, i.e., $(x_1^{(n)}, x_2^{(n)}, \ldots) \longrightarrow (x_1, x_2, \ldots)$ if $x_1^{(n)} \longrightarrow x_1$, $x_2^{(n)} \longrightarrow x_2$, ... This space we shall denote by R^∞.

In connection with the results of our investigation, Professor B. Jessen asked us whether in the space R^∞ a structure theorem existed for closed modules, analogous to the well-known structure theorem for closed modules in the usual finite-dimensional space. By help of our results we could show that this is really the case, but later on we found an independent and more perspicuous proof of the structure theorem in question. Now, conversely, it turned out that our main result about the congruences could be proved more simply when the structure theorem in R^∞ was used, and this we took advantage of in the paper mentioned. In the present article we shall treat, separately, this structure theorem.

---

1) By infinite we mean here and in the following enumerably infinite.

2. In the usual m-dimensional space R_m closed modules are characterized by the following well-known

Structure Theorem R_m. A closed module in the m-dimensional space R_m is a point set E which by a "substitution" can be transformed into a point set of the form $\{(\xi_1, \dots, \xi_p, h_1, \dots, h_q, 0, 0, \dots, 0)\}$ where p and q are fixed indices ≥ 0 with $p + q \leq m$ which only depend on the point set, while $\xi_1, \dots, \xi_p$ independently run through all numbers and $h_1, \dots, h_q$ independently run through all integers. Conversely, each such point set E is a closed module.

A substitution is here a transformation of the form

$$(1) \qquad \begin{Bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ y_m \end{Bmatrix} = \begin{Bmatrix} t_{11} & t_{12} & \dots & t_{1m} \\ t_{21} & t_{22} & \dots & t_{2m} \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ t_{m1} & t_{m2} & \dots & t_{mm} \end{Bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_m \end{Bmatrix}$$

where the determinant of the matrix is $\neq 0$. The substitutions can also be characterized as the linear, one-to-one, bicontinuous transformations of the space onto itself.

Our purpose is to prove the following analogous structure theorem for closed modules in R^∞.

Structure Theorem R^∞. A closed module in the infinite-dimensional space R^∞ is a point set E which by a "substitution" can be transformed into a point set of a special form, in the following denoted by S^∞, namely a point set $\{(x_1, x_2, \dots, x_n, \dots)\}$ of the following structure: The indices 1, 2, ..., n, ... can be divided into three fixed classes $\{n_r\}$, $\{n_s\}$, $\{n_t\}$ which depend only on the point set, such that the coordinates x_{n_r} independently run through all numbers, and the coordinates x_{n_s} independently run through all integers, while all the remaining coordinates x_{n_t} are

all constantly zero. Conversely, each such point set E *is a closed module*.

Here, a *substitution* in R^∞ is defined as a linear, one-to-one, bicontinuous transformation of the space onto itself. The latter part of the structure theorem is then obvious.

3. In order to prove the structure theorem in R^∞ we shall use the analogous structure theorem in R_m, $m = 1, 2, \ldots$. The transition from the finite-dimensional case is, however, not a trivial one. We shall have to put in an intermediate space R_∞ between the finite-dimensional spaces R_m and R^∞. The space R_∞ is, like R^∞, an infinite-dimensional space, but while a point $X = (x_1, x_2, \ldots)$ in R^∞ may have quite arbitrary coordinates, a point $A = (a_1, a_2, ***)$ in R_∞ always has coordinates which from a certain step (depending on the point) is 0, i.e., $a_n = 0$ for $n \geqslant N = N(A)$.[1]

Between the spaces R_∞ and R^∞ there exists, when a convergence notion in R_∞ is suitably chosen, a *duality*. Once established, this duality permits us to get at the structure theorem for closed modules in R^∞ from an analogous structure theorem for closed modules in R_∞. Now, as mentioned, the space R_∞ lies "nearer to" the finite-dimensional spaces R_m than does R^∞, in fact it can be exhausted by the $a_1 \ldots a_m$-space for $m \to \infty$. This is the reason why, as we shall see, the structure theorem in R_∞ can easily be obtained from the finite-dimensional case.

4. The duality, mentioned above, between R_∞ and R^∞ is analogous to a duality considered by *M. Riesz* between two m-dimensional spaces $R_m = \{(x_1, \ldots, x_m)\}$ and $R_m = \{(a_1, \ldots, a_m)\}$.

---

1) For points in R_∞ we use the notation $(a_1, a_2, ***)$ in order to make it apparent that their coordinates are all 0 from a certain step on.

Riesz establishes his duality by help of the structure theorem R_m and uses it to prove Kronecker's general theorem on diophantine approximation. In order to establish our duality we need only the structure theorem in R_∞ together with the duality of Riesz. We can then use it to prove the structure theorem in R^∞ (and also, following Riesz, to a certain generalization of Kronecker's theorem into which, however, we shall not enter here).

5. We begin by establishing the duality of Riesz. Denoting the column vectors $(x_1, x_2, \ldots, x_m)$ and $(y_1, y_2, \ldots, y_m)$ in (1) by X and Y (for convenience we usually write them horizontally) and denoting the quadratic matrix $\{t_{rs}\}$ by T we may write the substitution (1) as $Y = TX$. We also denote the substitution itself by T and may then as well write the connection between X and Y as $Y = T(X)$.

We define the scalar product between two points $X = (x_1, \ldots, x_m)$ from R_m and $A = (a_1, \ldots, a_m)$ from R_m as

$$A \cdot X = X \cdot A = a_1x_1 + a_2x_2 + \ldots + a_mx_m.$$

Using matrix multiplication the scalar product may be written A^*X.[1)]

To a substitution in R_m with matrix T we let correspond the substitution in R_m with matrix $T' = (T^*)^{-1}$. These two substitutions are connected by the fundamental relation

(2) $A \cdot X = T(A) \cdot T'(X)$ for every A from R_m and every X from R_m.

In fact,

$$A \cdot X = A^*X = A^*T^*[T^*]^{-1}X = [TA]^*T'X = T(A) \cdot T'(X).$$

We call T' the dual substitution of T. Since $(T^*)^{-1} = (T^{-1})^*$ it follows that the dual substitution of T' is T, i.e.,

$$T'' = T. \tag{3}$$

6. If M is an arbitrary module in R_m we call the point set

1) The star denotes the operation of transposing a matrix.

(eo ipso closed module) in R_m which consists of all points X in R_m for which

$$A \cdot X \equiv 0 \pmod 1) \text{ for every } A \text{ from } M$$

the dual module of M and denote it by M'.

<u>Theorem 1</u>. <u>If we subject a module</u> M <u>in</u> R_m <u>to a substitution</u> T <u>and subject the dual module</u> M' <u>in</u> R_m <u>to the dual substitution</u> T' <u>then the resulting module</u> $T'(M')$ <u>in the latter case is the dual module of the resulting module</u> $T(M)$ <u>in the former case, i.e.</u>,

$$T'(M') = (T(M))'.$$

This is an immediate consequence of the relation (2) when we only observe that $T(A)$ runs through $T(M)$ and $T'(X)$ runs through $T'(M')$ when A runs through M and X through M'.

<u>Theorem 2</u>. <u>For an arbitrary module</u> M <u>in</u> R_m <u>the dual module</u> M'' <u>of its dual module</u> M' <u>is the closure</u> $\bar{M}$ <u>of</u> M, <u>i.e.</u>,

$$M'' = \bar{M}.$$

Since, from continuity reasons, $M' = (\bar{M})'$, it is enough to show that for a closed module H the simpler relation

$$H'' = H \tag{4}$$

is valid. If T is an arbitrary substitution, (4) is equivalent to $T(H'') = T(H)$, and replacing the first T by T'' by help of (3) and using theorem 1 twice on the left-hand side of the equation we see that (4) is equivalent to

$$(T(H))'' = T(H).$$

This is the relation (4) for the module $T(H)$. Now, on account of the structure theorem R_m we may choose the substitution T such that

$$T(H) = \{(\xi_1,\ldots,\xi_p,\ h_1,\ldots,\ h_q,0,\ldots,\ 0)\}$$

and thus we have only to show that for such a simple closed module the dual module of its dual module is the module itself. This,

however, is an immediate consequence of the following almost evident

Theorem 3. For a closed module H of the special form $\{(\xi_1, \dots, \xi_p, h_1, \dots, h_q, 0, \dots, 0)\}$ (see structure theorem R_m) the dual module H' is the module $\{(0, \dots, 0, n_1, \dots, n_q, \alpha_1, \dots \alpha_{m-p-q})\}$ with zero's on the coordinate places which before had arbitrary values, arbitrary values $\alpha_1, \dots, \alpha_{m-p-q}$ on the coordinate places which before had the value zero, and arbitrary integers $n_1, \dots, n_q$ on the places which also before had arbitrary integral values.

In order to prove theorem 3 we first observe that obviously all the points $(0, \dots, 0, n_1, \dots, n_q, \alpha_1, \dots, \alpha_{m-p-q})$ lie in H'. Conversely, we have to show that all points in H' have the form mentioned. Since the points A in H' have to fulfill

$$(\xi_1, 0, 0, \dots, 0) \cdot A \equiv 0 \pmod 1 \text{ for all values } \xi_1$$

it follows that the first coordinate in A must be zero, and analogously for the p first coordinates in A (we suppose that $p > 0$). Since

$$(\underset{1}{0}, \underset{2}{0}, \dots, \underset{p}{0}, \underset{p+1}{1}, 0, \dots, 0) \cdot A \equiv 0 \pmod 1$$

it follows that the $(p+1)^{th}$ coordinate in A must be integral, and analogously for the $(p+2)^{th}, \dots, (p+q)^{th}$ coordinate (we suppose that $q > 0$). This proves theorem 3.

7. We now pass to the establishment of the duality between R_∞ and R^∞. Let T be an arbitrary linear transformation in R_∞. Let the fundamental points $(1, 0, 0, \dots), (0, 1, 0, \dots)$, ... by the transformation be taken into the points

$$T(1, 0, 0, ***) = S_1 = (t_{11}, t_{21}, ***)$$
$$T(0, 1, 0, ***) = S_2 = (t_{12}, t_{22}, ***)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

from R_∞. The arbitrary point $A = (a_1, a_2, ***)$ will then be carried into the point

$$B = T(A) = a_1 S_1 + a_2 S_2 + ***.$$

Introducing the matrix $T = \{t_{rs}\}$ the linear transformation may be written $B = TA$. In the following we denote a linear transformation and the corresponding matrix by the same letter T.

Conversely, each such matrix equation

$$\begin{Bmatrix} b_1 \\ b_2 \\ * \\ * \\ * \end{Bmatrix} = \begin{Bmatrix} t_{11} & t_{12} & \cdots \\ t_{21} & t_{22} & \cdots \\ * & * & \\ * & * & \\ * & * & \end{Bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ * \\ * \\ * \end{Bmatrix}$$

where the column vectors are arbitrary points from R_∞ is a linear transformation in R_∞.

We now define the scalar product between two points $A = (a_1, a_2, ***)$ and $X = (x_1, x_2, \ldots)$ from R_∞ and R^∞, respectively. We put

$$A \cdot X = X \cdot A = a_1 x_1 + a_2 x_2 + *** .$$

Evidently, R_∞ consists of all points A from R^∞ for which the scalar product $A \cdot X = a_1 x_1 + a_2 x_2 + \ldots$ has a meaning for every X from R^∞. In matrix notation the scalar product is expressed by A^*X or X^*A.

For a given linear transformation T from R_∞ and two arbitrary points X and Y from R^∞ we now set up the condition

(5) $$A \cdot X = T(A) \cdot Y \text{ for every } A \text{ from } R_\infty .$$

We shall show that this condition is equivalent to a continuous linear transformation in R^∞ of Y into X (and thus, in particular, that to any given Y there exists one and only one X satisfying (5)).

In matrix notation the condition runs as follows

$$A^*X = (TA)^*Y \quad \text{or} \quad A^*X = A^*T^*Y.$$

Putting successively $A^* = (1, 0, 0, \cdots), (0, 1, 0, \cdots), \ldots$ in this relation we get

(6) $$X = T^*Y$$

and conversely the former condition follows from (6) by left multiplication with A^*.

That such a transformation in R^∞, expressed by help of a matrix T^* whose row vectors are arbitrary points from R_∞, is really a continuous, linear transformation in R^∞, follows at once.

Putting (6) into (5) and changing Y to X we get the relation

(7) $A \cdot T^*(X) = T(A) \cdot X$ for every A from R_∞ and every X from R^∞.

8. We now define <u>a substitution in</u> R_∞ as a linear, one-to-one transformation of R_∞ onto R_∞.

If T is a substitution in R_∞ the condition (5) is equivalent to the condition

(8) $$A \cdot Y = T^{-1}(A) \cdot X \text{ for every A from } R_\infty,$$

in fact, we have only substituted $T^{-1}(A)$ for A and interchanged the two sides of the equation (5). Here T^{-1} denotes the inverse substitution of T. Since (5) was equivalent to (6) we see that (8) is equivalent to

(9) $$Y = (T^{-1})^*X.$$

Hence also the relations (6) and (9) are equivalent which shows that T^* is a one-to-one, linear, bicontinuous transformation of

R^∞ onto R^∞, i.e., a substitution in R^∞, and that $(T^*)^{-1} = (T^{-1})^*$. Putting $T' = (T^*)^{-1}$, and replacing X by $T'(X)$ in (7), we obtain the following

Theorem 4. *If T is a substitution in R_∞ then T^* is a substitution in R^∞ and there exists a uniquely determined substitution T' in R^∞ such that*

$$A \cdot X = T(A) \cdot T'(X) \tag{10}$$

for every A from R_∞ and every X from R^∞, viz., the substitution $T' = (T^*)^{-1} = (T^{-1})^*$.

We call T' the *dual substitution* of T.

9.[1] We now interchange the two spaces R_∞ and R^∞ and start by considering an arbitrary *continuous*, linear transformation T in R^∞. Let the fundamental points $(1, 0, 0, \ldots)$, $(0, 1, 0, \ldots)$, ... be taken by the transformation into the points

$$\begin{aligned} T\{(1, 0, 0, \ldots)\} &= S_1 = (t_{11}, t_{21}, \ldots) \\ T\{(0, 1, 0, \ldots)\} &= S_2 = (t_{12}, t_{22}, \ldots) \\ &\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

from R^∞. In order to obtain the image by the mapping of an arbitrary point $X = (x_1, x_2, \ldots)$ in R^∞ we first consider the point $(x_1, x_2, \ldots, x_n, 0, 0, \ldots)$. This point will be taken into $x_1S_1 + x_2S_2 + \ldots + x_nS_n$ and since $(x_1, x_2, \ldots, x_n, 0, 0, \ldots) \to (x_1, x_2, \ldots)$ for $n \to \infty$ it follows that $x_1S_1 + x_2S_2 + \ldots$ must be convergent for every $(x_1, x_2, \ldots)$ from R^∞ and that

$$Y = T(X) = x_1S_1 + x_2S_2 + \ldots.$$

In matrix form the transformation is

1) For the proof of our main result, viz., the structure theorem R^∞, section 9 is unnecessary.

$$\begin{Bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix} = \begin{Bmatrix} t_{11} & t_{12} & \cdots \\ t_{21} & t_{22} & \cdots \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{Bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix}$$

where the row vectors, in order that $x_1S_1 + x_2S_2 + \ldots$ be convergent for every X, are points from R_∞. Conversely, as also mentioned above, each such transformation is a continuous, linear transformation in R^∞.

We now proceed by setting up a condition analogous to (5), only with the spaces R_∞ and R^∞ interchanged, and continuing in this way we obtain the analogous form of theorem 4:

<u>Theorem 4a</u>. <u>If</u> T <u>is a substitution in</u> R^∞ <u>then</u> T^* <u>is a substitution in</u> R_∞ <u>and there exists a uniquely determined substitution</u> T' <u>in</u> R_∞ <u>such that</u>

(10a) $$X \cdot A = T(X) \cdot T'(A)$$

<u>for every</u> X <u>from</u> R^∞ <u>and every</u> A <u>from</u> R_∞, <u>viz., the substitution</u> $T' = (T^*)^{-1} = (T^{-1})^*$.

We call T' the <u>dual substitution</u> of T.

From the theorems 4 and 4a immediately follows

<u>Theorem 5</u>. <u>For an arbitrary substitution</u> T <u>from</u> R^∞ <u>or</u> R_∞ <u>the dual substitution</u> T" <u>of its dual substitution</u> T' <u>is the substitution itself, i.e.,</u>

$$T'' = T.$$

<u>10</u>. We have defined a substitution in R^∞ and a substitution in R_∞ in different ways, in fact, we did not claim the bicontinuity of a substitution in R_∞, such as we did in R^∞. So far, we cannot speak of continuity in R_∞ at all, since we have not yet introduced any notion of convergence in R_∞. We now introduce

such a convergence notion in R_∞ which is essential for us in the following (and which is not the one obtained by considering R_∞ as a subspace of R^∞).

11. We first observe that our notion of convergence in R^∞ on account of which a sequence $X^{(n)} = (x_1^{(n)}, x_2^{(n)}, \dots)$ is said to converge towards $X = (x_1, x_2, \dots)$ if $x_1^{(n)} \longrightarrow x_1$, $x_2^{(n)} \longrightarrow x_2$, ... may also be stated as follows:

A sequence $X^{(n)}$ converges towards X if and only if

$$A \cdot X^{(n)} \longrightarrow A \cdot X \quad \text{for every } A \text{ from } R_\infty .$$

In fact, since a point A from R_∞ only contains a finite number of non-zero coordinates the former condition involves the latter, and conversely, the former condition is obtained from the latter by putting successively $A = (1, 0, 0, \cdots), (0, 1, 0, \cdots), \dots$.

In the new form the notion of convergence in R^∞ has a dual notion of convergence in R_∞:

A sequence $A^{(n)}$ of points from R_∞ is said to converge towards a point A from R_∞ if and only if

$$X \cdot A^{(n)} \longrightarrow X \cdot A \quad \text{for every } X \text{ from } R^\infty .$$

This will be our convergence notion in R_∞.[1)]

With this convergence notion in R_∞, let us first show that a linear transformation T in R_∞ (in contrast to a linear transformation T in R^∞) is always continuous, i.e., that $T(A^{(n)}) \longrightarrow T(A)$ for $A^{(n)} \longrightarrow A$. In fact, from (7), and $A^{(n)} \longrightarrow A$, it follows that for an arbitrary X from R^∞

1) In the following we shall use only the definition of convergence in R_∞ in the above form; we may, however, mention that this definition, as easily seen, is equivalent with the following (more direct) one: Convergence of a sequence in R_∞ means convergence in every coordinate and moreover the existence of a number p depending only on the sequence, such that all points of the sequence have 0 on the coordinate places with higher number than p.

$$X\cdot T(A^{(n)}) = T^*(X)\cdot A^{(n)} \longrightarrow T^*(X)\cdot A = X\cdot T(A)$$

which proves the statement. Hence, also in R_∞ the substitutions are just the linear, one-to-one, bicontinuous transformations of R_∞ onto R_∞.

12. If M is an arbitrary point set in R_∞ we may consider the "dual" point set M' in R^∞ which consists of all points X for which

$$A\cdot X \equiv 0 \pmod 1 \text{ for every } A \text{ from } M.$$

Obviously the set M' is a module. Furthermore M' is closed, for if $X^{(n)} \longrightarrow X$ in R^∞ and all $X^{(n)}$ lie in M', then for every A from M we have $0 \equiv A\cdot X^{(n)} \longrightarrow A\cdot X$ so that $A\cdot X \equiv 0$. That M' is closed may also be expressed by

$$\overline{(M')} = M'.$$

Analogously, we may prove that

$$(\overline{M})' = M', \tag{11}$$

for if $A^{(n)} \longrightarrow A$ in R_∞ and all $A^{(n)}$ lie in M, then for every X from M' we have $0 \equiv X\cdot A^{(n)} \longrightarrow X\cdot A$ so that $X\cdot A \equiv 0$.

Similarly, if M is a point set in R^∞ we define its dual point set M' in R_∞. Evidently M' is also here a closed module and (11) is valid.

In particular, every <u>module</u> M in R_∞ or R^∞ has a (closed) <u>dual module</u> in R^∞ or R_∞ respectively.

With the notations introduced, we see at once that <u>theorem 1</u> is also true if the space R_m is replaced by R_∞ or R^∞.

13. Starting from an arbitrary point set M in R_∞ or R^∞, and repeating the process above of passing to the dual set, we get a point set (closed module) $M'' = (M')'$ lying in the original space. It is plain that $M'' \supseteq M$. Since moreover, on account of (11)', we have $M'' = (\overline{M})''$ we see that

(12) $$M'' \geqq \bar{M}.$$

We shall prove that when M is a module the sign of equality holds in (12). In fact, analogously to theorem 2 in the finite-dimensional case we have the following two important theorems in our infinite-dimensional spaces.

Theorem 6. For an arbitrary module M in R_∞ the dual module M" of its dual module M' is the closure $\bar{M}$ of M, i.e.,

$$M'' = \bar{M}.$$

Theorem 6a. For an arbitrary module M in R^∞ the dual module M" of its dual module M' is the closure $\bar{M}$ of M, i.e.,

$$M'' = \bar{M}.$$

We begin by proving theorem 6a which, as we shall see, may easily be deduced from the analogous theorem 2 for the m-dimensional case, m = 1, 2, ... As to theorem 6 (which will not be needed for the establishment of our main result) we postpone its proof to 16; it follows the same line as the proof of theorem 2 and will be based upon a structure theorem for closed modules in R_∞. This structure theorem R_∞, which will also be an important tool in the proof of our main result, i.e., the structure theorem R^∞, will be given below in 14.

Proof of theorem 6a. We have already observed that $M'' \geqq \bar{M}$. Thus we have only to prove that $M'' \subseteqq \bar{M}$. Let then $Y = (y_1, y_2, \ldots)$ be an arbitrary point in M". In order to show that Y lies in $\bar{M}$, let m be an arbitrary positive integer. We consider the points $(a_1, a_2, \ldots, a_m, 0, 0, \ldots) = (a_1, a_2, \ldots, a_m)$ from the common part L of M' and the $a_1a_2\ldots a_m$-space. Then for every point in L we have

(13) $$(y_1, y_2, \ldots, y_m)\cdot(a_1, a_2, \ldots, a_m) \equiv 0 \pmod 1.$$

Next, let $\mathcal{M}$ denote the projection of M on the $x_1x_2\ldots x_m$-space

(i.e., the set of points $(x_1, x_2, \ldots, x_m)$ arising from the points $(x_1, x_2, \ldots)$ of M by cancelling all coordinates with indices $> m$). Then plainly $L = \mathcal{M}'$, and thus on account of (13) the point $(y_1, y_2, \ldots, y_m)$ belongs to $\mathcal{M}''$. Now, according to theorem 2,

$$\mathcal{M}'' = \bar{\mathcal{M}},$$

and hence $(y_1, y_2, \ldots, y_m)$ can be approximated by points $(x_1, x_2, \ldots, x_m)$ from $\mathcal{M}$. Since m is arbitrary it follows that $Y = (y_1, y_2, \ldots)$ can be approximated by points $(x_1, x_2, \ldots)$ from M, i.e., Y must lie in $\bar{M}$, q.e.d.

14. We shall now prove the following structure theorem for closed modules in R_∞ which (like the structure theorem R^∞ to be proved later on) is quite analogous to the structure theorem R_m.

Structure theorem R_∞. A closed module in the infinite-dimensional space R_∞ is a point set E which by a substitution can be transformed into a point set of a special form, denoted by R_∞ namely, a point set $\{(a_1, a_2, \ldots, a_n, \ldots)\}$ of the following structure: The indices 1, 2, ..., n, ... can be divided into three fixed classes $\{n_r\}$, $\{n_s\}$, $\{n_t\}$ which depend only on the point set, such that the coordinates a_{n_r} independently run through all numbers, and the coordinates a_{n_s} independently run through all integers, while all the remaining coordinates a_{n_t} are constantly zero. Only, of course, the simultaneous variation of the a_{n_r} and the a_{n_s} in the set is limited by the obvious demand that $(a_1, a_2, \ldots)$ always shall lie in R_∞, i.e., have 0 from a certain coordinate place (depending on the point). Conversely, each such point set E is a closed module.

Again the latter part of the theorem is obvious.

In order to prove the theorem (i.e., the first part of it) we shall use the structure theorem R_m in the following form:

Structure Theorem R_m. A closed module H in the m-dimensional space R_m is a point set to which can be found linearly independent vectors $F_1, \ldots, F_p, V_1, \ldots, V_q$ $(p + q \leqq m)$ such that the point set consists of all points of the form

$$X = \xi_1 F_1 + \ldots + \xi_p F_p + h_1 V_1 + \ldots + h_q V_q$$

where $\xi_1, \ldots, \xi_p$ independently run through all numbers, and $h_1, \ldots, h_q$ independently run through all integers.

We shall say that the vectors $F_1, \ldots, F_p, V_1, \ldots, V_q$ generate H with arbitrary and integral coefficients, respectively.

Moreover, we also need the following recursive information on the freedom with which we can choose the generating vectors $F_1, \ldots, F_p, V_1, \ldots, V_q$ for a given closed module in R_m.

Additional Theorem R_m. If H is a closed module in R_m and S is a subspace (vector space) of R_m we can choose the generating vectors of H by choosing first in an arbitrary way generating vectors of the closed module $S \cap H$[1] and next supplementing these vectors with certain other vectors (if necessary). The generating vectors of $S \cap H$ shall be taken with the same type of coefficients (arbitrary ξ or integral h) used in the generation of H.

We are now able to prove the structure theorem R_∞. We denote by H_m the common part of H and the $x_1 \ldots x_m$-space. Then, obviously, H_m is in the usual sense a closed module in the

1) $S \cap H$ denotes the common part of S and H.

$x_1 \ldots x_m$-space. Furthermore, H_m is the common part of H_{m+1} and the $x_1 \ldots x_m$-space. Hence it follows from the additional theorem R_m, $m = 1, 2, \ldots$, that we can generate successively the closed modules H_1, H_2, ... by linearly independent vectors with arbitrary and integral coefficients in such a way that the generating vectors of H_{m+1} are the generating vectors of H_m with the same type of coefficients, in connection with other vectors (if necessary). In this way we get a sequence of linearly independent vectors G_1, G_2, ... which provided with suitable types of coefficients (integral or arbitrary) will generate H (generation, of course, in the sense that for each vector of H only a finite number of generators is used). With arbitrary coefficients the vectors span a subspace Q(H) of R_∞. Let Q_1 denote the common part of Q(H) and the x_1-axis. If the space Q_1 is not the whole x_1-axis, but only the 0-vector, we place a non-zero vector on the x_1-axis. Then this vector together with Q(H) will span a space $Q^{(1)}$ which contains the x_1-axis. If Q(H) itself contains the x_1-axis we put $Q^{(1)} = Q(H)$. Next, let Q_2 denote the common part of $Q^{(1)}$ and the x_1x_2-plane. If the space Q_2 is not the whole x_1x_2-plane, but only the x_1-axis we place a vector in the x_1x_2-plane outside the x_1-axis. Then this vector together with $Q^{(1)}$ will span a space $Q^{(2)}$ which contains the x_1x_2-plane. If $Q^{(1)}$ itself contains the x_1x_2-plane we put $Q^{(2)} = Q^{(1)}$. In this way we continue. If the vectors thus found in one way or the other are put into a sequence with the vectors G_1, G_2, ... we get a sequence of linearly independent vectors U_1, U_2, ... which provided with suitable types of coefficients (zero, integral or arbitrary) will generate H and with mere arbitrary coefficients the whole space R_∞. The linear independence of U_1, U_2, ...

secures that each point in R_∞ has only one representation by this generation. Hence

$$B = a_1U_1 + a_2U_2 + \cdots$$

is a substitution in R_∞ of $A = (a_1, a_2, \cdots)$ into B. It takes the fundamental vectors (1, 0, 0, $\cdots$), (0, 1, 0, $\cdots$), ... into the vectors U_1, U_2, ... Therefore the inverse substitution, which takes U_1, U_2, ... into the fundamental vectors, will take the closed module H into a set $\{(a_1, a_2, \cdots)\}$ determined by $a_i = 0$ for certain i, a_i arbitrary integral for certain i, and a_i arbitrary for the remaining i. This proves structure theorem R_∞.

15. We have now obtained all the means necessary to prove the main theorem of this paper, i.e., the structure theorem R^∞. Let first H be an arbitrary closed module in R_∞. Then, on account of the structure theorem R_∞, there exists a substitution T in R_∞ such that T(H) has the special form S_∞ mentioned in the structure theorem R_∞. The dual module H' of H is a closed module in R^∞. We shall show that H' by a substitution can be taken into a closed module of the special form S^∞ mentioned in the structure theorem R^∞. In fact, the dual transformation T' of T has this property, for on account of theorem 1 (for R_∞) $T'(H') = (T(H))'$ and $(T(H))'$ is, as the dual module of a closed module of the special type S_∞, itself of the special type S^∞ as appears from the following

Theorem 7. The dual module of a module of the type S_∞, and more explicitly, of a point set $\{(a_1, a_2, \cdots)\}$ where the a_{n_r} are arbitrary, the a_{n_s} integral, and the a_{n_t} zero, is a module of the type S^∞, namely more explicitly the point set $\{(x_1, x_2, \ldots)\}$

<u>where the</u> x_{n_r} <u>are zero, the</u> x_{n_s} <u>integral, and the</u> x_{n_t} <u>arbitrary</u>.

The proof is quite analogous to the proof of theorem 3 and may therefore be omitted.

We saw above that the dual module of a closed module in R_∞ by a substitution can be taken into a point set of the special type S^∞. The structure theorem R^∞ will therefore follow if we can show that <u>every</u> closed module H in R^∞ is the dual module of a closed module K in R_∞. This, however, is an immediate consequence of theorem 6a, on account of which $H = (H')'$ so that we may use H' for K.

<u>16</u>. We now pass to the proof of theorem 6. Let M be an arbitrary module in R_∞. We shall show that $M'' = \bar{M}$. Since from (11) we have $(\bar{M})' = M'$ it is enough to prove that $H'' = H$ for an arbitrary closed module H in R_∞. If T is an arbitrary substitution in R_∞ this is equivalent to $T(H'') = T(H)$. Replacing the first T by T" by help of theorem 5 and using theorem 1 twice on the left-hand side of the equation (for the spaces R^∞ and R_∞) we see that it can be written $(T(H))'' = T(H)$. This is the original equation for the transformed module T(H). Now by help of structure theorem R_∞ we can choose T such that T(H) becomes a set of the type S_∞. The equation $(T(H))'' = T(H)$ is then an immediate consequence of theorem 7 and its analogous theorem.

<u>17</u>. We finally remark that the duality considered above in connection with the space R_∞ and R^∞ is the sort of duality which Pontrjagin has established between an abelian group and its character group. It is, moreover, a simple example where Pontrjagin's assumption of local compactness is not fulfilled.

FUNCTION THEORY

A THEOREM CONCERNING POWER SERIES

By Harald Bohr.

Communicated by G. H. Hardy.*

[Received February 5th, 1913.—Read February 13th, 1913.]

1. The theory of Diophantine approximation has, as I have shown in a series of previous papers,† important applications to the theory of Dirichlet's series. It becomes clear in the course of these investigations that the theory of the absolute convergence of Dirichlet's series of the type $\Sigma a_n n^{-s}$ is very closely connected with the theory of power-series in an infinite number of variables. As an illustration of this I may mention the following theorem:—

Let $\Sigma a_n n^{-s}$ be a Dirichlet's series absolutely convergent for $\sigma = \mathfrak{R}(s) = \sigma_0$, and let

$$P(x_1, x_2, \ldots) = c_0 + \sum_{\alpha} c_\alpha x_\alpha + \sum_{\alpha, \beta} c_{\alpha, \beta} x_\alpha x_\beta + \ldots$$

be the power-series which may be obtained formally from the Dirichlet's series by writing

$$p_1^{-s} = x_1, \quad p_2^{-s} = x_2, \quad \ldots,$$

where p_r denotes the r-th prime number. Let θ denote the set of points

* Extracted from letters of the author.

† See, for example, *Acta Mathematica*, Vol. 36, pp. 197 240.

in the plane of the complex variable corresponding to the values taken by the Dirichlet's series when the variable $s=\sigma+it$ *describes the line* $\sigma=\sigma_0$, *and* Θ *the set of points corresponding to the values taken by the power-series when the variables* $x_1, x_2, \ldots$ *describe independently the circles*

$$|x_1| = p_1^{-\sigma_0}, \quad |x_2| = p_2^{-\sigma_0}, \quad \ldots .$$

Then the set θ, *which is obviously part of* Θ, *is everywhere dense in* Θ.

In particular the solution of what is called the "absolute convergence problem" for Dirichlet's series of the type $\Sigma a_n n^{-s}$ must be based upon a study of the relations between the absolute value of a power-series in an infinite number of variables on the one hand, and the sum of the absolute values of the individual terms of this series on the other. It was in the course of this investigation that I was led to consider a problem concerning power-series in one variable only, which we discussed last year, and which seems to be of some interest in itself.

2. The question which we discussed was as follows. *Let* x_1 *be a positive number between* 0 *and* 1. *Is it always possible to find a power-series* $\Sigma a_n x^n$ *such that*

(1) $f(x) = \Sigma a_n x^n$ *is regular for* $|x| < 1$ *and continuous for* $|x| \leqslant 1$,

(2) $|f(x)| < 1$ *for* $|x| \leqslant 1$,

(3) $\Sigma |a_n| x_1^n > 1$?

It is obvious that the question is to be answered in the affirmative when x_1 is sufficiently near to 1: in fact, the hypotheses (1) and (2) are perfectly consistent with

$$\Sigma\, |a_n|\, x_1^n \rightarrow \infty,$$

as $x_1 \rightarrow 1$. I can, however, now prove that the *general* question must be answered *negatively*. This is shown by the following theorem.

THEOREM.—*If the conditions* (1) *and* (2) *are satisfied, then*

$$\Sigma\, |a_n x^n| < 1,$$

for $|x| \leqslant \frac{1}{6}$; *i.e.*, $\Sigma\, |a_n|\, 6^{-n} < 1$.

It is plain that we may, without loss of generality, suppose a_0 real and positive. Then

$$a_0 = f(0) < 1.$$

Let
$$g(x) = f(x) - a_0 = \sum_1^\infty a_n x^n,$$
$$R = \max_{|x|=1} \mathfrak{R}\{g(x)\},$$
$$\mathbf{m} = \max_{|x|=\frac{1}{2}} |g(x)|.$$

We use the inequality of Carathéodory*
$$\max_{|x|=\rho} |F(x)| \leqslant |\gamma| + |\beta| \frac{r-\rho}{r+\rho} + \frac{2\rho}{r-\rho} \max_{|x|=r} \mathfrak{R}\{F(x)\},$$
in which $F(x)$ is supposed regular for $|x| \leqslant r$,† and $0 < \rho < r$, and
$$F(0) = \beta + i\gamma.$$

Taking $F(x) = g(x)$, so that $\beta = \gamma = 0$, and $r = 1$, $\rho = \frac{1}{2}$, we obtain
$$\mathbf{m} \leqslant 2R.$$
Now $R \geqslant 0$, and so
$$a_0 + R = a_0 + \max_{|x|=1} \mathfrak{R}\{g(x)\} \leqslant \max_{|x|=1} |a_0 + g(x)| = \max_{|x|=1} |f(x)| < 1;$$
i.e.,
$$R < 1 - a_0,$$
and
$$\mathbf{m} = \max_{|x|=\frac{1}{2}} |g(x)| < 2(1-a_0).$$
But
$$a_n = \frac{1}{2\pi i} \int_{|x|=\frac{1}{2}} \frac{g(x)}{x^{n+1}}\, dx \quad (n \geqslant 1),$$
and so
$$|a_n| \leqslant 2^n \mathbf{m} < 2^{n+1}(1-a_0).$$

Accordingly, for $|x| \leqslant \frac{1}{6}$, we have
$$\sum_0^\infty |a_n x^n| = a_0 + \sum_1^\infty |a_n x^n| < a_0 + 2(1-a_0) \sum_1^\infty 2^n |x|^n$$
$$= a_0 + \frac{4(1-a_0)|x|}{1-2|x|} \leqslant a_0 + (1-a_0) = 1.$$
Thus the theorem is proved.

3.‡ If k is any positive number less than unity, it either is or is not

* See, for example, Landau, *Handbuch*, p. 299.

† The inequality obviously still holds if $F(x)$ is regular for $|x| < r$, and continuous for $|x| \leqslant r$.

‡ This section is extracted from a later letter dated June 19th, 1913.

true that the hypotheses (1) and (2) involve

$$\Sigma \mid a_n x^n \mid < 1$$

for $|x| \leqslant k$. It follows from my theorem that the numbers k for which the implication holds have a positive upper limit K, and that $K \geqslant \frac{1}{6}$.

The problem remains of the exact determination of the value of K. I have learnt recently that Messrs. M. Riesz, Schur, and Wiener, whose attention had been drawn to the subject by my theorem, have succeeded independently in solving this problem completely. Their solutions show that $K = \frac{1}{3}$. Mr. Wiener has very kindly given me permission to reproduce here his very simple and elegant proof of this result.

Wiener proves first that if the hypotheses (1) and (2) are satisfied, and if, as above, we assume that $0 \leqslant a_0 < 1$, then the inequality

$$|a_n| < 2^{n+1}(1-a_0)$$

established above may be replaced by

$$|a_n| < 1-a_0^2. \tag{1}$$

For $n = 1$ this is a known result, which follows at once from the fact that the function

$$\frac{f(x)-a_0}{x\{1-a_0 f(x)\}}$$

is regular for $|x| < 1$, continuous for $|x| \leqslant 1$, numerically less than 1 for $|x| = 1$, and takes for $x = 0$ the value $a_1/(1-a_0^2)$. Now let n be any integer greater than 1, and let ρ be a primitive n-th root of 1. Then

$$F(x) = f(x)+f(\rho x)+\ldots+f(\rho^{n-1}x) = \sum_{m=0}^{\infty} n a_{mn} x^{mn}$$

is regular for $|x| < 1$ and continuous and numerically less than n for $|x| \leqslant 1$. Hence it follows that the function

$$\phi(x) = a_0 + a_n x + a_{2n} x^2 + \ldots$$

satisfies the conditions originally imposed upon $f(x)$; and thus (1) is proved.

A fortiori, for $n \geqslant 1$,

$$|a_n| < (1+a_0)(1-a_0) < 2(1-a_0), \tag{2}$$

and so $$\sum_1^{\infty} 3^{-n} |a_n| < a_0 + 2(1-a_0) \sum_1^{\infty} 3^{-n} = a_0 + 1 - a_0 = 1.$$

Thus $$K \geqslant \tfrac{1}{3}.$$

That, on the other hand, $K \leqslant \frac{1}{3}$, follows immediately from the con-

sideration of the special function

$$\psi(x) = \frac{1-x}{1-ax} = 1+x(a-1)+x^2(a^2-a)+\ldots = \Sigma a_n x^n,$$

where $0 < a < 1$. Here

$$\max_{|x|=1} |\psi(x)| = \frac{2}{1+a},$$

and
$$\Sigma |a_n x^n| = 1+\frac{(1-a)|x|}{1-a|x|}.$$

Hence
$$\Sigma |a_n x^n| > \max_{|x|=1} \left| \frac{1-x}{1-ax} \right|,$$

for
$$1+\frac{(1-a)|x|}{1-a|x|} > \frac{2}{1+a},$$

i.e., for
$$|x| > \frac{1}{1+2a}.$$

As $a \to 1$, the last expression tends to the limit $\frac{1}{3}$. Thus $K \leqslant \frac{1}{3}$, and so $K = \frac{1}{3}$.

*

En funktionsteoretisk Bemærkning.

Af Harald Bohr, København.

1. Blandt Mængden af hele transcendente Funktioner

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots$$

findes der som bekendt, og som man meget let kan vise, saadanne, hvis absolute Værdi $|f(z)|$ vokser vilkaarlig stærkt, naar den uafhængige Variable $z = x + iy$ fjerner sig bort i det Uendelige langs en opgiven Radius-Vector, f. Eks. langs den reelle positive Akse; herved forstaas i nøjagtig Formulering, at dersom $\varphi(x)$ er en vilkaarligt opgiven, positiv, kontinuert, stedse voksende og med x i det Uendelige voksende Funktion af den positive Variable x, vil der eksistere en dertil svarende hel transcendent Funktion $f(z)$, som for alle positive x opfylder Betingelsen

$$|f(x)| > \varphi(x).$$

Denne Sætning kan man f. Eks. bevise saaledes: Lad a_0 være et positivt Tal og $n_1, n_2, \cdots n_p, \cdots$ en voksende Følge af hele positive Tal. Da er Potensrækken med positive Koefficienter

$$f(z) = a_0 + a_1 z + \cdots = a_0 + \left(\frac{z}{1}\right)^{n_1} + \left(\frac{z}{2}\right)^{n_2} + \cdots + \left(\frac{z}{p}\right)^{n_p} + \cdots$$

aabenbart konvergent for alle z (idet dens Koefficienter a_n jo opfylder Betingelsen $\sqrt[n]{a_n} \to 0$ for $n \to \infty$) og fremstiller følgelig en hel Transcendent. Jeg vil bevise, at Koefficienten a_0 samt Talfølgen $n_1, n_2, \cdots n_p \cdots$ kan vælges saaledes, at Funk-

tionen $f(z)$ opfylder den stillede Fordring: $f(x) > \varphi(x)$ for $x > 0$. Hertil vælger jeg $a_0 = \varphi(2)$, og bestemmer n_1 saaledes, at $2^{n_1} > \varphi(3)$, derefter $n_2 > n_1$ saaledes, at $\left(\frac{3}{2}\right)^{n_2} > \varphi(4), \cdots$ i Almindelighed $n_p > n_{p-1}$ saaledes, at

$$\left(\frac{p+1}{p}\right)^{n_p} > \varphi(p+2).$$

For dette Valg af Tallene $a_0, n_1, n_2, \cdots$ vil $f(z)$ øjensynlig opfylde Betingelsen $f(x) > \varphi(x)$ for $x > 0$. Thi 1) er $0 < x < 2$, finder vi

$$f(x) > a_0 = \varphi(2) > \varphi(x),$$

og 2) er $x \geqq 2$, og bestemmer vi det hele positive Tal q saaledes, at $q+1 \leqq x < q+2$, finder vi

$$f(x) > \left(\frac{x}{q}\right)^{n_q} \geqq \left(\frac{q+1}{q}\right)^{n_q} > \varphi(q+2) > \varphi(x).$$

Hermed er den omhandlede Sætning bevist.

2. I Tilknytning til den ovenfor nævnte Sætning vedrørende en hel Transcendents Vækst langs en enkelt Radius-Vector omtalte Professor *Nørlund* i en Samtale med mig følgende Problem, som han var stødt paa ved en Undersøgelse: Eksisterer der en hel Transcendent $f(z)$, som ikke blot langs en enkelt Radius-Vector, men indenfor en hel Parallelstrimmel, f. Eks. indenfor den ved Ulighederne $x > 0$, $-1 < y < 1$ bestemte Strimmel, vokser vilkaarlig stærkt i det Uendelige, d. v. s. som i ethvert Punkt $z = x + iy$ indenfor den nævnte Strimmel opfylder Betingelsen

$$|f(z)| = |f(x+iy)| > \varphi(x),$$

hvor $\varphi(x)$ som ovenfor er en vilkaarligt given, positiv, kon-

tinuert, stedse voksende og i det Uendelige voksende Funktion af den positive Variable x? Hensigten med denne lille Afhandling er i første Linie at meddele Løsningen af dette Problem, nemlig at vise, hvorledes man ved Anvendelse af en Sætning indenfor Læren om konform Afbildning i Forbindelse med en kendt funktionsteoretisk Sætning af *Carathéodory* kan indse, at det stillede Spørgsmaal maa besvares benægtende.

3. Jeg skal imidlertid ikke indskrænke mig til at bevise den i 2 opstillede Paastand om, at en hel Transcendent ikke kan vokse vilkaarlig stærkt i det Uendelige indenfor en Parallelstrimmel, men skal udlede et væsentligt almindeligere Resultat, hvis Bevis ikke er vanskeligere og benytter ganske den samme Tankegang, som Beviset for den ovenfor nævnte speciellere Sætning. Det almindelige Resultat, jeg vil udlede, lyder saaledes:

Lad der i den komplekse z-Plan være givet et Omraade Ω bestemt ved Ulighederne

$$x > 0, \quad -\omega(x) < y < \omega(x),$$

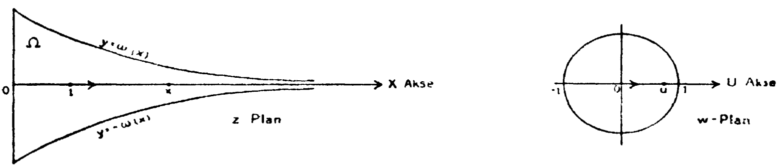
hvor $\omega(x)$ er en for $x \geqq 0$ defineret, positiv, kontinuert Funktion, som aftager mod 0 for $x \to \infty$. Der eksisterer da en til dette Omraade hørende (d. v. s. alene ved Omraadet Ω bestemt) positiv, kontinuert Funktion $\psi(x)$ af den positive Variable x med følgende Egenskab: Enhver Funktion $f(z) = f(x + iy)$, der er analytisk indenfor Omraadet Ω, og hvis numeriske Værdi $|f(z)|$ i hele dette Omraade forbliver større end en positiv Konstant k, vil for alle tilstrækkelig store Værdier af den positive Variable x, d. v. s. for $x > x_0$, opfylde Betingelsen

$$|f(x)| < \psi(x).$$

[Denne Sætning udsiger aabenbart i løs Formulering, at hvis en i Omegnen af en Radius-Vector analytisk Funktion $f(z)$

vokser meget stærkt i numerisk Værdi, naar z fjerner sig i det Uendelige langs denne Radius-Vector, da vil $f(z)$ nødvendigvis »svinge« meget stærkt i den umiddelbare Nærhed af Radius-Vectoren, og vil specielt i denne umiddelbare Nærhed antage numerisk vilkaarlig smaa Værdier.]

Bevis: Jeg afbilder det indre af Omraadet Ω i z-Planen een-eentydigt og konformt paa det Indre af Enhedscirklen $|w| < 1$ i en ny kompleks Plan, hvis Variable jeg betegner med $w = u + iv$; denne Afbildning vælger jeg saaledes, hvad der i Følge en Hovedsætning indenfor Læren om konform Afbildning er



muligt paa en og kun een Maade, at Punktet $z = 1$ indenfor Omraadet Ω afbildes i Centrum $w = 0$ af Enhedscirklen, og at de paa Figurerne ved Pile betegnede Retninger udfra henholdsvis Punkterne $z = 1$ og $w = 0$ svarer til hinanden. Med andre Ord, jeg afbilder Omraadet Ω paa Enhedscirklen $|w| < 1$ ved Hjælp af en analytisk Funktion $w = \alpha(z)$, der opfylder Betingelserne: $\alpha(1) = 0$ og $\alpha'(1)$ reel positiv Af Symmetrigrunde vil ved denne Afbildning den positive Akse $0 < x < \infty$ i z-Planen aabenbart afbildes paa Liniestykket $-1 < u < 1$, d. v. s. Funktionen $u = \alpha(x)$ vil være en reel, kontinuert og stedse voksende Funktion af den positive Variable x, som naar x vokser fra 0 til ∞ vil vokse fra -1 til $+1$. — Jeg paastaar nu, at den (alene ved Omraadet Ω bestemte) stedse voksende og i det Uendelige voksende Funktion af den positive Variable x

$$\psi(x) = e^{\frac{1}{(\alpha(x)-1)^2}}$$

opfylder den i Sætning opstillede Fordring*).

*) Jeg fremhæver, at jeg ikke har søgt at angive den »bedst mulige« Funktion $\psi(x)$, men kun at bevise Eksistensen af en Funktion $\psi(x)$ med den omhandlede Egenskab.

Dette indses saaledes: Lad $f(z) = f(x+iy)$ være en Funktion af den i Sætningen omhandlede Art, d. v. s. en Funktion, som er analytisk indenfor Omraadet Ω, og hvis numeriske Værdi $|f(z)|$ indenfor hele Ω er $> k > 0$. Idet $f(z)$ altsaa specielt er $\neq 0$ indenfor Ω, deler den uendeligtydige Funktion $\log f(z)$ sig op indenfor Omraadet Ω i uendelig mange eentydige regulære Grene; jeg betegner en vilkaarlig valgt af dem med $F(z)$. Der gælder da indenfor hele Ω Uligheden*)

$$\Re(F(z)) = \operatorname{Log}|f(z)| > \operatorname{Log} k;$$

Funktionen $-\Re(F(z))$ er altsaa indenfor Omraadet Ω mindre end en positiv Konstant K. Endvidere vil jeg til Afkortning betegne den absolute Værdi af Tallet $\Re(F(1))$ med K_1. Samtidig med den indenfor Ω analytiske Funktion $F(z)$ betragter jeg den »tilsvarende« indenfor Enhedscirklen $|w| < 1$ analytiske Funktion $G(w)$, d. v. s. den Funktion $G(w)$, der er saaledes forbundet med $F(z)$, at $F(z)$ og $G(w)$ antager samme Værdi i henholdsvis det ene og det andet af to vilkaarlige Punkter z og w, som svarer til hinanden ved den ovenfor betragtede konforme Afbildning; med andre Ord, der gælder Ligningen $G(w) = G(\alpha(z)) = F(z)$. Specielt har vi da indenfor hele Enhedscirklen $|w| < 1$ Uligheden $-\Re(G(w)) < K$, ligesom vi har $|\Re(G(0))| = K_1$.

Jeg anvender nu følgende kendte Sætning af *Carathéodory*: Opfylder den for $|w| < 1$ regulære Funktion $G(w)$ indenfor hele Enhedscirklen Uligheden $-\Re(G(w)) < K$, samt er $|\Re(G(0))| = K_1$, da gælder der i ethvert Punkt w indenfor Enhedscirklen Uligheden

$$\Re(G(w)) \leqq K_1 \frac{1+|w|}{1-|w|} + 2K \frac{|w|}{1-|w|}.$$

Betydningen af denne Ulighed ligger deri, at den i det vil-

*) Her og i det følgende betegner $\Re(z)$ den reelle Del af det komplekse Tal z, medens $\operatorname{Log} x$ angiver den reelle Logarithme af det positive Tal x.

kaarlige Punkt w indenfor Enhedscirklen giver os en højere Grænse for den positive reelle Del $+\Re(G(w))$ udfra Kendskabet til 1) en højere Grænse for den negative reelle Del $-\Re(G(w))$ for $|w| < 1$ og 2) den absolute Værdi af $\Re(G(w))$ i Centrum $w = 0$. Anvender vi *Carathéodory*'s Ulighed paa vor Funktion $G(w)$, finder vi specielt, idet vi sætter $w = u$, hvor u er et vilkaarligt reelt Tal beliggende paa Radien $0 < u < 1$, Uligheden

$$\Re(G(u)) \leq K_1 \frac{1+u}{1-u} + 2K \frac{u}{1-u} < \frac{2K_1 + 2K}{1-u},$$

altsaa (ved grov Regning) for alle u tilstrækkelig nær ved 1, d. v. s for $u_0 < u < 1$

$$\Re(G(u)) < \frac{1}{(1-u)^2}.$$

Sætter vi her $u = \alpha(x)$, finder vi for alle tilstrækkelig store Værdier af x, d. v. s. for $x > x_0$, hvor x_0 er det Punkt indenfor Ω, der svarer til Punktet u_0 indenfor Enhedscirklen, Uligheden

$$\Re(G(\alpha(x))) = \Re(F(x)) < \frac{1}{(1-\alpha(x))^2}.$$

Hermed er Beviset for den opstillede Sætning fuldført; thi idet $|f(x)| = e^{\Re(F(x))}$, er den sidst fundne Ulighed ensbetydende med

$$|f(x)| < e^{\frac{1}{(1-\alpha(x))^2}} = \psi(x),$$

hvilket netop er den Ulighed, vi skulde bevise.

Über die Koeffizientensumme einer beschränkten Potenzreihe.

Von

Harald Bohr in Kopenhagen.

Vorgelegt in der Sitzung vom 11. November 1916 durch Herrn Landau.

Einleitung.

Es sei

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

im Einheitskreise $|x| < 1$ regulär, und es werde

$$s_n = \sum_{\nu=0}^{n} a_\nu$$

gesetzt. Es bezeichne G_n, bei festem n, die obere Grenze von $|s_n|$ für alle im Einheitskreise $|x| < 1$ regulären Funktionen $f(x)$, die der Bedingung $|f(x)| < 1$ für $|x| < 1$ genügen. Durch eine sinnreiche Methode ist es bekanntlich Landau[1]) gelungen, für jedes $n \geqq 1$[2]) die Zahl G_n zu bestimmen, und zwar mit dem Ergebnis

$$G_n = \sum_{\nu=0}^{n} \binom{-\frac{1}{2}}{\nu}^2 = 1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 + \cdots + \left(\frac{1 \cdot 3 \ldots (2n-1)}{2 \cdot 4 \ldots 2n}\right)^2.$$

Hiernach ist, wie mit Hilfe der Stirlingschen Formel unmittelbar zu beweisen ist[3]),

---

1) Vergl. z. B. das schöne Buch von Landau. Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie. 110 S Berlin 1916, worauf ich betreffs der Literatur verweise.

2) Für $n = 0$ ist offenbar $G_n = 1$.

3) Landau l. c. S. 22—23.

$$G_n \sim \frac{1}{\pi} \log n,$$

d. h. es ist

$$\lim_{n=\infty} \frac{G_n}{\log n} = \frac{1}{\pi}.$$

Wir wenden uns nun zu einer wesentlich andererartigen Fragestellung, nämlich der folgenden: Was läßt sich bei einer festen für $|x| < 1$ regulären und beschränkten Funktion $f(x)$ über die „Maximalgrößenordnung" der Koeffizientensumme s_n bei wachsendem n aussagen? Das erste und wichtigste hierhergehörige Problem, nämlich die Frage, ob die Zahl $|s_n|$ bei jedem festen $f(x)$ eine beschränkte Funktion von n ist oder nicht, ist bekanntlich von Fejér[1]) gelöst worden, und zwar mit dem Resultate, daß $|s_n|$ bei festem $f(x)$ nicht beschränkt zu sein braucht. — Durch einen Blick auf die Methode[2]), mittels welcher Fejér eine im Einheitskreise $|x| < 1$ reguläre und beschränkte Funktion $f(x)$ mit nicht beschränkten $|s_n|$ konstruiert hat, ersieht man sofort, daß diese Methode nicht nur zum Existenzbeweis einer solchen Funktion $f(x)$ angewendet werden kann, sondern sogar imstande ist, das folgende genauere Resultat zu liefern: Es sei $\psi(n)$ eine beliebige positive Funktion von n, die $= o(\log n)$ ist, d. h. die der Bedingung $\lim_{n=\infty} \frac{\psi(n)}{\log n} = 0$ genügt. Dann gibt es zu diesem $\psi(n)$ eine im Einheitskreise $|x| < 1$ reguläre und beschränkte Funktion $f(x)$ derart, daß die Ungleichung

$$|s_n| > \psi(n)$$

für unendlich viele n erfüllt ist. — Die Fejérsche Methode erlaubt dagegen nicht, zu entscheiden, ob es eine für $|x| < 1$ reguläre und beschränkte Funktion $f(x)$ derart gibt, daß

$$\limsup_{n=\infty} \frac{|s_n|}{\log n} > 0$$

ist, oder anders ausgedrückt, ob die (z. B. aus dem Landauschen Resultate $G_n \sim \frac{1}{\pi} \log n$ unmittelbar folgende) Relation $s_n = O(\log n)$ für jede feste im Einheitskreise $|x| < 1$ reguläre und beschränkte Funktion $f(x)$ durch die schärfere Relation $s_n = o(\log n)$ ersetzt

1) Vergl. z. B. Landau, l. c. S. 8 und S. 23—25.

2) Siehe Landau, l. c. S. 24—25.

werden kann. Wie Landau in seinem Buche hervorhebt [1]), ist diese letzte Frage bisher ungelöst.

Der Zweck dieser Abhandlung ist die Erledigung dieses Problems. Es wird sich das Resultat ergeben, daß die Relation $s_n = o(\log n)$ nicht für jede feste im Einheitskreise reguläre und beschränkte Funktion gilt, mit anderen Worten:

Es existiert eine für $|x| < 1$ reguläre und beschränkte Funktion $f(x)$, für die

$$\limsup_{n=\infty} \frac{|s_n|}{\log n} > 0$$

ist.

Wegen $G_n \sim \frac{1}{\pi} \log n$ ist dieser Satz offenbar mit dem folgenden identisch:

Es gibt eine positive absolute Konstante c derart, daß bei passender Wahl einer für $|x| < 1$ regulären Funktion $f(x)$ mit $|f(x)| < 1$ für $|x| < 1$ die Ungleichung

$$|s_n| > c \,.\, G_n$$

für unendlich viele n erfüllt ist.

Ich werde übrigens ein wesentlich genaueres Resultat ableiten, nämlich gleichzeitig die obere Grenze C aller absoluten Konstanten c im Sinne des letzten Satzes bestimmen. Nach dem (für alle $f(x)$ und alle n gültigen) Satze $|s_n| \leqq G_n$ folgt sofort, daß $C \leqq 1$ ist. Ich werde beweisen, daß C tatsächlich $= 1$ ist, d. h. ich werde den folgenden Satz beweisen:

Hauptsatz: Zu jedem $0 < \lambda < 1$ gibt es eine für $|x| < 1$ reguläre Funktion $f(x)$ mit $|f(x)| < 1$ für $|x| < 1$ derart, daß die Ungleichung

$$|s_n| > (1-\lambda)\, G_n$$

für unendlich viele n besteht.

Bei dem Beweis dieses Satzes wird eine bei der Landauschen Bestimmung von G_n auftretende, von Landau herrührende Darstellungsformel für s_n sowie eine von Landau eingeführte Hilfsfunktion $f_n(x)$ eine fundamentale Rolle spielen; ich werde daher in § 1 die Landausche Bestimmung von G_n reproduzieren. In § 2 werde ich einige Hilfsbetrachtungen vorausschicken, die für den Beweis des Hauptsatzes in § 3 erforderlich sind.

---

1) Landau, l. c. S. 9.

§ 1.
Die Landausche obere Grenze von $|s_n|$[1]).

Die Landausche Darstellungsformel der Koeffizientensumme: Es werde

$$P_n(x) = \sum_{\nu=0}^{n} \binom{-\frac{1}{2}}{\nu}(-x)^\nu = 1+\frac{1}{2}x+\frac{1.3}{2.4}x^2+\cdots+\frac{1.3\ldots(2n-1)}{2.4\ldots 2n}x^n$$

und

$$Q_n(x) = \frac{1}{x^{n+1}}(P_n(x))^2$$

gesetzt. Dann gilt für jede im Einheitskreise $|x|<1$ reguläre Funktion $f(x)$ die Formel

$$s_n = \frac{1}{2\pi i}\int Q_n(x)\,f(x)\,dx,$$

wobei das Integral über einen (beliebigen) Kreis $|x| = r$ mit $0<r<1$ zu erstrecken ist.

Die Richtigkeit dieser Formel folgt sofort daraus, daß $(P_n(x))^2$ ein mit $1+x+\cdots+x^n$ beginnendes Polynom

$$1+x+\cdots+x^n+b_{n+1}x^{n+1}+\cdots+b_{n+k}x^{n+k}$$

ist, also der Faktor $Q_n(x)$ von der Form

$$Q_n(x) = \frac{1}{x}+\frac{1}{x^2}+\cdots+\frac{1}{x^{n+1}}+b_{n+1}+\cdots+b_{n+k}x^{k-1}$$

ist.

Hilfssatz: Die Nullstellen des obigen Polynoms $P_n(x)$ liegen sämtlich außerhalb des Einheitskreises $|x| = 1$.

Denn nach einem leicht beweisbaren Satz von Kakeya[2]) hat eine Gleichung

$$P(x) = c_0+c_1x+\cdots+c_nx^n = 0$$

mit $c_0>c_1>\ldots>c_n>0$ alle ihre Wurzeln absolut >1.

Satz von Landau: Für die Menge aller $f(x)$, die im Kreise $|x|<1$ regulär und absolut <1 sind, hat bei festem $n\geqq 1$ die obere Grenze von $|s_n| = |a_0+\cdots+a_n|$ den Wert

$$1+\left(\frac{1}{2}\right)^2+\left(\frac{1.3}{2.4}\right)^2+\cdots+\left(\frac{1.3\ldots(2n-1)}{2.4\ldots 2n}\right)^2 = G_n.$$

1) Der Inhalt dieses Paragraphen ist dem Landauschen Buche S. 20—23 entnommen.

2) Vergl. Landau l. c. S. 20.

Beweis: 1) Es sei $f(x)$ für $|x| < 1$ regulär und absolut < 1. Aus der Formel

$$s_n = \frac{1}{2\pi i}\int_{|x|=r} Q_n(x)\, f(x)\, dx \qquad (0 < r < 1)$$

folgt sofort, $x = re^{i\varphi}$ gesetzt,

$$|s_n| < \frac{1}{2\pi}\int_0^{2\pi} |Q_n(x)|\, r d\varphi = \frac{1}{2\pi r^n}\int_0^{2\pi} |P_n(x)|^2\, d\varphi =$$

$$\frac{1}{r^n}\sum_{\nu=0}^{n}\binom{-\frac{1}{2}}{\nu}^2 r^{2\nu}$$

also, da die linke Seite von r frei ist,

$$|s_n| \leqq \sum_{\nu=0}^{n}\binom{-\frac{1}{2}}{\nu}^2 = \sum_{\nu=0}^{n}\left(\frac{1\,.\,3\ldots(2\nu-1)}{2\,.\,4\ldots 2\nu}\right)^2 = G_n.$$

2) Ich werde (stets nach Landau) beweisen: Es existiert eine für $|x| < 1$ (übrigens sogar für $|x| \leqq 1$) reguläre Funktion $f_n(x)$ derart, daß $|f_n(x)| < 1$ für $|x| < 1$ (also $|f_n(x)| \leqq 1$ für $|x| \leqq 1$) und

$$s_n = \sum_{\nu=0}^{n}\binom{-\frac{1}{2}}{\nu}^2 = G_n$$

ist. Um die Idee des Beweises deutlich hervortreten zu lassen, sei die folgende Bemerkung vorausgeschickt: Damit eine für $|x| \leqq 1$ reguläre Funktion $f_n(x)$ die Bedingungen $s_n = G_n$ und $|f_n(x)| < 1$ für $|x| < 1$, d. h. (weil $f_n(x)$, wegen $G_n > 1$, nicht konstant sein kann) $|f_n(x)| \leqq 1$ für $|x| \leqq 1$ erfüllt, ist es offenbar, auf Grund der beiden Formeln

$$s_n = \frac{1}{2\pi i}\int_{|x|=1} Q_n(x)\, f_n(x)\, dx = \frac{1}{2\pi}\int_0^{2\pi} Q_n\left(e^{i\varphi}\right) f_n\left(e^{i\varphi}\right) e^{i\varphi}\, d\varphi$$

und

$$G_n = \frac{1}{2\pi}\int_{|x|=1} |Q_n(x)|\,.\,|dx| = \frac{1}{2\pi}\int_0^{2\pi} \left|Q_n\left(e^{i\varphi}\right)\right| d\varphi,$$

notwendig und hinreichend, daß erstens $|f_n(x)| = 1$ für $|x| = 1$ und zweitens $Q_n\left(e^{i\varphi}\right) f_n\left(e^{i\varphi}\right) e^{i\varphi} = \left|Q_n\left(e^{i\varphi}\right)\right|$ ist, also daß auf dem ganzen Einheitskreise $|x| = 1$ der absolute Wert von $f_n(x)$ gleich 1 ist und das Produkt der Funktion $f_n(x) = f_n\left(e^{i\varphi}\right)$ mit der Funktion $x\,.\,Q_n(x) = e^{i\varphi} Q_n\left(e^{i\varphi}\right) = \Phi_n(\varphi)$ die Amplitude 0 besitzt. Eine solche für $|x| \leqq 1$ reguläre Funktion $f_n(x)$ existiert aber und ist durch den Ausdruck

$$f_n(x) = \frac{x^n P_n\left(\frac{1}{x}\right)}{P_n(x)} = \frac{\frac{1.3\ldots(2n-1)}{2.4\ldots 2n} + \cdots + \frac{1}{2}x^{n-1} + x^n}{1 + \frac{1}{2}x + \cdots + \frac{1.3\ldots(2n-1)}{2.4\ldots 2n}x^n}$$

gegeben; denn es ist ja für $|x| = 1$ einerseits

$$|f_n(x)| = \frac{\left|P_n\left(\frac{1}{x}\right)\right|}{|P_n(x)|} = \frac{\left|P_n\left(e^{-i\varphi}\right)\right|}{\left|P_n\left(e^{i\varphi}\right)\right|} = 1,$$

während andererseits für $|x| = 1$

$$x \cdot Q_n(x) \cdot f_n(x) = x\,\frac{(P_n(x))^2}{x^{n+1}}\,\frac{x^n P_n\left(\frac{1}{x}\right)}{P_n(x)} = P_n(x) \cdot P_n\left(\frac{1}{x}\right) =$$
$$|P_n(x)|^2 = |Q_n(x)|$$

ist. Für dies $f_n(x)$ ist also $s_n = G_n$, womit der Satz bewiesen ist.

§ 2.
Beweis einiger Hilfssätze.

Es seien durchweg im Folgenden (wie in § 1) mit $P_n(x)$, $Q_n(x)$, $\Phi_n(\varphi)$ und $f_n(x)$ die vier Funktionen

$$P_n(x) = \sum_{\nu=0}^{n} \binom{-\frac{1}{2}}{\nu}(-x)^\nu \qquad (\text{für } |x| \leqq 1),$$

$$Q_n(x) = \frac{1}{x^{n+1}}(P_n(x))^2 \qquad (\text{für } 0 < |x| \leqq 1),$$

$$\Phi_n(\varphi) = e^{i\varphi} Q_n\left(e^{i\varphi}\right) \qquad (\text{für alle reellen } \varphi),$$

$$f_n(x) = \frac{x^n P_n\left(\frac{1}{x}\right)}{P_n(x)} \qquad (\text{für } |x| \leqq 1)$$

bezeichnet. Dann ist speziell $\left|f_n\left(e^{i\varphi}\right)\right| = 1$ und $\Phi_n(\varphi) f_n\left(e^{i\varphi}\right) = |\Phi_n(\varphi)|$ für $0 \leqq \varphi < 2\pi$, sowie

$$\frac{1}{2\pi}\int_0^{2\pi} \Phi_n(\varphi) f_n\left(e^{i\varphi}\right) d\varphi = \frac{1}{2\pi}\int_0^{2\pi} |\Phi_n(\varphi)|\, d\varphi = G_n,$$

wo G_n die Landausche Zahl

$$1 + \left(\frac{1}{2}\right)^2 + \cdots + \left(\frac{1.3\ldots(2n-1)}{2.4\ldots 2n}\right)^2$$

bezeichnet. Es sei ferner durchweg λ eine feste Zahl im Intervalle $0 < \lambda < 1$, nämlich die im Hauptsatze auftretende Zahl.

Hilfssatz 1: Es sei δ eine beliebige Zahl im Intervalle $0 < \delta < \pi$. Dann gilt für alle hinreichend großen n, d. h. für $n > N = N(\delta, \lambda)$, die Ungleichung

$$\frac{1}{2\pi}\int_{\delta}^{2\pi-\delta} |\Phi_n(\varphi)|\, d\varphi < \frac{\lambda}{50}\cdot G_n,$$

also $\left(\text{wegen } \frac{1}{2\pi}\int_0^{2\pi} |\Phi_n(\varphi)|\, d\varphi = G_n\right)$

$$\frac{1}{2\pi}\int_{-\delta}^{\delta} |\Phi_n(\varphi)|\, d\varphi > \left(1-\frac{\lambda}{50}\right)\cdot G_n.$$

Beweis: Es konvergiert auf dem Bogen $\delta \leqq \varphi \leqq 2\pi - \delta$ des Einheitskreises $x = e^{i\varphi}$ die Potenzreihe

$$P(x) = \sum_{\nu=0}^{\infty} \binom{-\frac{1}{2}}{\nu}(-x)^{\nu}$$

gleichmäßig gegen die Funktion $(1-x)^{-\frac{1}{2}}$, d. h. es konvergiert dort die Funktion $P_n(x)$ für $n \to \infty$ gleichmäßig gegen $(1-x)^{-\frac{1}{2}}$. Folglich konvergiert auf demselben Bogen die Funktion

$$|\Phi_n(\varphi)| = |Q_n(x)| = |P_n(x)|^2$$

gleichmäßig gegen $\frac{1}{|1-x|}$. Hieraus folgt aber sofort, daß für alle hinreichend großen n, d. h. für $n > N_0(\delta)$,

$$\frac{1}{2\pi}\int_{\delta}^{2\pi-\delta} |\Phi_n(\varphi)|\, d\varphi < \frac{1}{\pi}\int_{\delta}^{2\pi-\delta} \frac{d\varphi}{|1-e^{i\varphi}|} = K$$

ist, wo K nur von δ abhängt, also (wegen $G_n \to \infty$), daß für $n > N(> N_0)$

$$\frac{1}{2\pi}\int_{\delta}^{2\pi-\delta} |\Phi_n(\varphi)|\, d\varphi < \frac{\lambda}{50}\, G_n$$

ist.

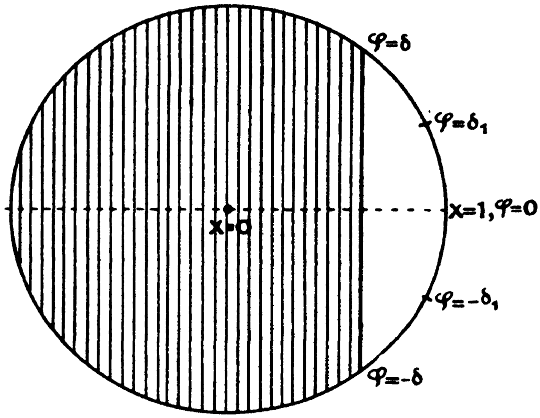
Hilfssatz 2: Es sei N eine beliebige positive ganze Zahl, und $0 < \delta \leqq \pi$, $0 < \varepsilon < \frac{1}{2}$ beliebig gegeben. Dann gibt es dazu ein ganzes $n > N$ und eine für $|x| \leqq 1$ reguläre Funktion $g(x)$ mit den folgenden Eigenschaften:

1) $$\frac{1}{2\pi}\int_{-\delta}^{\delta}|\Phi_n(\varphi)|\,d\varphi > \left(1-\frac{\lambda}{50}\right)G_n$$

oder (was damit gleichbedeutend ist)

2) $$\frac{1}{2\pi}\int_{\delta}^{2\pi-\delta}|\Phi_n(\varphi)|\,d\varphi < \frac{\lambda}{50}\,G_n.$$

$$|g(x)| \leqq 1-\frac{\lambda}{7} \quad \text{für } |x| \leqq 1.$$



Figur 1.

3) $|g(x)| < \varepsilon$ in demjenigen Teil des Einheitskreises $|x| \leqq 1$, für den $\Re(x) \leqq \cos\delta$[1] ist (siehe Figur 1).

4) $$\frac{1}{2\pi}\left|\int_{-\delta}^{\delta}\Phi_n(\varphi)\,g\left(e^{i\varphi}\right)d\varphi\right| > \left(1-\frac{\lambda}{3}\right)G_n.$$

Beweis: Ich bestimme zunächst eine positive Zahl $m = m(\delta, \varepsilon)$ so, daß $e^{m(\cos\delta - 1)} < \varepsilon$ ist, also die Funktion $e^{m(x-1)}$ im Gebiete $|x| \leqq 1$, $\Re(x) \leqq \cos\delta$ absolut $< \varepsilon$ ist. Danach bestimme ich (was offenbar, da $e^{m(x-1)} = 1$ für $x = 1$ ist, möglich ist) eine positive Zahl $\delta_1 < \delta$ derart, daß auf dem Bogen $-\delta_1 \leqq \varphi \leqq \delta_1$ des Einheitskreises $x = e^{i\varphi}$ die Ungleichung

$$\Re\left(e^{m(x-1)}\right) > 1-\frac{\lambda}{7}$$

1) $\Re(z)$ bedeute stets den reellen Teil der komplexen Zahl z.

besteht. Zu diesem δ_1 bestimme ich schließlich, auf Grund des Hilfssatzes 1, eine ganze Zahl $n > N$ derart, daß

$$\frac{1}{2\pi}\int_{-\delta_1}^{\delta_1} |\Phi_n(\varphi)|\,d\varphi > \left(1-\frac{\lambda}{50}\right) G_n$$

ist. Ich beweise nun: diese Zahl $n > N$ und die dazu gehörige für $|x| \leqq 1$ reguläre Funktion $g(x) = \left(1-\frac{\lambda}{7}\right).\,e^{m(x-1)} f_n(x)$ erfüllen die sämtlichen Bedingungen des Hilfssatzes 2.

1) Wegen $\delta_1 < \delta$ folgt sofort aus

$$\frac{1}{2\pi}\int_{-\delta_1}^{\delta_1} |\Phi_n(\varphi)|\,d\varphi > \left(1-\frac{\lambda}{50}\right) G_n,$$

daß

$$\frac{1}{2\pi}\int_{-\delta}^{\delta} |\Phi_n(\varphi)|\,d\varphi > \left(1-\frac{\lambda}{50}\right) G_n$$

ist.

2) Weil im Einheitskreise $|x| \leqq 1$

$$\left|e^{m(x-1)}\right| \leqq 1 \quad , \quad |f_n(x)| \leqq 1$$

ist, gilt dort die Ungleichung

$$|g(x)| = \left(1-\frac{\lambda}{7}\right)\left|e^{m(x-1)}\right| .\, |f_n(x)| \leqq 1-\frac{\lambda}{7}.$$

3) Aus $\left|e^{m(x-1)}\right| < \varepsilon$ für $|x| \leqq 1$, $\Re(x) \leqq \cos\delta$ folgt in diesem Gebiete

$$|g(x)| = \left(1-\frac{\lambda}{7}\right)\left|e^{m(x-1)}\right| .\, |f_n(x)| < \left(1-\frac{\lambda}{7}\right).\,\varepsilon\,.\,1 < \varepsilon.$$

4) Schließlich ergibt sich

$$\begin{aligned}\frac{1}{2\pi}\left|\int_{-\delta}^{\delta} \Phi_n(\varphi)\, g\left(e^{i\varphi}\right) d\varphi\right| &> \frac{1}{2\pi}\left|\int_{-\delta_1}^{\delta_1} \Phi_n(\varphi)\, g\left(e^{i\varphi}\right) d\varphi\right| \\ &\quad - \frac{1}{2\pi}\left\{\int_{-\delta}^{-\delta_1} + \int_{\delta_1}^{\delta} |\Phi_n(\varphi)|\,d\varphi\right\} \\ &\geqq \frac{1}{2\pi}\left(1-\frac{\lambda}{7}\right)\left|\int_{-\delta_1}^{\delta_1} \Phi_n(\varphi)\, e^{m\left(e^{i\varphi}-1\right)} f_n\left(e^{i\varphi}\right) d\varphi\right| \\ &\quad - \frac{1}{2\pi}\int_{\delta_1}^{2\pi-\delta_1} |\Phi_n(\varphi)|\,d\varphi,\end{aligned}$$

also, da $\Phi_n(\varphi) f_n(e^{i\varphi}) = |\Phi_n(\varphi)|$ und $\Re e^{m(e^{i\varphi}-1)} > 1 - \frac{\lambda}{7}$ für $-\delta_1 \leqq \varphi \leqq \delta_1$ ist,

$$\frac{1}{2\pi}\left|\int_{-\delta}^{\delta} \Phi_n(\varphi)\, g(e^{i\varphi})\, d\varphi\right| > \left(1-\frac{\lambda}{7}\right)^2 \cdot \frac{1}{2\pi}\int_{-\delta_1}^{\delta_1} |\Phi_n(\varphi)|\, d\varphi$$

$$-\frac{1}{2\pi}\int_{\delta_1}^{2\pi-\delta_1} |\Phi_n(\varphi)|\, d\varphi > \left(1-\frac{\lambda}{7}\right)^2\left(1-\frac{\lambda}{50}\right) G_n - \frac{\lambda}{50} G_n$$

$$> \left\{1-\lambda\left(\frac{2}{7}+\frac{1}{50}+\frac{1}{50}\right)\right\} \cdot G_n > \left(1-\frac{\lambda}{3}\right) \cdot G_n.$$

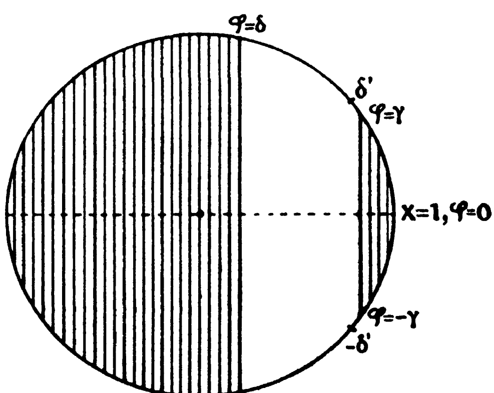
Hilfssatz 3: Es sei N eine beliebige positive ganze Zahl und $0 < \delta \leqq \pi$, $0 < \varepsilon < \frac{1}{2}$ beliebig gegeben. Dann gibt es ein ganzes $n > N$, ein positives $\gamma < \frac{\delta}{2}$ und eine für $|x| < 1$ reguläre, für $|x| \leqq 1$ stetige Funktion $h(x)$ mit den folgenden Eigenschaften:

1) $$\frac{1}{2\pi}\left\{\int_{-\delta}^{-\gamma} + \int_{\gamma}^{\delta} |\Phi_n(\varphi)|\, d\varphi\right\} > \left(1-\frac{\lambda}{25}\right) G_n,$$

also

$$\frac{1}{2\pi}\left\{\int_{\delta}^{2\pi-\delta} + \int_{-\gamma}^{\gamma} |\Phi_n(\varphi)|\, d\varphi\right\} < \frac{\lambda}{25} G_n.$$

2) $$|h(x)| \leqq 1 - \frac{\lambda}{7} \text{ für } |x| \leqq 1.$$



Figur 2.

3) $|h(x)| < \varepsilon$ in denjenigen beiden Teilen des Einheitskreises $|x| \leqq 1$ (siehe Figur 2), welche die Ungleichung $\Re(x) \leqq \cos\delta$ resp. die Ungleichung $\Re(x) \geqq \cos\gamma$ befriedigen.

4) $$\frac{1}{2\pi}\left|\int_{-\delta}^{-\gamma} + \int_{\gamma}^{\delta} \Phi_n(\varphi)\, h\left(e^{i\varphi}\right) d\varphi\right| > \left(1 - \frac{\lambda}{2}\right) G_n.$$

Beweis: Ich bestimme zunächst zu den gegebenen Zahlen N, δ und ε eine ganze Zahl $n > N$ und eine dazu gehörige für $|x| \leqq 1$ reguläre Funktion $g(x)$ im Sinne des Hilfssatzes 2. Dann ist

$$\frac{1}{2\pi}\int_{-\delta}^{\delta} |\Phi_n(\varphi)|\, d\varphi > \left(1 - \frac{\lambda}{50}\right) G_n$$

und

$$\frac{1}{2\pi}\left|\int_{-\delta}^{\delta} \Phi_n(\varphi)\, g\left(e^{i\varphi}\right) d\varphi\right| > \left(1 - \frac{\lambda}{3}\right) G_n.$$

Danach bestimme ich ein $\delta' < \frac{\delta}{2}$ so, daß

$$\frac{1}{2\pi}\int_{-\delta'}^{\delta'} |\Phi_n(\varphi)|\, d\varphi < \frac{\lambda}{50} G_n,$$

also a fortiori $\left(\text{wegen } |g(x)| \leqq 1 - \frac{\lambda}{7} < 1\right)$

$$\frac{1}{2\pi}\left|\int_{-\delta'}^{\delta'} \Phi_n(\varphi)\, g\left(e^{i\varphi}\right) d\varphi\right| < \frac{\lambda}{50} G_n$$

ist. Dann ist

$$\frac{1}{2\pi}\left\{\int_{-\delta}^{-\delta'} + \int_{\delta'}^{\delta} |\Phi_n(\varphi)|\, d\varphi\right\} > \left(1 - \frac{\lambda}{50}\right) G_n - \frac{\lambda}{50} G_n = \left(1 - \frac{\lambda}{25}\right) G_n$$

und

$$\frac{1}{2\pi}\left|\int_{-\delta}^{-\delta'} + \int_{\delta'}^{\delta} \Phi_n(\varphi)\, g\left(e^{i\varphi}\right) d\varphi\right| > \left(1 - \frac{\lambda}{3}\right) G_n - \frac{\lambda}{50} G_n > \left(1 - \frac{\lambda}{2{,}5}\right) G_n.$$

Ich wähle nunmehr die positive Zahl μ so klein, daß auf den beiden Bogen $-\delta \leqq \varphi \leqq -\delta'$, $\delta' \leqq \varphi \leqq \delta$ des Einheitskreises $x = e^{i\varphi}$ die für $|x| < 1$ reguläre, für $|x| \leqq 1$ stetige Funktion

$$\left(\frac{1-x}{2}\right)^{\mu} = e^{\mu \operatorname{Log}\left(\frac{1-x}{2}\right)}$$

$\left(\text{wo Log}\left(\frac{1-x}{2}\right)\right.$ für $\Re(x) < 1$ den Hauptwert bedeutet$\left.\right)$ um so wenig von 1 abweicht, daß

$$\frac{1}{2\pi}\left|\int_{-\delta}^{-\delta'} + \int_{\delta'}^{\delta} \Phi_n(\varphi)\, g\left(e^{i\varphi}\right)\left(\frac{1-e^{i\varphi}}{2}\right)^{\mu} d\varphi\right| > \left(1 - \frac{\lambda}{2{,}5}\right) G_n$$

ist. Es sei

$$h(x) = g(x)\left(\frac{1-x}{2}\right)^{\mu}$$

gesetzt. Dann ist $h(x)$ regulär für $|x| < 1$, stetig für $|x| \leqq 1$, und es gilt, da $\left|\left(\frac{1-x}{2}\right)^{\mu}\right| \leqq 1$ für $|x| \leqq 1$ ist, im ganzen Einheitskreise $|x| \leqq 1$ die Ungleichung

$$|h(x)| \leqq |g(x)|,$$

also speziell im Gebiete $|x| \leqq 1$, $\Re(x) \leqq \cos\delta$ die Ungleichung

$$|h(x)| \leqq |g(x)| < \varepsilon.$$

Die somit definierte für $|x| \leqq 1$ stetige Funktion $h(x)$ ist ferner Null im Punkte $x = 1$; folglich läßt sich ein positives $\gamma < \delta'$ so wählen, daß $|h(x)| < \varepsilon$ für $|x| \leqq 1$, $\Re(x) \geqq \cos\gamma$ ist. Ich behaupte nun: die obigen Zahlen $n > N$ und $\gamma < \frac{\delta}{2}$, sowie die obige für $|x| < 1$ reguläre für $|x| \leqq 1$ stetige Funktion $h(x)$ erfüllen die sämtlichen Bedingungen des Hilssatzes 3. Denn:

1) Wegen $\gamma < \delta'$ ist

$$\frac{1}{2\pi}\left\{\int_{-\delta}^{-\gamma} + \int_{\gamma}^{\delta} |\Phi_n(\varphi)|\, d\varphi\right\} > \frac{1}{2\pi}\left\{\int_{-\delta}^{-\delta'} + \int_{\delta'}^{\delta} |\Phi_n(\varphi)|\, d\varphi\right\}$$
$$> \left(1 - \frac{\lambda}{25}\right) G_n.$$

2) Es ist für $|x| \leqq 1$

$$|h(x)| \leqq |g(x)| \leqq 1 - \frac{\lambda}{7}.$$

3) Es ist $|h(x)| < \varepsilon$ für $|x| \leqq 1$, $\Re(x) \leqq \cos\delta$ und für $|x| \leqq 1$, $\Re(x) \geqq \cos\gamma$.

4) Es ist

$$\frac{1}{2\pi}\left|\int_{-\delta}^{-\gamma} + \int_{\gamma}^{\delta} \Phi_n(\varphi)\, h\left(e^{i\varphi}\right) d\varphi\right| > \frac{1}{2\pi}\left|\int_{-\delta}^{-\delta'} + \int_{\delta'}^{\delta} \Phi_n(\varphi)\, h\left(e^{i\varphi}\right) d\varphi\right|$$
$$- \frac{1}{2\pi}\int_{-\delta'}^{\delta'} |\Phi_n(\varphi)|\, d\varphi > \left(1 - \frac{\lambda}{2{,}5}\right) G_n - \frac{\lambda}{50} G_n > \left(1 - \frac{\lambda}{2}\right) G_n.$$

§ 3.
Beweis des Hauptsatzes.

Satz: Zu jedem $0 < \lambda < 1$ gibt es eine für $|x| < 1$ reguläre Funktion $f(x)$ mit $|f(x)| < 1$ für $|x| < 1$ derart, daß die Ungleichung

$$|s_n| > (1 - \lambda)\, G_n$$

für unendlich viele n besteht.

Beweis: Es sei zunächst eine Folge positiver Zahlen

$$\varepsilon_1,\ \varepsilon_2,\ \varepsilon_3,\ \ldots\ \varepsilon_p,\ \ldots$$

so gewählt, daß die Reihe $\sum\limits_{p=1}^{\infty} \varepsilon_p$ konvergiert und ihr Wert $< \frac{\lambda}{8}$ ist. Ich bestimme nunmehr, unter Benutzung des Hilfssatzes 3 des § 2, eine Folge für $|x| < 1$ regulärer und für $|x| \leqq 1$ stetiger Funktionen

$$h_1(x),\ h_2(x),\ \ldots\ h_p(x),\ \ldots,$$

eine Folge positiver ganzer Zahlen

$$1 < n_1 < n_2 \cdots < n_p \cdots$$

und eine Folge positiver Zahlen

$$\pi > \delta_1 > \delta_2 \cdots > \delta_p \cdots \qquad (\text{mit } \delta_p \to 0)$$

durch das folgende Verfahren: Zunächst wende ich den Hilfssatz 3 an auf die Zahlen $N = 1$, $\delta = \pi$, $\varepsilon = \varepsilon_1$; der Hilfssatz ergibt alsdann eine ganze Zahl $n = n_1 > 1$, eine positive Zahl $\gamma = \delta_1 < \frac{\pi}{2}$ und eine für $|x| < 1$ reguläre, für $|x| \leqq 1$ stetige Funktion $h(x) = h_1(x)$, welche den vier Bedingungen des Hilfssatzes genügen. Nachdem n_1, δ_1 und $h_1(x)$ somit festgelegt sind, wende ich wieder den Hilfssatz 3 an, jetzt aber auf die Zahlen $N = n_1$, $\delta = \delta_1$ und $\varepsilon = \varepsilon_2$, und bekomme dadurch, im Sinne des Hilfssatzes, ein ganzes $n = n_2 > n_1$, ein positives $\gamma = \delta_2 < \frac{\delta_1}{2}$ und eine für $|x| < 1$ reguläre, für $|x| \leqq 1$ stetige Funktion $h(x) = h_2(x)$. Nachdem hiermit die Zahlen n_2 und δ_2, sowie die Funktion $h_2(x)$ bestimmt sind, wende ich wiederum den Hilfssatz 3 an, nun aber auf die Zahlen $N = n_2$, $\delta = \delta_2$, $\varepsilon = \varepsilon_3$ und bekomme dadurch ein ganzes $n = n_3 > n_2$, ein positives $\gamma = \delta_3 < \frac{\delta_2}{2}$ und eine für $|x| < 1$ reguläre, für $|x| \leqq 1$ stetige Funktion $h(x) = h_3(x)$. Durch Fortsetzung dieser sukzessiven Bestimmungen erhalte ich offenbar eine Folge

für $|x| < 1$ regulärer, für $|x| \leqq 1$ stetiger Funktionen $h_1(x)$, $h_2(x)$, ... $h_p(x)$, ..., eine Folge positiver ganzer Zahlen $1 < n_1 < n_2 \cdots < n_p \cdots$ und eine Folge positiver Zahlen $\pi = \delta_0 > \delta_1 > \delta_2 \cdots > \delta_p \cdots$ (wo $\delta_p \to 0$ wegen $\delta_p < \frac{1}{2}\delta_{p-1}$) mit den folgenden Eigenschaften: Für jedes $p = 1, 2, 3, \ldots$ ist

$$1)\quad \frac{1}{2\pi}\left\{\int_{-\delta_{p-1}}^{-\delta_p} + \int_{\delta_p}^{\delta_{p-1}} |\Phi_{n_p}(\varphi)|\, d\varphi\right\} > \left(1 - \frac{\lambda}{25}\right) G_{n_p},$$

also

$$2)\quad \begin{gathered} \frac{1}{2\pi}\left\{\int_{-\delta_p}^{\delta_p} + \int_{\delta_{p-1}}^{2\pi - \delta_{p-1}} |\Phi_{n_p}(\varphi)|\, d\varphi\right\} < \frac{\lambda}{25} G_{n_p}, \\ |h_p(x)| \leqq 1 - \frac{\lambda}{7} \quad \text{für} \quad |x| \leqq 1, \end{gathered}$$

3) $|h_p(x)| < \varepsilon_p$ in den beiden Gebieten $|x| \leqq 1$, $\Re(x) \leqq \cos\delta_{p-1}$ und $|x| \leqq 1$, $\Re(x) \geqq \cos\delta_p$,

$$4)\quad \frac{1}{2\pi}\left|\int_{-\delta_{p-1}}^{-\delta_p} + \int_{\delta_p}^{\delta_{p-1}} \Phi_{n_p}(\varphi)\, h_p\left(e^{i\varphi}\right) d\varphi\right| > \left(1 - \frac{\lambda}{2}\right) G_{n_p}.$$

Ich betrachte nunmehr die unendliche Reihe

$$\sum_{q=1}^{\infty} h_q(x) = h_1(x) + h_2(x) + \cdots.$$

Es ist diese Reihe offenbar, **bei jedem** p, **im Gebiete** $|x| \leqq 1$, $\Re(x) \leqq \cos\delta_p$ **gleichmäßig konvergent**; denn es gilt ja in diesem Gebiete für jedes $q > p$ die Ungleichung $|h_q(x)| < \varepsilon_q$. Die Reihe $\sum h_q(x)$ definiert somit **im ganzen Einheitskreise** $|x| \leqq 1$ **mit Ausnahme des einzigen Punktes** $x = 1$ **eine stetige Funktion** $f(x)$, **welche für** $|x| < 1$ **regulär ist.** Ich werde beweisen, daß diese Funktion $f(x) = \sum a_n x^n$ die Bedingungen des Satzes erfüllt. Zunächst **ist** $f(x)$ **im ganzen Gebiete** $|x| \leqq 1$, $x \neq 1$ **(also speziell für** $|x| < 1$**) absolut** < 1. Denn es sei $x \neq 1$ ein beliebiger Punkt des Einheitskreises $|x| \leqq 1$, und die ganze Zahl $p = p(x) \geqq 1$ so bestimmt, daß $\cos\delta_{p-1} \leqq \Re(x) < \cos\delta_p$ ist. Dann gelten ja die Ungleichungen

$$|h_q(x)| \begin{cases} \leqq 1 - \dfrac{\lambda}{7} & \text{für} \quad q = p \\ < \varepsilon_q & \text{für} \quad q \neq p, \end{cases}$$

also

$$|f(x)| \leqq \sum_{q=1}^{\infty} |h_q(x)| < \left(1 - \frac{\lambda}{7}\right) + \sum_{q=1}^{\infty} \varepsilon_q < \left(1 - \frac{\lambda}{7}\right) + \frac{\lambda}{8} < 1.$$

Es bleibt zu beweisen, daß die Ungleichung

$$|s_n| > (1-\lambda)\, G_n$$

für unendlich viele n erfüllt ist; ich werde zeigen, daß diese Ungleichung für jedes $n = n_p$ besteht. Dies ergibt sich folgendermaßen: Es ist nach der Landauschen Formel für jedes $0 < r < 1$

$$s_{n_p} = \frac{1}{2\pi i}\int_{|x|=r} Q_{n_p}(x) f(x)\, dx.$$

also, weil die für $|x| \leqq 1$, $x \neq 1$ definierte Funktion $f(x)$ stetig und beschränkt ist, und daher das Integral alsbald über den Einheitskreis erstreckt werden kann,

$$s_{n_p} = \frac{1}{2\pi}\int_0^{2\pi} Q_{n_p}\left(e^{i\varphi}\right) f\left(e^{i\varphi}\right) e^{i\varphi}\, d\varphi = \frac{1}{2\pi}\int_0^{2\pi} \Phi_{n_p}(\varphi) f\left(e^{i\varphi}\right) d\varphi,$$

wo das Integral ein uneigentliches Integral ist $\left(\text{d. h. } \int_0^{2\pi} = \lim \int_\alpha^{2\pi-\beta} \text{ für } \alpha \to 0,\ \beta \to 0\right)$. Hieraus folgt

$$|s_{n_p}| > \frac{1}{2\pi}\left|\int_{-\delta_{p-1}}^{-\delta_p} + \int_{\delta_p}^{\delta_{p-1}} \Phi_{n_p}(\varphi) f\left(e^{i\varphi}\right) d\varphi\right|$$

$$- \frac{1}{2\pi}\left\{\int_{-\delta_p}^{\delta_p} + \int_{\delta_{p-1}}^{2\pi-\delta_{p-1}} |\Phi_{n_p}(\varphi)|\, d\varphi\right\}$$

$$> \frac{1}{2\pi}\left|\int_{-\delta_{p-1}}^{-\delta_p} + \int_{\delta_p}^{\delta_{p-1}} \Phi_{n_p}(\varphi) \sum_{q=1}^{\infty} h_q\left(e^{i\varphi}\right) d\varphi\right| - \frac{\lambda}{25} G_{n_p}.$$

Wegen der gleichmäßigen Konvergenz von $\sum h_q\left(e^{i\varphi}\right)$ in den beiden Intervallen $-\delta_{p-1} \leqq \varphi \leqq -\delta_p$, $\delta_p \leqq \varphi \leqq \delta_{p-1}$ ist das erste Glied auf der rechten Seite gleich

$$\frac{1}{2\pi}\left|\sum_{q=1}^{\infty}\left\{\int_{-\delta_{p-1}}^{-\delta_p} + \int_{\delta_p}^{\delta_{p-1}} \Phi_{n_p}(\varphi)\, h_q\left(e^{i\varphi}\right) d\varphi\right\}\right|$$

$$\geqq \frac{1}{2\pi}\left|\int_{-\delta_{p-1}}^{-\delta_p} + \int_{\delta_p}^{\delta_{p-1}} \Phi_{n_p}(\varphi)\, h_p\left(e^{i\varphi}\right) d\varphi\right|$$

$$- \frac{1}{2\pi}\sum_{q=1}^{\infty}{}' \left\{\int_{-\delta_{p-1}}^{-\delta_p} + \int_{\delta_p}^{\delta_{p-1}} \left|\Phi_{n_p}(\varphi)\, h_q\left(e^{i\varphi}\right)\right| d\varphi\right\},$$

wo q in $\sum'$ die sämtlichen positiven ganzen Zahlen außer p durchläuft. Nun ist aber für jedes $q \neq p$ in den beiden Intervallen $-\delta_{p-1} \leqq \varphi \leqq -\delta_p$, $\delta_p \leqq \varphi \leqq \delta_{p-1}$ die Funktion $h_q\left(e^{i\varphi}\right)$ absolut $< \varepsilon_q$, also

$$\frac{1}{2\pi} \sum_{q=1}^{\infty}{}' \left\{ \int_{-\delta_{p-1}}^{-\delta_p} + \int_{\delta_p}^{\delta_{p-1}} \left| \Phi_{n_p}(\varphi)\, h_q\left(e^{i\varphi}\right) \right| d\varphi \right\}$$
$$< \sum_{q=1}^{\infty}{}' \varepsilon_q \frac{1}{2\pi} \int_0^{2\pi} \left| \Phi_{n_p}(\varphi) \right| d\varphi < \frac{\lambda}{8} G_{n_p}.$$

Wir bekommen somit schließlich

$$|s_{n_p}| > \frac{1}{2\pi} \left| \int_{-\delta_{p-1}}^{-\delta_p} + \int_{\delta_p}^{\delta_{p-1}} \Phi_{n_p}(\varphi)\, h_p\left(e^{i\varphi}\right) d\varphi \right| - \frac{\lambda}{8} G_{n_p} - \frac{\lambda}{25} G_{n_p}$$
$$> \left(1 - \frac{\lambda}{2}\right) G_{n_p} - \frac{\lambda}{8} G_{n_p} - \frac{\lambda}{25} G_{n_p} > (1-\lambda)\, G_{n_p}.$$

Hiermit ist der Hauptsatz bewiesen.

Über die Koeffizientensumme einer beschränkten Potenzreihe.

Von

Harald Bohr in Kopenhagen.

(Zweite Mitteilung.)

Vorgelegt durch Herrn Landau in der Sitzung vom 25. November 1916

Einleitung.

Es sei

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

eine im Einheitskreise $|x| < 1$ reguläre Funktion, die der Bedingung $|f(x)| < 1$ für $|x| < 1$ genügt, und es sei

$$s_n = \sum_{\nu=0}^{n} a_\nu$$

gesetzt. Nach Landau ist, bei festem n, die obere Grenze G_n von $|s_n|$ für die Menge aller solcher Funktionen $f(x)$ gleich

$$G_n = \sum_{\nu=0}^{n} \binom{-\frac{1}{2}}{\nu}^2 = 1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1.3}{2.4}\right)^2 + \cdots + \left(\frac{1.3\ldots(2n-1)}{2.4\ldots 2n}\right)^2.$$

In einer früheren Abhandlung in diesen Nachrichten[1]) habe ich den Satz bewiesen:

Es sei $0 < \lambda < 1$. Dann gibt es dazu eine feste im Einheitskreise $|x| < 1$ reguläre Funktion $f(x)$ mit $|f(x)|$

---

1) H. Bohr: Über die Koeffizientensumme einer beschränkten Potenzreihe, 1916; diese Abhandlung werde ich im Folgenden einfach als I zitieren. Das Verständnis der vorliegenden Arbeit setzt die Kenntnis von I voraus.

< 1 für $|x| < 1$ derart, daß die Ungleichung

$$|s_n| > (1-\lambda)\, G_n$$

für unendlich viele n erfüllt ist.

In § 1 der vorliegenden Abhandlung werde ich beweisen, daß dieser Satz aus I durch den folgenden schärferen Satz ersetzt werden kann.

Satz A: Es existiert eine im Einheitskreise $|x| < 1$ reguläre Funktion $f(x)$ mit $|f(x)| < 1$ für $|x| < 1$, derart, daß

$$\limsup_{n=\infty} \frac{|s_n|}{G_n} = 1$$

ist.

Durch diesen Satz ist für die betreffende Fragestellung ein gewisser Abschluß erreicht. Es erhebt sich jedoch die Frage, ob man nicht den Satz dahin verschärfen kann, daß er die Existenz einer für $|x| < 1$ regulären Funktion $f(x)$ mit $|f(x)| < 1$ für $|x| < 1$ derart besagt, daß

$$\liminf_{n=\infty} \{G_n - |s_n|\} < \infty$$

ist, vielleicht sogar, daß $\liminf \{G_n - |s_n|\} = 0$ ist. Diese Frage werde ich in § 2 erledigen und zwar mit dem Resultate, daß eine solche Funktion $f(x)$ nicht existiert. Ich werde nämlich in § 2 den Satz beweisen.

Satz B: Für jede feste im Einheitskreise $|x| < 1$ reguläre Funktion $f(x)$, die der Bedingung $|f(x)| < 1$ für $|x| < 1$ genügt, ist

$$\lim_{n=\infty} \{G_n - |s_n|\} = \infty.$$

Der Beweis dieses Satzes B führe ich unter wesentlicher Benutzung eines sehr bemerkenswerten Satzes von Fatou über beschränkte Potenzreihen sowie einiger bekannten Sätze aus der Theorie des Lebesgueschen Integrals.

§ 1.

Es bezeichnen durchweg im Folgenden (wie in I) $P_n(x)$, $Q_n(x)$, $\Phi_n(\varphi)$ und $f_n(x)$ die vier Funktionen

$$P_n(x) = \sum_{\nu=0}^{n} \binom{-\frac{1}{2}}{\nu} (-x)^\nu \qquad (\text{für } |x| \leqq 1)$$

$$Q_n(x) = \frac{1}{x^{n+1}} (P_n(x))^2 \qquad (\text{für } 0 < |x| \leqq 1)$$

$$\Phi_n(\varphi) = e^{i\varphi} Q_n\left(e^{i\varphi}\right) \qquad (\text{für alle reellen } \varphi)$$

und

$$f_n(x) = \frac{x^n P_n\left(\frac{1}{x}\right)}{P_n(x)} \qquad (\text{für } |x| \leqq 1).$$

Dann ist (vergl. I) $\left|f_n\left(e^{i\varphi}\right)\right| = 1$ und $\Phi_n(\varphi) f_n\left(e^{i\varphi}\right) = |\Phi_n(\varphi)|$ für $0 \leqq \varphi < 2\pi$, sowie

$$\frac{1}{2\pi} \int_0^{2\pi} \Phi_n(\varphi) f_n\left(e^{i\varphi}\right) d\varphi = \frac{1}{2\pi} \int_0^{2\pi} |\Phi_n(\varphi)| \, d\varphi = G_n,$$

wo G_n die Landausche Zahl $\sum_{\nu=0}^{n} \binom{-\frac{1}{2}}{\nu}^2$ bezeichnet.

Hilfssatz: Es sei N eine beliebige positive ganze Zahl und $0 < \delta \leqq \pi$, sowie $0 < \varepsilon < 1$ beliebig gegeben. Dann gibt es ein ganzes $n > N$ und eine für $|x| < 1$ reguläre, für $|x| \leqq 1$ stetige Funktion $h(x)$ mit den folgenden Eigenschaften:

1) $$\frac{1}{2\pi} \int_{\delta}^{2\pi - \delta} |\Phi_n(\varphi)| \, d\varphi < \varepsilon \,.\, G_n,$$

also $\left(\text{wegen } \frac{1}{2\pi} \int_0^{2\pi} |\Phi_n(\varphi)| \, d\varphi = G_n\right)$

$$\frac{1}{2\pi} \int_{-\delta}^{\delta} |\Phi_n(\varphi)| \, d\varphi > (1 - \varepsilon) G_n.$$

2) $$|h(x)| \leqq 1 - \varepsilon \quad \text{für } |x| \leqq 1.$$

3) $|h(x)| < \varepsilon$ in demjenigen Teil des Einheitskreises $|x| \leqq 1$, für den $\Re(x) \leqq \cos\delta$ ist.

4) $$h(1) = 0.$$

5) $$\frac{1}{2\pi} \left| \int_{-\delta}^{\delta} \Phi_n(\varphi) \, h\left(e^{i\varphi}\right) d\varphi \right| > (1 - 2\varepsilon) G_n.$$

Beweis: Es ergibt sich fast wörtlich wie bei dem Beweise eines ganz entsprechenden Hilfssatzes in I, § 2, daß bei passender Wahl eines (hinreichend großen) positiven m, eines (hinreichend großen) ganzen $n > N$ und eines (hinreichend kleinen) positiven μ

die Funktion

$$h(x) = (1-\varepsilon)\, e^{m(x-1)} \left(\frac{1-x}{2}\right)^{\mu} f_n(x)$$

und die Zahl n den Bedingungen des Hilfssatzes genügen. Die erforderlichen Modificationen in der Darstellung in I sind so unwesentlich und naheliegend, daß ich die Ausführung derselben dem Leser überlassen werde.

Satz A: Es existiert eine im Einheitskreise $|x| < 1$ reguläre Funktion $f(x)$ mit $|f(x)| < 1$ für $|x| < 1$ derart, daß

$$\limsup_{n=\infty} \frac{|s_n|}{G_n} = 1$$

ist.

Beweis: Es sei zunächst eine Folge positiver Zahlen $1 > \varepsilon_1 > \varepsilon_2 \cdots > \varepsilon_q \cdots$ so gewählt, daß für jedes $p \geqq 1$

$$\sum_{q=p+1}^{\infty} \varepsilon_q < \frac{1}{2}\varepsilon_p$$

ist $\left(\text{z. B. } \varepsilon_q = \frac{1}{4^q}\right)$. Ich bestimme nunmehr, von obigem Hilfssatze ausgehend, eine Folge für $|x| < 1$ regulärer, für $|x| \leqq 1$ stetiger Funktionen $h_1(x), h_2(x), \ldots h_p(x), \ldots$ sowie eine dazu gehörige Folge positiver ganzer Zahlen $1 < n_1 < n_2 \cdots < n_p \cdots$ und eine Folge positiver Zahlen $\pi = \delta_1 > \delta_2 \cdots > \delta_p \cdots$ (mit $\delta_p \to 0$) durch folgendes Verfahren: Zunächst wende ich den obigen Hilfssatz an auf die Zahlen $N = 1$, $\delta = \delta_1 = \pi$, $\varepsilon = \varepsilon_1$. Der Hilfssatz ergibt alsdann eine ganze Zahl $n = n_1 > 1$ und eine für $|x| < 1$ reguläre, für $|x| \leqq 1$ stetige Funktion $h(x) = h_1(x)$, welche den fünf Bedingungen des Hilfssatzes genügen. Zu dieser Zahl n_1 und dieser Funktion $h_1(x)$ bestimme ich (was offenbar aus Stetigkeitsgründen möglich ist) eine positive Zahl $\delta_2 < \frac{\delta_1}{2}$ derart, daß erstens (wegen $h_1(1) = 0$)

$$|h_1(x)| < \tfrac{1}{2}\varepsilon_2 \quad \text{für} \quad |x| \leqq 1,\ \Re(x) \geqq \cos\delta_2$$

und zweitens

$$\frac{1}{2\pi}\int_{-\delta_2}^{\delta_2} |\Phi_{n_1}(\varphi)|\, d\varphi < \varepsilon_1 G_{n_1}$$

ist. Nachdem n_1, δ_2 und $h_1(x)$ somit festgelegt sind, wende ich wieder den obigen Hilfssatz an, jetzt aber auf die Zahlen $N = n_1$, $\delta = \delta_2$, $\varepsilon = \varepsilon_2$ und bekomme dadurch, im Sinne des Hilfssatzes,

ein ganzes $n = n_2 > n_1$ und eine für $|x| < 1$ reguläre, für $|x| \leqq 1$ stetige Funktion $h(x) = h_2(x)$. Zu dieser Zahl n_2 und den beiden Funktionen $h_1(x)$ und $h_2(x)$ bestimme ich (aus Stetigkeitsgründen) eine positive Zahl $\delta_3 < \frac{\delta_2}{2}$ derart, daß erstens (wegen $h_1(1) = h_2(1) = 0$)

$$|h_1(x)| + |h_2(x)| < \tfrac{1}{2}\varepsilon_3 \quad \text{für} \quad |x| \leqq 1,\ \Re(x) \geqq \cos\delta_3$$

und zweitens

$$\frac{1}{2\pi}\int_{-\delta_3}^{\delta_3} |\Phi_{n_2}(\varphi)|\,d\varphi < \varepsilon_2 G_{n_2}$$

ist. Nachdem hiermit die Zahlen n_2 und δ_3, sowie die Funktion $h_2(x)$ festgelegt sind, wende ich wiederum den Hilfssatz an, jetzt aber auf die Zahlen $N = n_2$, $\delta = \delta_3$, $\varepsilon = \varepsilon_3$ und bekomme dadurch ein ganzes $n = n_3 > n_2$ und eine für $|x| < 1$ reguläre, für $|x| \leqq 1$ stetige Funktion $h(x) = h_3(x)$. Zu dieser Zahl n_3 und den drei Funktionen $h_1(x)$, $h_2(x)$, $h_3(x)$ bestimme ich (aus Stetigkeitsgründen) eine positive Zahl $\delta_4 < \frac{\delta_3}{2}$ derart, daß erstens (wegen $h_1(1) = h_2(1) = h_3(1) = 0$)

$$|h_1(x)| + |h_2(x)| + |h_3(x)| < \tfrac{1}{2}\varepsilon_4 \quad \text{für} \quad |x| \leqq 1,\ \Re(x) \geqq \cos\delta_4$$

und zweitens

$$\frac{1}{2\pi}\int_{-\delta_4}^{\delta_4} |\Phi_{n_3}(\varphi)|\,d\varphi < \varepsilon_3 G_{n_3}$$

ist. Durch Fortsetzung dieser sukzessiven Bestimmungen erhalte ich offenbar eine unendliche Folge für $|x| < 1$ regulärer, für $|x| \leqq 1$ stetiger Funktionen $h_1(x), h_2(x), \ldots h_p(x), \ldots$, eine Folge positiver ganzer Zahlen $1 < n_1 < n_2 \cdots < n_p \cdots$ und eine Folge positiver Zahlen $\pi = \delta_1 > \delta_2 \cdots > \delta_p \cdots$ (wo $\delta_p \to 0$ wegen $\delta_p < \frac{1}{2}\delta_{p-1}$) mit den folgenden Eigenschaften: Für jedes $p \geqq 1$ ist

$$1) \qquad \frac{1}{2\pi}\int_{\delta_p}^{2\pi-\delta_p} |\Phi_{n_p}(\varphi)|\,d\varphi < \varepsilon_p G_{n_p}$$

$$2) \qquad |h_p(x)| \leqq 1 - \varepsilon_p \quad \text{für} \quad |x| \leqq 1$$

$$3) \qquad |h_p(x)| < \varepsilon_p \quad \text{für} \quad |x| \leqq 1,\ \Re(x) \leqq \cos\delta_p$$

$$4) \qquad \frac{1}{2\pi}\left|\int_{-\delta_p}^{\delta_p} \Phi_{n_p}(\varphi)\, h_p\!\left(e^{i\varphi}\right) d\varphi\right| > (1 - 2\varepsilon_p)\, G_{n_p}$$

$$5) \qquad |h_1(x)| + |h_2(x)| + \cdots + |h_p(x)| < \tfrac{1}{2}\varepsilon_{p+1} \quad \text{für} \quad |x| \leqq 1,$$
$$\Re(x) \geqq \cos\delta_{p+1}$$

$$6) \qquad \frac{1}{2\pi}\int_{-\delta_{p+1}}^{\delta_{p+1}} |\Phi_{n_p}(\varphi)|\, d\varphi < \varepsilon_p G_{n_p}.$$

Ich betrachte nunmehr die unendliche Reihe

$$\sum_{q=1}^{\infty} h_q(x) = h_1(x) + h_2(x) + \cdots.$$

Es ist diese Reihe offenbar, bei jedem p, im Gebiete $|x| \leqq 1$, $\Re(x) \leqq \cos\delta_p$ gleichmäßig konvergent; denn es gilt ja in diesem Gebiete für jedes $q \geqq p$ die Ungleichung $|h_q(x)| < \varepsilon_q$. Die Reihe $\sum h_q(x)$ definiert somit im ganzen Einheitskreise $|x| \leqq 1$ mit Ausnahme des einzigen Punktes $x = 1$ eine stetige Funktion $f(x)$, welche für $|x| < 1$ regulär ist. Ich werde beweisen, daß diese Funktion $f(x) = \sum a_n x^n$ die Bedingungen des Satzes A erfüllt. Zunächst ist $f(x) = \sum h_q(x)$ im ganzen Gebiete $|x| \leqq 1$, $x \neq 1$ (also speziell für $|x| < 1$) absolut < 1, ja es ist sogar (was ich später verwenden werde)

$$\sum_{q=1}^{\infty} |h_q(x)| < 1 \quad \text{für } |x| \leqq 1,\ x \neq 1.$$

Denn es sei $x \neq 1$ ein beliebiger Punkt des Einheitskreises $|x| \leqq 1$ und die ganze Zahl $p = p(x)$ so bestimmt, daß $\cos\delta_p \leqq \Re(x) < \cos\delta_{p+1}$; dann gelten ja die Ungleichungen

$$\sum_{q=1}^{p-1} |h_q(x)| < \tfrac{1}{2}\varepsilon_p, \quad |h_p(x)| \leqq 1 - \varepsilon_p, \quad |h_q(x)| < \varepsilon_q \quad (\text{für } q > p),$$

also

$$\sum_{q=1}^{\infty} |h_q(x)| < \tfrac{1}{2}\varepsilon_p + (1-\varepsilon_p) + \sum_{q=p+1}^{\infty} \varepsilon_q < \tfrac{1}{2}\varepsilon_p + (1-\varepsilon_p) + \tfrac{1}{2}\varepsilon_p = 1.$$

Es bleibt zu beweisen, daß $\limsup \frac{|s_n|}{G_n} = 1$ ist. Den Beweis hierfür führe ich dadurch, daß ich zeige: es besteht für jedes $p = 1, 2, 3, \ldots$ die Ungleichung

$$|s_{n_p}| > (1 - 5\varepsilon_p)\, G_{n_p}.$$

Die Richtigkeit dieser Ungleichung ergibt sich folgendermaßen: Es ist nach Landau (vergl. I) für jedes $0 < r < 1$

$$s_{n_p} = \frac{1}{2\pi i}\int_{|x|=r} Q_{n_p}(x) f(x)\, dx,$$

also, weil die für $|x| \leqq 1$, $x \neq 1$ definierte Funktion $f(x)$ stetig

und beschränkt ist, und daher das Integral alsbald auf den Einheitskreis bezogen werden kann,

$$s_{n_p} = \frac{1}{2\pi}\int_0^{2\pi} Q_{n_p}\left(e^{i\varphi}\right) f\left(e^{i\varphi}\right) e^{i\varphi}\, d\varphi = \frac{1}{2\pi}\int_0^{2\pi} \Phi_{n_p}(\varphi) f\left(e^{i\varphi}\right) d\varphi.$$

Hieraus folgt

$$\begin{aligned}
|s_{n_p}| &\geqq \frac{1}{2\pi}\left|\int_{-\delta_p}^{\delta_p} \Phi_{n_p}(\varphi) f\left(e^{i\varphi}\right) d\varphi\right| - \frac{1}{2\pi}\int_{\delta_p}^{2\pi-\delta_p} |\Phi_{n_p}(\varphi)|\, d\varphi \\
&> \frac{1}{2\pi}\left|\int_{-\delta_p}^{\delta_p} \Phi_{n_p}(\varphi) \sum_{q=1}^{\infty} h_q\left(e^{i\varphi}\right) d\varphi\right| - \varepsilon_p G_{n_p} \\
&\geqq \frac{1}{2\pi}\left|\int_{-\delta_p}^{\delta_p} \Phi_{n_p}(\varphi)\, h_p\left(e^{i\varphi}\right) d\varphi\right| \\
&\quad - \frac{1}{2\pi}\int_{-\delta_p}^{\delta_p} |\Phi_{n_p}(\varphi)| \cdot \sum_{q=1}^{p-1} \left|h_q\left(e^{i\varphi}\right)\right| d\varphi \\
&\quad - \frac{1}{2\pi}\left\{\int_{-\delta_p}^{-\delta_{p+1}} + \int_{\delta_{p+1}}^{\delta_p} |\Phi_{n_p}(\varphi)| \cdot \sum_{q=p+1}^{\infty} \left|h_q\left(e^{i\varphi}\right)\right| d\varphi\right\} \\
&\quad - \frac{1}{2\pi}\int_{-\delta_{p+1}}^{\delta_{p+1}} |\Phi_{n_p}(\varphi)| \cdot \sum_{q=p+1}^{\infty} \left|h_q\left(e^{i\varphi}\right)\right| d\varphi - \varepsilon_p G_{n_p} \\
&> (1-2\varepsilon_p)\, G_{n_p} - \frac{\varepsilon_p}{2}\cdot\frac{1}{2\pi}\int_0^{2\pi} |\Phi_{n_p}(\varphi)|\, d\varphi \\
&\quad - \sum_{q=p+1}^{\infty} \varepsilon_q \cdot \frac{1}{2\pi}\int_0^{2\pi} |\Phi_{n_p}(\varphi)|\, d\varphi \\
&\quad - \frac{1}{2\pi}\int_{-\delta_{p+1}}^{\delta_{p+1}} |\Phi_{n_p}(\varphi)|\, d\varphi - \varepsilon_p G_{n_p} \\
&> (1-2\varepsilon_p)\, G_{n_p} - \frac{\varepsilon_p}{2} G_{n_p} - \frac{\varepsilon_p}{2} G_{n_p} - \varepsilon_p G_{n_p} - \varepsilon_p G_{n_p} \\
&= (1-5\varepsilon_p)\, G_{n_p}.
\end{aligned}$$

Hiermit ist der Satz A bewiesen.

§ 2.

Satz B: Es sei $f(x) = \sum a_n x^n$ im Einheitskreise $|x| < 1$ regulär und absolut < 1. Dann ist

$$\lim_{n=\infty} \left\{ G_n - |s_n| \right\} = \infty,$$

d. h. nach Annahme einer beliebigen positiven Konstanten K gibt es ein $N = N(K)$ derart, daß für $n > N$

$$G_n - |s_n| > K$$

ist.

Beweis: Es ist nach Landau (vergl. I) für jedes $0 < r < 1$ und jedes n

$$s_n = \frac{1}{2\pi i}\int_{|x|=r} Q_n(x) f(x)\,dx = \frac{1}{2\pi i}\int_{|x|=r} \frac{(P_n(x))^2}{x^{n+1}} f(x)\,dx,$$

also

$$s_n = \frac{1}{2\pi r^n}\int_0^{2\pi} \left(P_n(re^{i\varphi})\right)^2 e^{-in\varphi} f(re^{i\varphi})\,d\varphi.$$

Weil aber $f(x)$ für $|x| < 1$ regulär und beschränkt ist, existiert nach einem bekannten Satz von Fatou[1]) für alle φ des Intervalles $0 \leqq \varphi < 2\pi$, höchstens mit Ausnahme einer Punktmenge vom Maße Null, der Grenzwert

$$\lim_{r=1} f(re^{i\varphi}) = F(\varphi),$$

wo r durch wachsende Werte gegen 1 strebt; also konvergiert (da $P_n(x)$ für $|x| \leqq 1$ stetig ist) für dieselbe Werte von φ die Funktion

$$\left(P_n(re^{i\varphi})\right)^2 e^{-in\varphi} f(re^{i\varphi})$$

gegen die Grenzfunktion $\left(P_n(e^{i\varphi})\right)^2 e^{-in\varphi} F(\varphi)$. Diese letzte (im ganzen Intervall $0 \leqq \varphi < 2\pi$, höchstens mit Ausnahme einer Punktmenge vom Maße Null, definierte) Funktion $\left(P_n(e^{i\varphi})\right)^2 e^{-in\varphi} F(\varphi)$ ist als Grenzfunktion einer gleichmäßig beschränkten und stetigen Funktion im Lebesgueschen Sinne integrierbar, und es gilt nach einem Hauptsatze in der Lebesgueschen Theorie die Gleichung

$$\lim_{r=1}\int_0^{2\pi} \left(P_n(re^{i\varphi})\right)^2 e^{-in\varphi} f(re^{i\varphi})\,d\varphi$$
$$= \underset{(L)}{\int_0^{2\pi}} \left(P_n(e^{i\varphi})\right)^2 e^{-in\varphi} F(\varphi)\,d\varphi,$$

1) P. Fatou: Séries trigonométriques et séries de Taylor, Acta Mathematica, Bd. 30, S. 335—400, 1906. Ein sehr einfach dargestellter Beweis des betreffenden Fatouschen Satzes steht bei Carathéodory: Über die gegenseitige Beziehung der Ränder bei der konformen Abbildung des Innern einer Jordanschen Kurve auf einen Kreis, Mathematische Annalen, Bd 73, S. 305—320.

wo das (L) unter dem Integralzeichen rechts bezeichnet, daß es sich um ein Lebesguesches Integral handelt. Aus der obigen Formel

$$s_n = \frac{1}{2\pi r^n} \int_0^{2\pi} \left(P_n(re^{i\varphi})\right)^2 e^{-in\varphi} f(re^{i\varphi})\, d\varphi,$$

wo die linke Seite s_n von r frei ist, bekommen wir somit durch den Grenzübergang $r \to 1$ die Darstellungsformel

$$s_n = \frac{1}{2\pi} \underset{(L)}{\int_0^{2\pi}} \left(P_n(e^{i\varphi})\right)^2 e^{-in\varphi} F(\varphi)\, d\varphi,$$

wo das Integral nunmehr auf den Einheitskreis bezogen ist. Wegen

$$G_n = \frac{1}{2\pi} \int_0^{2\pi} |P\ (e^{i\varphi})|^2\, d\varphi$$

ist also

$$G_n - |s_n| = \frac{1}{2\pi} \int_0^{2\pi} |P_n(e^{i\varphi})|^2\, d\varphi$$

$$- \frac{1}{2\pi} \left| \underset{(L)}{\int_0^{2\pi}} \left(P_n(e^{i\varphi})\right)^2 e^{-in\varphi} F(\varphi)\, d\varphi \right|.$$

Ich bestimme nunmehr zu der gegebenen Zahl K die positive Zahl δ so klein, daß

$$\frac{1}{2\pi} \int_\delta^{2\pi-\delta} \frac{d\varphi}{|1 - e^{i\varphi}|} > K + 3$$

ist. Wegen $|F(\varphi)| \leqq 1$ für alle φ im Definitionsbereich von $F(\varphi)$ folgt sofort aus der obigen Formel für $G\ - |s_n|$, daß

$$G_n - |s_n| \geqq \frac{1}{2\pi} \int_\delta^{2\pi-\delta} |P_n(e^{i\varphi})|^2\, d\varphi$$

$$- \frac{1}{2\pi} \left| \underset{(L)}{\int_\delta^{2\pi-\delta}} \left(P_n(e^{i\varphi})\right)^2 e^{-in\varphi} F(\varphi)\, d\varphi \right|.$$

Im Integrationsintervall $\delta \leqq \varphi \leqq 2\pi - \delta$ strebt

$$P_n(e^{i\varphi}) = \sum_{\nu=0}^{n} \binom{-\frac{1}{2}}{\nu} (-e^{i\varphi})^\nu$$

für $n \to \infty$ gleichmäßig gegen

$$\sum_{\nu=0}^{\infty} \binom{-\frac{1}{2}}{\nu} (-e^{i\varphi})^{\nu} = (1-e^{i\varphi})^{-\frac{1}{2}},$$

also strebt $\left(P_n(e^{i\varphi})\right)^2$ gleichmäßig gegen $\dfrac{1}{1-e^{i\varphi}}$; es gilt folglich für alle hinreichend großen n, d. h. für $n > N_1$ die Ungleichung

$$\left| \left(P_n(e^{i\varphi})\right)^2 - \frac{1}{1-e^{i\varphi}} \right| < 1 \quad \text{für} \quad \delta \leqq \varphi \leqq 2\pi - \delta.$$

Hieraus folgt aber sofort, daß für $n > N_1$

$$\begin{aligned} G_n - |s_n| &> \frac{1}{2\pi} \int_{\delta}^{2\pi-\delta} \frac{d\varphi}{|1-e^{i\varphi}|} \\ &\quad - \frac{1}{2\pi} \left| \underset{(L)}{\int_{\delta}^{2\pi-\delta}} \frac{F(\varphi)}{1-e^{i\varphi}} \cdot e^{-in\varphi} \, d\varphi \right| - 2 \\ &> K + 1 - \frac{1}{2\pi} \left| \underset{(L)}{\int_{\delta}^{2\pi-\delta}} G(\varphi) e^{-in\varphi} \, d\varphi \right| \end{aligned}$$

ist, wo $G(\varphi) = \dfrac{F(\varphi)}{1-e^{i\varphi}}$ eine feste (d. h. von n unabhängige) im Intervalle $(\delta, 2\pi - \delta)$, höchstens mit Ausnahme einer Punktmenge vom Maße Null, definierte, meßbare und beschränkte Funktion ist. Für eine solche Funktion $G(\varphi)$ gilt aber nach einem bekannten Satze über Fourierkonstanten [1]) die Gleichung

$$\lim_{n=\infty} \underset{(L)}{\int_{\delta}^{2\pi-\delta}} G(\varphi) e^{-in\varphi} \, d\varphi = 0.$$

Es ist also für alle hinreichend grossen n, d. h. für $n > N_2$

$$\frac{1}{2\pi} \left| \underset{(L)}{\int_{\delta}^{2\pi-\delta}} G(\varphi) e^{-in\varphi} \, d\varphi \right| < 1.$$

Für $n > N = \operatorname{Max}(N_1, N_2)$ gilt daher die Ungleichung

$$G_n - |s_n| > K + 1 - 1 = K.$$

Hiermit ist der Satz B bewiesen.

---

1) Vergl. z. B. de la Vallée Poussin, Cours d'Analyse Infinitésimale, Bd. II, 2 Aufl. (1912), S. 140.

Über streckentreue und konforme Abbildung.

Von

Harald Bohr in Kopenhagen.

In einer Ebene mit gegebenem Umlaufssinn, deren Punkte wir durch eine komplexe Variable $z = x + iy$ charakterisieren, sei ein einfach zusammenhängendes Gebiet G gegeben, das wir der Einfachheit halber als das Innere einer Jordanschen Kurve annehmen werden. In einer anderen Ebene mit gegebenem Umlaufssinn, deren Punkte wir durch die komplexe Variable $\zeta = \xi + i\eta$ charakterisieren, sei ebenfalls ein im Endlichen gelegenes, durch eine Jordansche Kurve begrenztes Gebiet Γ gegeben, und es sei das Gebiet G *eineindeutig* und *stetig* auf das Gebiet Γ abgebildet. Die (im Gebiete G definierte) Funktion, welche G auf Γ abbildet, werden wir mit $\zeta = f(z)$ bezeichnen, während wir die inverse (im Gebiete Γ definierte) Funktion, welche Γ auf G abbildet, mit $z = \varphi(\zeta)$ bezeichnen.

Unter den eineindeutigen und stetigen Abbildungen des Gebietes G auf das Gebiet Γ sind die *konformen Abbildungen* von ganz besonderer Wichtigkeit. Das sind Abbildungen, bei denen die im Gebiete G definierte Abbildungsfunktion $f(z)$ eine *analytische Funktion* ist, mit anderen Worten Abbildungen, die so beschaffen sind, daß bei jedem festen z_0 in G der Differenzenquotient

$$\frac{\zeta - \zeta_0}{z - z_0} = \frac{f(z) - f(z_0)}{z - z_0}$$

einem bestimmten (wegen der Eineindeutigkeit der Abbildung von selbst von Null verschiedenen) Grenzwerte $f'(z_0)$ zustrebt, wenn der Punkt z in beliebiger Weise gegen z_0 konvergiert, was wir durch $z \to z_0$ bezeichnen werden.

Wir wollen im folgenden eine eineindeutige und stetige Abbildung von G auf Γ eine *streckentreue Abbildung* nennen, wenn bei jedem festen z_0 in G der positive Bruch

$$\frac{|\zeta-\zeta_0|}{|z-z_0|} = \frac{|f(z)-f(z_0)|}{|z-z_0|}$$

für $z \to z_0$ einem bestimmten positiven Grenzwerte $g(z_0)$ zustrebt.

Ferner soll eine eineindeutige und stetige Abbildung von G auf Γ eine *winkeltreue Abbildung* heißen, wenn bei jedem festen z_0 in G die auf dem Einheitskreise gelegene Zahl

$$\frac{\operatorname{sign}(\zeta-\zeta_0)}{\operatorname{sign}(z-z_0)} = \frac{\operatorname{sign}(f(z)-f(z_0))}{\operatorname{sign}(z-z_0)},$$

unter $\operatorname{sign} u$ für $u \neq 0$ die Zahl $\frac{u}{|u|}$ verstanden, sich für $z \to z_0$ einem bestimmten Grenzwerte $h(z_0)$ nähert. Offenbar ist $|h(z_0)| = 1$.

Weil die *eine* Gleichung

$$\lim_{z \to z_0} \frac{\zeta-\zeta_0}{z-z_0} = u_0 (\neq 0)$$

mit den *zwei* Gleichungen

$$\lim_{z \to z_0} \frac{|\zeta-\zeta_0|}{|z-z_0|} = |u_0|, \qquad \lim_{z \to z_0} \frac{\operatorname{sign}(\zeta-\zeta_0)}{\operatorname{sign}(z-z_0)} = \operatorname{sign} u_0$$

dem Inhalte nach übereinstimmt, sind die beiden Aussagen: 1. die Abbildung von G auf Γ ist *konform*, und 2. die Abbildung von G auf Γ ist *sowohl streckentreu als winkeltreu*, offenbar als gleichbedeutend anzusehen.

Ich werde nunmehr in der vorliegenden Abhandlung den Satz beweisen:

Hauptsatz: *Jede eineindeutige und stetige Abbildung des Gebietes G auf das Gebiet Γ, die streckentreu ist und den Umlaufssinn ungeändert läßt, ist eine konforme Abbildung. Sie ist also von selbst winkeltreu.*

Dieser Satz läßt sich offenbar (aus Symmetriegründen) auch folgendermaßen aussprechen:

Es sei $f(z)$ eine im Gebiete G stetige Funktion, die das Gebiet G schlicht auf das Gebiet Γ abbildet, und zwar derart, daß die Abbildung streckentreu ist. Dann ist $f(z)$ entweder eine analytische Funktion oder die konjugierte einer analytischen Funktion in G. Der eine oder andere Fall tritt ein, je nachdem der Umlaufssinn bei der eineindeutigen und stetigen Abbildung erhalten bleibt oder nicht.

Ich hebe noch zur Orientierung hervor, daß in diesem Satze die Voraussetzung, daß die im Gebiete G definierte stetige Funktion $\zeta = f(z)$ dieses *schlicht* auf ein Gebiet Γ in der ζ-Ebene abbildet, *wesentlich* ist. Der Satz gilt *nicht* mehr, wenn diese Voraussetzung weggelassen wird. Mit anderen Worten: *Eine in einem Gebiete G stetige Funktion $f(z)$,*

von der nur vorausgesetzt wird, daß sie überall in G einen „absoluten Differentialquotienten" besitzt, d. h. die so beschaffen ist, daß bei jedem festen z_0 *in G der Quotient*

$$\frac{|f(z)-f(z_0)|}{|z-z_0|}$$

für $z \to z_0$ *einem bestimmten positiven Grenzwerte zustrebt, braucht weder analytisch in G noch konjugiert zu einer analytischen Funktion in G zu sein.* Um dies einzusehen, genügt schon ein so einfaches Beispiel, wie es durch die etwa im Einheitskreise G durch die Festsetzungen

$$f(z) = f(x+iy) = \begin{cases} x+iy & \text{für } y \geqq 0, \\ x-iy & \text{für } y \leqq 0 \end{cases}$$

definierte, stetige Funktion $f(z)$ geliefert wird. Denn es hat offenbar diese Funktion in jedem Punkte z in G einen absoluten Differentialquotienten (gleich $+1$), und sie ist dennoch weder analytisch in G noch zu einer analytischen Funktion in G konjugiert. — Durch diese letzte Bemerkung ist eine Frage aus den Grundlagen der Theorie der analytischen Funktionen, welche neben anderen ähnlichen Problemen in einer Abhandlung von Herrn Remak[1]) aufgestellt ist, erledigt.

Bei dem folgenden Beweis des Hauptsatzes wird der von Lebesgue eingeführte Maßbegriff sowie der Lebesguesche Integralbegriff eine sehr wesentliche Rolle spielen[2]). Wir teilen den Beweis in drei Abschnitte ein.

In § 1 werden wir, ausgehend von einem bekannten Satze aus der Theorie der analytischen Funktionen, den Beweis des Hauptsatzes auf den Beweis des folgenden Satzes zurückführen:

Satz A: *Es bilde die stetige Funktion* $\zeta = f(z)$ *das Gebiet G eineindeutig, streckentreu und unter Beibehaltung des Umlaufssinnes auf das Gebiet* Γ *ab. Dann gehört zu jedem* $\delta > 0$ *ein* $\varepsilon > 0$, *so daß für ein jedes System im Gebiete G gelegener und nicht übereinandergreifender Quadrate* $q_1, q_2, \ldots, q_N$, *deren Gesamtflächeninhalt kleiner als* ε *ist, die Summe*

$$\sum_{n=1}^{N} \left| \int_{q_n} f(z)\,dz \right|$$

[1]) R. Remak, „Über winkeltreue und streckentreue Abbildung an einem Punkte und in der Ebene", Rendiconti del Circolo Matematico di Palermo, 38 (1914), S. 193 bis 246.

[2]) Betreffs der im folgenden zu verwendenden Sätze aus der Lebesgueschen Theorie werde ich an den einzelnen Stellen auf das soeben erschienene Buch von C. Carathéodory, „Vorlesungen über reelle Funktionen", Leipzig 1918, verweisen, in welchem ein systematischer Aufbau der Lebesgueschen Theorie gegeben wird.

kleiner als δ ist. Unter $\int_{q_n} f(z)\,dz$ wird das komplexe Integral von $f(z)$ längs des Umfanges des Quadrates q_n verstanden.

Der § 2 enthält eine für den Beweis des Satzes A benötigte mengentheoretische Betrachtung über streckentreue Abbildungen. Von den in § 2 bewiesenen Hilfssätzen sei an dieser Stelle der folgende erwähnt. *Bei einer streckentreuen Abbildung des Gebietes G auf das Gebiet Γ hat jede Menge M in Γ, die einer im Gebiete G gelegenen Menge M vom Maße Null entspricht, selbst das Maß Null.*

Schließlich werden wir in § 3 unter Benutzung der Resultate von § 2 den Beweis des Satzes A und damit denjenigen des Hauptsatzes in wenigen Worten führen können.

§ 1.

Es bedeute überall im folgenden $\zeta = f(z)$ eine Funktion, die das Gebiet G in der z-Ebene *eineindeutig* und *stetig* auf das Gebiet Γ in der ζ-Ebene abbildet, und zwar derart, daß die Abbildung *streckentreu* ist und *den Umlaufssinn nicht ändert*. Es handelt sich darum zu beweisen, daß die Abbildung *konform* ist, d. h. *daß $f(z)$ eine im Gebiete G analytische Funktion ist.*

Eine notwendige und hinreichende Bedingung dafür, daß die nach Voraussetzung in G stetige Funktion $f(z)$ daselbst analytisch ist, besteht nach einem bekannten funktionentheoretischen Satze darin, *daß das komplexe Integral $\int f(z)\,dz$, erstreckt über den Rand eines beliebigen in G gelegenen Quadrates, verschwindet.* Wir werden den Beweis des Hauptsatzes dadurch erbringen, daß wir das Erfülltsein der zuletzt genannten Bedingung nachweisen.

Es sei in der z-Ebene ein beliebiges Quadrat mit der Seitenlänge S (also dem Flächeninhalt S^2) gegeben, dessen Rand ganz im Innern des Gebietes G verläuft, und es bezeichne Q die Menge aller Punkte z, die im Innern oder auf dem Rande des Quadrates liegen. Der Abstand der Menge Q von der Jordanschen Kurve, die G begrenzt, möge $H\,(H > 0)$ heißen. Wir betrachten alsdann für alle z in Q und für $0 < |h| < H$ die positive, stetige Funktion der beiden komplexen Variablen z und h

$$F(z,h) = \frac{|f(z+h)-f(z)|}{|h|}.$$

Nach Voraussetzung existiert für jedes feste z in Q der Grenzwert

$$\lim_{h\to 0} F(z,h) = \lim_{h\to 0} \frac{|f(z+h)-f(z)|}{|h|}.$$

Wir bezeichnen diesen mit $g(z)$. Wäre hierbei noch außerdem voraus-

gesetzt, daß der Grenzübergang *für alle z in Q gleichmäßig* ist, so wäre der Beweis, daß das Integral $\int f(z)\,dz$, über den Umfang von Q erstreckt, gleich Null ist, unschwer zu führen. Eine solche Voraussetzung findet sich aber im Hauptsatze nicht. Nun hat sich aber bekanntlich, nachdem man dazu übergegangen ist, Punktmengen mittels des von Lebesgue eingeführten verfeinerten Maßbegriffes zu messen, das sehr bemerkenswerte Resultat ergeben, daß *jeder* Grenzübergang, bei welchem ein Parameter auftritt, in bezug auf diesen Parameter „im wesentlichen" gleichmäßig ist[3]). Indem wir speziell die vorhin eingeführte Funktion $F(z, h)$ betrachten, sprechen wir den soeben erwähnten Satz der Lebesgueschen Theorie wie folgt aus.

Hilfssatz 1: *Jedem (beliebig kleinen) $\lambda > 0$ läßt sich in Q eine meßbare Teilmenge T mit einem Maß größer als $S^2 - \lambda$ zuordnen, so daß für alle z in T die Funktion*

$$F(z, h) = \frac{|f(z+h) - f(z)|}{|h|}$$

für $h \to 0$ gegen $g(z)$ gleichmäßig strebt. Es gibt also zu jedem $\delta_1 > 0$ ein (von z unabhängiges) $\delta_2 > 0$ derart, daß für alle z in T und für $0 < |h| < \delta_2$ die Ungleichheit

$$\left| \frac{|f(z+h) - f(z)|}{|h|} - g(z) \right| < \delta_1$$

besteht.

Ehe wir dazu übergehen, diesen Hilfssatz auf unser Problem anzuwenden, schicken wir zwei äußerst einfache geometrische Hilfssätze voraus, von denen der erste evident ist, während der Beweis des zweiten so nahe liegt, daß wir diesen dem Leser überlassen können.

Hilfssatz 2: *Es sei q ein Quadrat (mit Einschluß des Randes) in der z-Ebene, das durch Vermittelung einer Funktion $\zeta = F(z)$ eineindeutig und stetig auf ein Gebiet (mit Einschluß des Randes) in der ζ-Ebene abgebildet wird. Es mögen im Quadrate q zwei Punkte z_1 und $z_2 \neq z_1$ existieren, so daß für jedes $z \neq z_1$ in q*

$$\frac{|F(z) - F(z_1)|}{|z - z_1|} = k_1,$$

für jedes $z \neq z_2$ in q

$$\frac{|F(z) - F(z_2)|}{|z - z_2|} = k_2$$

gilt, unter k_1 und k_2 zwei positive Konstanten verstanden. Dann ist $k_1 = k_2$. Die Abbildung von q auf das entsprechende Gebiet in der ζ-Ebene ist entweder eine Ähnlichkeitstransformation oder zu einer Ähnlichkeits-

[3]) Vgl. Carathéodory, l. c. [2]) S. 382, Satz 12.

transformation symmetrisch, d. h. die in q definierte Funktion $F(z)$ ist entweder eine lineare Funktion $F(z) = c_1 + c_2 z$, oder es ist $F(z) = c_1 + c_2 \bar{z}$, wo $\bar{z}$ die zu z konjugierte Zahl bezeichnet. Der eine oder der andere Fall tritt ein, je nachdem bei der Abbildung der Umlaufssinn erhalten bleibt oder nicht.

Aus diesem Hilfssatze ergibt sich sehr leicht die Richtigkeit des folgenden ähnlichen Hilfssatzes, in dem wir der Gleichmäßigkeit halber drei Punkte z_1, z_2, z_3 betrachten.

Hilfssatz 3: *Es sei q ein Quadrat (mit Einschluß des Randes) in der z-Ebene von der Seitenlänge s, und es sei q mittels einer Funktion $\zeta = F(z)$ eineindeutig und stetig auf ein Gebiet (mit Einschluß des Randes) in der ζ-Ebene abgebildet, und zwar so, daß bei der Abbildung der Umlaufssinn erhalten bleibt. Es mögen ferner ein Wert $\delta > 0$, drei positive Konstanten k_1, k_2, k_3 und drei feste in q gelegene Punkte z_1, z_2, z_3, welche von drei verschiedenen Eckpunkten Z_1, Z_2, Z_3 von q Abstände*

$$|Z_1 - z_1| < \frac{s}{4}, \quad |Z_2 - z_2| < \frac{s}{4}, \quad |Z_3 - z_3| < \frac{s}{4}$$

haben, existieren, so daß für jedes $i = 1, 2, 3$ die Ungleichung

$$k_i - \delta < \frac{|F(z) - F(z_i)|}{|z - z_i|} < k_i + \delta$$

für alle $z \neq z_i$ im Quadrate q besteht. Dann ist für ein sehr kleines δ die Abbildung „beinahe" eine Ähnlichkeitstransformation, d. h. genau gesprochen: zu jedem $\varepsilon_0 > 0$ gehört ein nur von ε_0 (und nicht etwa von k_1, k_2, k_3 und s) abhängiges $\delta_0 > 0$ derart, daß, wenn in den obigen Voraussetzungen $\delta \leqq \delta_0$ ist, die im Quadrate q definierte Abbildungsfunktion $F(z)$ sich auf die Form bringen läßt:

$$F(z) = c_1 + c_2 z + \psi(z),$$

wo c_1 und c_2 Konstanten bezeichnen, während $\psi(z)$ für alle z in q die Ungleichung $|\psi(z)| < \varepsilon_0 s$ befriedigt.

Nach diesen Vorbereitungen läßt sich nunmehr leicht der folgende Satz beweisen:

Hilfssatz 4: *Es sei Q ein beliebiges Quadrat (inkl. Rand), welches im Gebiete G gelegen ist, und es seien $\varepsilon_1 > 0$ und $\varepsilon_2 > 0$ beliebig gegeben. Dann läßt sich eine Zerlegung des Quadrates Q in eine endliche Anzahl von kleineren Quadraten $q_1, q_2, \ldots, q_N$ derart vornehmen, daß die folgende Bedingung erfüllt ist: Die Teilquadrate $q_1, q_2, \ldots, q_N$ können in zwei Klassen I und II eingeteilt werden, so daß*

1. *die Summe* $\sum\limits_{(I)} \left| \int f(z)\,dz \right|$ *der absoluten Beträge der Integrale* $\int f(z)\,dz$, *erstreckt längs der Quadrate der ersten Klasse, kleiner als* ε_1 *ist, und daß*

2. *die Summe der Flächeninhalte der zur zweiten Klasse gehörigen Quadrate kleiner als* ε_2 *ist.*

Beweis: Es sei S die Seitenlänge von Q, und es sei $\frac{\varepsilon_1}{4S^2} = \varepsilon_0$ gesetzt. Zu diesem $\varepsilon_0 > 0$ bestimmen wir ein $\delta_0 = \delta_0(\varepsilon_0) = \delta_0(\varepsilon_1)$ im Sinne des Hilfssatzes 3. Ferner bestimmen wir, was nach dem Hilfssatze 1 $\left(\text{für } \lambda = \frac{\varepsilon_2}{11}\right)$ möglich ist, eine meßbare Teilmenge T von Q mit einem Maß größer als $S^2 - \frac{\varepsilon_2}{11}$ derart, daß für alle z in T die Funktion

$$F(z,h) = \frac{|f(z+h)-f(z)|}{|h|}$$

für $h \to 0$ *gleichmäßig* gegen $g(z)$ strebt; dann können wir (wegen der Gleichmäßigkeit des Grenzüberganges) zu der obigen Zahl $\delta_0 > 0$ eine positive Zahl $\alpha = \alpha(\delta_0) = \alpha(\varepsilon_1)$ derart wählen, daß für jedes z in T und $0 < |h| < \alpha$ die Ungleichung

$$g(z) - \delta_0 < \frac{|f(z+h)-f(z)|}{|h|} < g(z) + \delta_0$$

besteht. Wir teilen nunmehr das Quadrat Q in $N = n^2$ kongruente Teilquadrate ein; die ganze Zahl n wird dabei so groß gewählt, daß die Seitenlänge $s = \frac{1}{n}S$ der Teilquadrate kleiner als $\frac{\alpha}{\sqrt{2}}$ ausfällt. Wir werden beweisen, daß bei dieser Zerlegung von Q in Teilquadrate die Bedingung des Hilfssatzes erfüllt ist. Dies ergibt sich folgendermaßen. Es sei q (vgl. die Fig.) ein beliebiges der n^2 Teilquadrate, und es sei um jeden der vier Eckpunkte ein Kreisbogen mit dem Radius $\frac{s}{4}$ gezogen, wodurch aus q vier kleine Kreisquadranten (die vier schraffierten Gebiete) herausgeschnitten werden. Wir unterscheiden nun zwei Fälle: 1. Unter den vier kleinen Kreisquadranten gibt es *mindestens drei*, welche im Innern einen Punkt der Menge T enthalten; in diesem Falle wird das Teilquadrat q in die Klasse I geschlagen. 2. Unter den vier kleinen Kreisquadranten gibt es *höchstens zwei*, welche im Innern einen Punkt der Menge T enthalten; in diesem Falle wird das Teilquadrat q zu der Klasse II gerechnet.

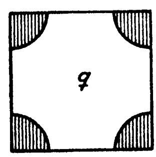


Fig 1

Es sei zunächst q ein beliebiges Quadrat der *ersten* Klasse. Dann gibt es nach Voraussetzung drei Punkte z_1, z_2, z_3 in q, welche der Menge T

angehören und so beschaffen sind, daß ihre Abstände von drei verschiedenen Eckpunkten Z_1, Z_2, Z_3 von q kleiner als $\frac{s}{4}$ sind. Da der Punkt $z_i\,(i=1,2,3)$ T angehört, und sein Abstand von jedem Punkte $z \neq z_i$ in q kleiner als die Diagonale $s\sqrt{2}$ von q, d. h. kleiner als α ist, so gelten für alle $z \neq z_i$ in q die Ungleichungen

$$k_i - \delta_0 < \frac{|f(z)-f(z_i)|}{|z-z_i|} < k_i + \delta_0,$$

unter k_i die positive Zahl $g(z_i)$ verstanden. Nach dem Hilfssatze 3 ist daher für alle z in dem betrachteten (zur ersten Klasse gehörigen) Quadrate q

$$f(z) = c_1 + c_2 z + \psi(z),$$

wo c_1 und c_2 von z unabhängig sind, während $|\psi(z)| < \varepsilon_0 s$ ist. Durch Integration längs des Umfanges von q erhalten wir somit

$$\int f(z)dz = \int (c_1 + c_2 z)dz + \int \psi(z)dz,$$

also, wegen

$$\int (c_1 + c_2 z)dz = 0,$$

$$\int f(z)dz = \int \psi(z)dz,$$

und hieraus weiter

$$\left|\int f(z)dz\right| \leqq \int |\psi(z)|\,|dz| < \varepsilon_0 s \cdot 4s = \frac{\varepsilon_1}{4S^2}\cdot 4s^2 = \frac{\varepsilon_1}{n^2} = \frac{\varepsilon_1}{N}.$$

Da die Anzahl der Quadrate der ersten Klasse höchstens gleich N ist, so haben wir hiermit die eine Hälfte unserer Behauptung, nämlich die Ungleichung

$$\sum_{(I)} \left|\int f(z)dz\right| < \frac{\varepsilon_1}{N}\cdot N = \varepsilon_1$$

bereits bewiesen.

Es erübrigt zu beweisen, daß die Summe der Flächeninhalte der Teilquadrate der *zweiten* Klasse kleiner als ε_2 ist. Zu diesem Zwecke bemerken wir zunächst, daß, wenn T' die Komplementärmenge zu T in bezug auf die Menge Q (d. h. die Menge, die aus allen Punkten in Q besteht, die nicht zu T gehören) bezeichnet, T' meßbar und von einem Maß kleiner als $\frac{\varepsilon_2}{11}$ ist. Nach Voraussetzung enthält jedes Teilquadrat q der zweiten Klasse mindestens zwei kleine Kreisquadranten vom Flächeninhalt $\frac{\pi}{4}\left(\frac{s}{4}\right)^2 = \frac{\pi}{64}s^2$, deren innere Punkte *sämtlich* der Menge T' angehören. Die Anzahl der Teilquadrate der zweiten Klasse ist folglich kleiner als

$$\frac{\varepsilon_2}{11} : \left(2\cdot\frac{\pi}{64}\cdot s^2\right) = \frac{\varepsilon_2}{s^2}\cdot\frac{32}{11\pi} < \frac{\varepsilon_2}{s^2},$$

d. h. die Summe der Flächeninhalte aller zu der zweiten Klasse gehörigen Teilquadrate ist kleiner als ε_2. Hiermit ist der Hilfsatz 4 bewiesen.

Es sei Q ein im Gebiete G gelegenes Quadrat, das bei vorgegebenen $\varepsilon_1 > 0$ und $\varepsilon_2 > 0$ in Teilquadrate q im Sinne des Hilfssatzes 4 zerlegt ist. Da das Integral $\int\limits_Q f(z)\,dz$ längs des Umfanges von Q gleich der Summe der Integrale erstreckt über den Umfang aller Teilquadrate $\sum \int\limits_q f(z)\,dz$ ist, so gilt

$$\left|\int\limits_Q f(z)\,dz\right| \leq \sum \left|\int\limits_q f(z)\,dz\right| = \sum_{(I)} \left|\int f(z)\,dz\right| + \sum_{(II)} \left|\int f(z)\,dz\right|,$$

wo in $\sum\limits_{(I)}$ bzw. $\sum\limits_{(II)}$ die Summation über die Teilquadrate der ersten bzw. der zweiten Klasse zu erstrecken ist. Es ist also

$$\left|\int\limits_Q f(z)\,dz\right| < \varepsilon_1 + \sum_{(II)} \left|\int f(z)\,dz\right|.$$

Aus dieser Ungleichung folgt sofort, daß der Beweis des Hauptsatzes, d. h. der Beweis der Gleichung $\int\limits_Q f(z)\,dz = 0$, geführt ist, wenn es gelingt zu zeigen, daß bei hinreichend kleinem ε_2 die Summe $\sum\limits_{(II)} \left|\int f(z)\,dz\right|$ beliebig klein wird. Der Beweis des Hauptsatzes ist also auf den Beweis des folgenden Satzes zurückgeführt.

Satz A: *Zu jedem $\delta > 0$ gibt es ein $\varepsilon > 0$ derart, daß für eine beliebige endliche Anzahl in G gelegener und nicht übereinander greifender Quadrate $q_1, q_2, \ldots, q_N$, deren Gesamtflächeninhalt kleiner als ε ist, die Summe $\sum \left|\int\limits_q f(z)\,dz\right|$, erstreckt über die N Quadrate $q_1, q_2, \ldots, q_N$, kleiner als δ ist.*

Bevor wir dazu übergehen, diesen Satz A (und damit den Hauptsatz) zu beweisen, werden wir in § 2 die gegebene streckentreue Abbildung von G auf Γ etwas näher studieren.

§ 2.

Wir beweisen zunächst den folgenden wichtigen Hilfssatz:

Hilfssatz 5: *Es sei M eine beliebige im Gebiete G gelegene Punktmenge vom Maße Null. Dann hat die M bei der gegebenen streckentreuen Abbildung im Gebiete Γ entsprechende Punktmenge M ebenfalls das Maß Null.*

Beweis: Es habe $g(z)$ die frühere Bedeutung, d. h. es bezeichne (wie überall im folgenden) $g(z)$ für jedes z in G den, nach Voraussetzung vorhandenen, Grenzwert

$$\lim_{h \to 0} \frac{|f(z+h)-f(z)|}{|h|}.$$

Es bezeichne ferner $M(K)$ für alle festen $K > 0$ die Menge der Punkte z in M, für die $g(z) < K$ ist. Es sei $\mathsf{M}(K)$ die $M(K)$ entsprechende Teilmenge von M. Da die Vereinigungsmenge einer abzählbaren Anzahl von Mengen (mit oder ohne gemeinsame Punkte), deren jede vom Maße Null ist, bekanntlich selbst vom Maße Null ist, so genügt es offenbar den Satz für $M(K)$ (bei beliebigem festem $K > 0$) statt für die gegebene Menge M selbst zu beweisen, d. h. es genügt zu beweisen, daß $\mathsf{M}(K)$ vom Maße Null ist; denn es ist ja M die Vereinigungsmenge der abzählbar unendlich vielen Mengen $\mathsf{M}(1)$, $\mathsf{M}(2)$, $\mathsf{M}(3)$, ... Für jeden Punkt z_0 der Menge $M(K)$ ist die Zahl $g(z_0) < K$. Es läßt sich daher um jeden Punkt z_0 in $M(K)$ eine kleine Kreisfläche $|z - z_0| < r \; (= r(z_0))$ im Gebiete G beschreiben, so daß für alle $z \neq z_0$ in dieser Kreisfläche die Ungleichung

$$|\zeta - \zeta_0| = |f(z) - f(z_0)| < K\,|z - z_0|$$

gilt. Man überzeugt sich leicht durch nochmaligen Gebrauch des Satzes, daß die Vereinigungsmenge von abzählbar unendlich vielen Mengen, deren jede vom Maße Null ist, selbst vom Maße Null ist wie vorhin, daß es beim Beweise unseres Satzes genügt, statt $M(K)$ eine Teilmenge L von $M(K)$ zu betrachten, deren Punkten z_0 eine *feste* (d. h. von z_0 unabhängige) positive Zahl R als Radius der vorerwähnten kleinen Kreisfläche um z_0 zugeordnet werden kann. Wir betrachten also (statt der ursprünglichen Menge M) eine in G gelegene Menge L *vom Maße Null* mit folgender Eigenschaft: *Es existieren zwei positive Zahlen K und R derart, daß für jedes z_0 in L die Ungleichung*

$$|f(z) - f(z_0)| < K\,|z - z_0|$$

für alle $z \neq z_0$ im Kreise $|z - z_0| < R$ besteht; zu beweisen ist, *daß die im Gebiete Γ gelegene Menge Λ, die L entspricht, ebenfalls vom Maße Null ist.* Der Beweis hierfür ist erbracht, wenn es gelingt zu zeigen, daß zu jedem $\varepsilon > 0$ eine meßbare Punktmenge ϱ vom Maße kleiner als ε existiert, welche Λ als Teilmenge enthält. Zu diesem Zwecke bestimmen wir zunächst zu der gegebenen Menge L vom Maße Null eine abzählbare Folge im Gebiete G gelegener Quadraten $q_1, q_2, \ldots, q_n, \ldots$, deren Seitenlängen $s_1, s_2, \ldots, s_n, \ldots$ sämtlich kleiner als $\frac{R}{\sqrt{2}}$ sind, und die folgende Eigenschaften haben: 1. Jeder Punkt von L ist ein innerer Punkt eines dieser Quadrate. Jedes Quadrat enthält mindestens einen Punkt von L. 2. Die Summe $\sum s_n^2$ der Flächeninhalte der Quadrate ist kleiner als $\frac{\varepsilon}{2\pi K^2}$. Wir bezeichnen diejenige Punktmenge im Gebiete Γ, welche bei der Ab-

bildung dem Innern des Quadrates q_n entspricht, mit ϱ_n; die Menge ϱ_n ist meßbar, weil sie aus lauter inneren Punkten besteht. Es sei nunmehr z_0 ein Punkt von L im Innern des Quadrates q_n und ζ_0 der entsprechende Punkt von Λ im Gebiete ϱ_n. Dann ist das Quadrat q_n ganz im Kreise $|z - z_0| < \sqrt{2}\, s_n < R$ gelegen, woraus folgt, daß die entsprechende Punktmenge ϱ_n ganz im Kreise $|\zeta - \zeta_0| < K\sqrt{2}\, s_n$ liegt. Ist μ_n das Maß der Menge ϱ_n, so ist also $\mu_n \leqq \pi\,(K\sqrt{2}\, s_n)^2 = 2\pi K^2 s_n^2$. Wir bilden schließlich die Vereinigungsmenge ϱ der abzählbar vielen Mengen $\varrho_1, \varrho_2, \ldots, \varrho_n, \ldots$. Die Menge ϱ enthält offenbar die Menge Λ als Teilmenge. Da ϱ_n meßbar ist und ihr Maß $\mu_n \leqq 2\pi K^2 s_n^2$ ist, so ist die Vereinigungsmenge ϱ ebenfalls meßbar und ihr Maß

$$\mu \leqq \sum 2\pi K^2 s_n^2 = 2\pi K^2 \sum s_n^2 < 2\pi K^2 \frac{\varepsilon}{2\pi K^2} = \varepsilon.$$

Hiermit ist der Beweis, daß Λ vom Maße Null ist, und damit der Beweis des Hilfssatzes vollendet.

Aus dem Hilfssatze 5 schließen wir zunächst, *daß, wenn M eine beliebige meßbare Menge im Gebiete G ist, die entsprechende Menge* M *im Gebiete* Γ *ebenfalls meßbar ist*[4]). Dies ergibt sich folgendermaßen: Weil M meßbar ist, können wir eine Folge von Mengen $M_1, M_2, \ldots, M_n, \ldots$ in G, deren jede aus lauter inneren Punkten besteht und M als Teilmenge enthält, derart wählen, daß für $n \to \infty$ das Maß von M_n gegen das Maß von M konvergiert. Es sei M' der Durchschnitt der Mengen $M_1, M_2, \ldots$, d. h. es bestehe M' aus denjenigen Punkten, welche sämtlichen Mengen $M_1, M_2, \ldots$ angehören; dann ist M' meßbar, und es ist das Maß von M' mindestens gleich dem Maß von M (weil M Teilmenge von M' ist) und zugleich höchstens gleich dem Maß von M_n (weil M' Teilmenge in M_n ist). Es ist also (weil das Maß von M_n für $n \to \infty$ gegen das Maß von M konvergiert) das Maß von M' gleich dem Maße von M. Hieraus folgt, daß die Differenzmenge $M_0 = M' - M$ vom Maße Null ist. Wir betrachten nunmehr die im Gebiete Γ gelegenen Mengen $\mathsf{M}_1, \mathsf{M}_2, \ldots, \mathsf{M}_n, \ldots$, welche den Mengen $M_1, M_2, \ldots, M_n, \ldots$ in G entsprechen; alle diese Mengen sind meßbar, weil sie aus lauter inneren Punkten bestehen. Folglich ist die Menge M', welche M' entspricht, gleichfalls meßbar, weil sie der Durchschnitt der meßbaren Mengen $\mathsf{M}_1, \mathsf{M}_2, \ldots$ ist. Ferner ist die Menge M_0, welche M_0 entspricht, meßbar, nämlich (nach dem Hilfssatze 5) vom Maße Null. Nachdem hiermit die Meßbarkeit von M' und M_0 bewiesen ist, folgt sofort aus der Gleichung $\mathsf{M} = \mathsf{M}' - \mathsf{M}_0$, daß M ebenfalls meßbar ist, womit die obige Behauptung bewiesen ist.

[4]) Vergleiche Carathéodory, l. c.[2]) S. 355, Satz 2.

Hilfssatz 6: *Zu jedem $\varepsilon > 0$ gibt es ein $\delta = \delta(\varepsilon) > 0$ derart, daß, wenn M eine im Gebiet G gelegene meßbare Menge vom Maße kleiner als δ ist, die entsprechende (nach dem vorhergehenden ebenfalls meßbare) Menge M im Gebiete Γ vom Maße kleiner als ε ist.*

Beweis: Wir führen den Beweis indirekt, d. h. wir nehmen an, daß der Satz unrichtig sei, daß es also ein $\varepsilon_0 > 0$ gibt, so daß jedem $\delta > 0$ eine in G gelegene Menge M vom Maße kleiner als δ zugeordnet werden kann, deren entsprechende Menge M in Γ vom Maße $\geqq \varepsilon_0$ ist. Wir werden zeigen, daß diese Annahme zu einem Widerspruch führt. Zu diesem Zwecke bestimmen wir (unter Benutzung der zu widerlegenden Annahme) eine Folge von meßbaren, im Gebiete G gelegenen Mengen $M_1, M_2, \ldots$ derart, daß bei jedem $n \geqq 1$ das Maß von M_n kleiner als $\frac{1}{n^2}$ ist, während die entsprechenden, in Γ gelegenen, meßbaren Mengen $\mathsf{M}_1, \mathsf{M}_2, \ldots$ sämtlich vom Maße $\geqq \varepsilon_0$ sind. Es sei M' der „Limes superior" der Folge $M_1, M_2, \ldots$, d. h. diejenige Menge, deren Punkte unendlich vielen der Mengen $M_1, M_2, \ldots$ angehören, und es sei M' der Limes superior der Folge $\mathsf{M}_1, \mathsf{M}_2, \ldots$. Dann sind sowohl M' als M' meßbare Mengen, und es ist offenbar M' diejenige Menge in Γ, welche der Menge M' in G entspricht. Da jede der Mengen $\mathsf{M}_1, \mathsf{M}_2, \ldots$ vom Maße $\geqq \varepsilon_0$ ist, und außerdem die sämtlichen Mengen $\mathsf{M}_1, \mathsf{M}_2, \ldots$ in einem beschränkten Teil der Ebene (nämlich im Gebiete Γ) liegen, so ist bekanntlich auch der Limes superior M' vom Maße $\geqq \varepsilon_0$ [5]). Dagegen ist M' vom Maße Null; denn die Menge M' ist für alle $n \geqq 1$ eine Teilmenge der Vereinigungsmenge $M^{(n)}$ der abzählbar vielen Mengen $M_{n+1}, M_{n+2}, \ldots$, welche Vereinigungsmenge $M^{(n)}$ meßbar ist mit einem Maß kleiner als $\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \ldots < \frac{1}{n}$, so daß auch das Maß von M' für alle n kleiner als $\frac{1}{n}$ ist. Die beiden soeben erhaltenen Resultate: 1. M' ist vom Maße Null, 2. die entsprechende Menge M' ist von einem Maße größer als Null, widersprechen aber nach dem Hilfssatze 5 einander, womit der Hilfssatz 6 bewiesen ist.

Es sei M eine beliebige meßbare Menge im Gebiete G und M die entsprechende meßbare Menge im Gebiete Γ. Ihr Maß bezeichnen wir mit μ. Diese Zahl μ können wir alsdann als *eine Funktion der Menge M* auffassen; diese „Mengenfunktion", welche für alle meßbaren Mengen M im Gebiete G definiert ist, und die wir mit $\mu = \omega(M)$ bezeichnen werden, besitzt offenbar die beiden folgenden Eigenschaften:

I. Sie ist *additiv*, d. h., wenn M_1 und M_2 zwei in G gelegene meß-

[5]) Carathéodory, l. c. [2]) S. 255, Satz 11.

bare Mengen *ohne gemeinsame Punkte* sind, und M die Vereinigungsmenge von M_1 und M_2 bedeutet, so gilt die Gleichung

$$\omega(M) = \omega(M_1) + \omega(M_2).$$

II. Sie ist *totalstetig*, d. h. zu jedem $\varepsilon > 0$ gehört ein $\delta = \delta(\varepsilon) > 0$ derart, daß für jede im Gebiete G gelegene meßbare Menge M, deren Maß kleiner als δ ist, die Ungleichung $\omega(M) < \varepsilon$ besteht. (Dies ist eine andere Form des Hilfssatzes 6.)

Eine derartige additive und totalstetige Mengenfunktion $\omega(M)$ besitzt aber nach einem bekannten Satz der Lebesgueschen Theorie die beiden wichtigen Eigenschaften[6]):

I. Sie besitzt „fast überall" in G, d. h. in jedem Punkte z_0 von G, höchstens mit Ausnahme der Punkte einer Menge vom Maße Null, einen endlichen „Differentialquotienten" $D(z_0)$. Dies besagt, daß, wenn $M = M(z_0, r)$ das Innere eines kleinen in G gelegenen Kreises um z_0 mit dem Radius r bedeutet, der positive Bruch

$$\frac{\omega(M)}{\pi r^2}$$

sich für $r \to 0$ einem bestimmten endlichen Grenzwert $D(z_0)$ nähert. — Für unsere spezielle Mengenfunktion $\omega(M)$ läßt sich übrigens unmittelbar direkt beweisen, daß sie sogar in *jedem* Punkte z_0 des Gebietes G (und nicht nur mit Ausnahme einer Punktmenge vom Maße Null) einen Differentialquotienten $D(z_0)$ besitzt, nämlich den Differentialquotienten $D(z_0) = \big(g(z_0)\big)^2$, wo $g(z_0)$ wie immer den Grenzwert

$$g(z_0) = \lim_{h \to 0} \frac{|f(z_0+h) - f(z_0)|}{|h|}$$

bedeutet. Es sei z_0 irgendein fester Punkt im Gebiete G, und es sei $\varepsilon > 0$ ein beliebig kleiner Wert. Für hinreichend kleine r wird jeder Punkt z auf der Kreisperipherie $|z - z_0| = r$ in einen Punkt ζ des Gebietes Γ abgebildet, welcher im Kreisringe

$$(1-\varepsilon) r g(z_0) < |\zeta - \zeta_0| < (1+\varepsilon) r g(z_0)$$

gelegen ist. Die Jordansche Kurve in Γ, welche dem Kreise $|z - z_0| = r$ in G entspricht, verläuft also ganz in diesem Kreisringe. Hieraus folgt aber sofort, daß die meßbare Punktmenge M, welche der Kreisfläche $|z - z_0| < r$ entspricht, also die Menge M, die aus den sämtlichen Punkten im Innern der erwähnten Jordanschen Kurve besteht, ein Maß $\omega(M)$ besitzt, das die Ungleichungen

$$\pi \{(1-\varepsilon) r g(z_0)\}^2 \leqq \omega(M) \leqq \pi \{(1+\varepsilon) r g(z_0)\}^2$$

[6]) Carathéodory, l. c. [2]) S. 496, Satz 1, 2.

befriedigt; es gelten also für alle hinreichend kleinen positiven r die Ungleichungen

$$(1-\varepsilon)^2\left(g(z_0)\right)^2 \leqq \frac{\omega(M)}{\pi r^2} \leqq (1+\varepsilon)^2\left(g(z_0)\right)^2,$$

und es existiert somit der Grenzwert

$$\lim_{r\to 0} \frac{\omega(M)}{\pi r^2}$$

und ist gleich $\left(g(z_0)\right)^2$.

II. Der Differentialquotient $D(z) = D(x+iy)$ ist eine (im Lebesgueschen Sinne) im zweidimensionalen Gebiete G summierbare Punktfunktion, und es gilt für jede meßbare Menge M in G die Relation

$$\omega(M) = \iint_M D(z)\, d\sigma.$$

Mit anderen Worten, *eine additive totalstetige Mengenfunktion* $\omega(M)$ *ist gleich dem „Flächenintegral" ihres Differentialquotienten, erstreckt über die Menge* M.

Für unsere Mengenfunktion $\omega(M)$, wo der Differentialquotient $D(z)$ sogar überall in G (und nicht nur mit Ausnahme einer Menge vom Maße Null) existiert und gleich $\left(g(z)\right)^2$ ist, erhalten wir also die Gleichung

$$\omega(M) = \iint_M \left(g(z)\right)^2 d\sigma,$$

d. h. es gilt der folgende Hilfssatz.

Hilfssatz 7: *Es sei M eine beliebige meßbare Menge in G und M die entsprechende (ebenfalls meßbare) Menge in Γ. Dann ist das Maß μ von M durch das Flächenintegral*

$$\mu = \iint_M \left(g(z)\right)^2 d\sigma$$

gegeben.

Wir beweisen schließlich den folgenden Satz, welcher das Ziel dieses Paragraphen bildet.

Hilfssatz 8: *Es sei q ein Quadrat, das (mit Einschluß des Randes) ganz in G liegt, und es bezeichne m den Flächeninhalt dieses Quadrates, während μ das Maß derjenigen Punktmenge im Gebiete Γ bezeichnet, welche dem Innern des Quadrates q entspricht. Dann ist der absolute Wert des komplexen Integrals $\int f(z)\,dz$ längs des Umfanges von q höchstens gleich $m+\mu$.*

Beweis: Wir dürfen offenbar ohne Beschränkung der Allgemeinheit beim Beweise annehmen, daß die Seiten von q der reellen bzw. der imaginären Achse parallel sind, also daß die Punkte $z = x+iy$ des Quadrates q durch Ungleichungen der Form $x_1 \leqq x \leqq x_2$, $y_1 \leqq y \leqq y_2$ charakterisiert

sind, wobei $x_2 - x_1 = y_2 - y_1$ ist. Dann lautet die zu beweisende Ungleichung

$$\left|\int\limits_{x_1}^{x_2}\{f(x+iy_1)-f(x+iy_2)\}dx + i\int\limits_{y_1}^{y_2}\{f(x_2+iy)-f(x_1+iy)\}dy\right| \leqq m+\mu.$$

Aus Symmetriegründen genügt es offenbar nachzuweisen, daß

$$\left|\int\limits_{y_1}^{y_2}\{f(x_2+iy)-f(x_1+iy)\}dy\right| \leqq \frac{m+\mu}{2}$$

ist. Wir werden sogar beweisen, daß

$$\int\limits_{y_1}^{y_2}|f(x_2+iy)-f(x_1+iy)|dy \leqq \frac{m+\mu}{2}$$

ist. Zu diesem Zwecke betrachten wir die Differenz $f(x_2+iy)-f(x_1+iy)$, wo $y_1 \leqq y \leqq y_2$ ist, und werden zunächst beweisen, *daß, wenn für ein $y = y'$ im Intervalle $y_1 \leqq y \leqq y_2$ die Funktion $g(x+iy')$ eine für $x_1 \leqq x \leqq x_2$ (im Lebesgueschen Sinne) summierbare Funktion von x ist, die Ungleichung*

$$|f(x_2+iy')-f(x_1+iy')| \leqq \int\limits_{x_1}^{x_2} g(x+iy')dx$$

gilt; mit anderen Worten, diese Ungleichung gilt für jedes $y = y'$ im Intervalle $y_1 \leqq y \leqq y_2$, für welches die rechte Seite einen Sinn hat. Wir betrachten hierzu (unter der Annahme, daß $g(x+iy')$ für $x_1 \leqq x \leqq x_2$ summierbar ist) die Funktion

$$h(x) = |f(x+iy')-f(x_1+iy')| \qquad (x_1 \leqq x \leqq x_2),$$

und bilden für einen kleinen (positiven oder negativen) Zuwachs $\varDelta x$ die Differenz $h(x+\varDelta x)-h(x)$. Es ist hierbei (weil $\big||b-a|-|c-a|\big| \leqq |b-c|$ ist)

$$|h(x+\varDelta x)-h(x)| \leqq |f(x+\varDelta x+iy')-f(x+iy')|,$$

also

$$\left|\frac{h(x+\varDelta x)-h(x)}{\varDelta x}\right| \leqq \frac{|f(x+\varDelta x+iy')-f(x+iy')|}{|\varDelta x|}.$$

Diese Ungleichung lehrt sofort, daß in einem beliebigen Punkt x des Intervalles $x_1 \leqq x \leqq x_2$ jede der vier Hauptderivierten der Funktion $h(x)$ (d. h. der obere und untere Differentialquotient von rechts und links) absolut genommen $\leqq g(x+iy')$ ist; denn es strebt ja für $|\varDelta x| \to 0$ die rechte Seite der letzten Ungleichung gegen $g(x+iy')$. Es werde eine (beliebig ausgewählte) der vier Hauptderivierten von $h(x)$, etwa der obere rechte Differentialquotient, mit $h'(x)$ bezeichnet. Dieser Differential-

quotient $h'(x)$ ist in jedem Punkte x des Intervalles $x_1 \leqq x \leqq x_2$ *endlich*, nämlich absolut genommen $\leqq g(x+iy')$, also (wie jede endliche Hauptderivierte) eine *meßbare* Funktion von x im Intervalle $x_1 \leqq x \leqq x_2$, und weil er dem absoluten Werte nach nicht größer ist als die positive *nach Voraussetzung* für $x_1 \leqq x \leqq x_2$ summierbare Funktion $g(x+iy')$, ist er selbst *summierbar*. Hieraus folgt aber nach einem bekannten Satze der Lebesgueschen Theorie[7]), daß

$$h(x_2) = h(x_1) + \int_{x_1}^{x_2} h'(x)\,dx,$$

also, weil $h(x_1) = 0$ ist, daß

$$h(x_2) = |f(x_2+iy') - f(x_1+iy')| = \int_{x_1}^{x_2} h'(x)\,dx$$

ist. Aus dieser Darstellung von $|f(x_2+iy') - f(x_1+iy')|$ durch ein Lebesguesches Integral folgt nunmehr sofort die Richtigkeit der obigen Ungleichung

$$|f(x_2+iy') - f(x_1+iy')| \leqq \int_{x_1}^{x_2} g(x+iy')\,dx;$$

denn es gilt ja im ganzen Intervalle $x_1 \leqq x \leqq x_2$ die Ungleichung $h'(x) \leqq g(x+iy')$. Wir wenden uns nunmehr zum Beweise des Hilfssatzes 8, d. h. zum Beweise der Ungleichung

$$\int_{y_1}^{y_2} |f(x_2+iy) - f(x_1+iy)|\,dy \leqq \frac{m+\mu}{2}.$$

Es ist

$$m = \iint_q d\sigma, \qquad \mu = \iint_q \left(g(x+iy)\right)^2 d\sigma;$$

die erste Gleichung ist trivial, die zweite ist im Hilfssatze 7 bewiesen. Hieraus folgt

$$\frac{m+\mu}{2} = \iint_q \tfrac{1}{2}\left\{\left(g(x+iy)\right)^2 + 1\right\} d\sigma,$$

also a fortiori, weil

$$0 < g(x+iy) \leqq \tfrac{1}{2}\left\{\left(g(x+iy)\right)^2 + 1\right\}$$

ist und $g(x+iy)$, als Wurzel der im Gebiete G meßbaren Funktion $\left(g(x+iy)\right)^2$, selbst in G *meßbar* ist,

$$\iint_q g(x+iy)\,d\sigma \leqq \frac{m+\mu}{2}.$$

Nun gilt aber, weil $g(x+iy)$ eine im Quadrate $x_1 \leqq x \leqq x_2$, $y_1 \leqq y \leqq y_2$ summierbare Funktion des reellen Wertepaares (x, y) ist, nach einem

---

[7]) Carathéodory, l. c.[3]) S. 597, Satz 4.

fundamentalen Satz über die Reduktion eines Flächenintegrals auf ein Doppelintegral[8]) die Gleichung

$$\iint_q g(x+iy)\,d\sigma = \int_{y_1}^{y_2} dy \int_{x_1}^{x_2} g(x+iy)\,dx.$$

Diese Gleichung ist folgendermaßen zu verstehen. Es existiert für jedes y im Intervalle $y_1 \leqq y \leqq y_2$, höchstens mit Ausnahme einer Punktmenge vom Maße Null, das Integral $\int_{x_1}^{x_2} g(x+iy)\,dx$, und es ist die durch dieses Integral gegebene Funktion $J(y)$ (die also für $y_1 \leqq y \leqq y_2$, höchstens mit Ausnahme einer Menge vom Maße Null, definiert ist) eine summierbare Funktion von y, deren Integral $\int_{y_1}^{y_2} J(y)\,dy$ gleich dem Flächenintegral $\iint_q g(x+iy)\,d\sigma$ ist. Hieraus folgt nun die zu beweisende Ungleichung; denn aus der für jedes y im Intervalle $y_1 \leqq y \leqq y_2$, für welches das Integral $\int_{x_1}^{x_2} g(x+iy)\,dx$ existiert, giltigen Ungleichung

$$J(y) = \int_{x_1}^{x_2} g(x+iy)\,dx \geqq |f(x_2+iy) - f(x_1+iy)|$$

ergibt sich sofort durch Integration nach y

$$\frac{m+\mu}{2} \geqq \iint_q g(x+iy)\,d\sigma = \int_{y_1}^{y_2} J(y)\,dy \geqq \int_{y_1}^{y_2} |f(x_2+iy) - f(x_1+iy)|\,dy.$$

Hiermit ist der Hilfssatz 8 bewiesen.

§ 3.

In § 1 haben wir den Beweis des Hauptsatzes auf den folgenden Satz zurückgeführt.

Satz A: *Zu jedem $\delta > 0$ gibt es ein $\varepsilon > 0$, so daß für eine beliebige endliche Anzahl in G gelegener und nicht übereinander greifender Quadrate $q_1, q_2, \ldots, q_N$, deren Gesamtflächeninhalt kleiner als ε ist, die Ungleichung*

$$\sum_{n=1}^{N} \left| \int_{q_n} f(z)\,dz \right| < \delta$$

besteht.

Mit Hilfe der Resultate des § 2 gelingt es nun unmittelbar den Satz A zu beweisen. Zu der gegebenen Zahl $\delta > 0$ bestimmen wir, was nach dem Hilfssatze 6 möglich ist, eine positive Zahl $\varepsilon < \frac{\delta}{2}$ derart, daß

[8]) Carathéodory, l. c.[2]) S. 632, Satz 4.

für jede in G gelegene meßbare Menge M vom Maße kleiner als ε die entsprechende Menge M im Gebiete Γ vom Maße $\omega(M) < \frac{\delta}{2}$ ist. Für dieses ε ist alsdann die Bedingung des Satzes A erfüllt. In der Tat ist nach dem Hilfssatze 8

$$\left|\int_{q_n} f(z)\,dz\right| \leqq m_n + \mu_n,$$

wo m_n den Flächeninhalt von q_n bezeichnet, während μ_n das Maß derjenigen Punktmenge im Gebiete Γ angibt, welche dem Innern des Quadrates q_n entspricht, und hieraus folgt durch Summation

$$\sum_{n=1}^{N}\left|\int_{q_n} f(z)\,dz\right| \leqq \sum_{n=1}^{N}(m_n + \mu_n) = \sum_{n=1}^{N} m_n + \sum_{n=1}^{N}\mu_n < \varepsilon + \frac{\delta}{2} < \delta.$$

Hiermit ist der Satz A und damit der Hauptsatz bewiesen.

(Eingegangen am 28. Februar 1918.)

Om den Hadamard'ske »Hulsætning«.

Af Harald Bohr.

1. Lad

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{1}$$

være en Potensrække i den komplekse Variable $z = x + iy = r \cdot e^{i\theta}$, om hvilken Række vi vil antage, at den har en endelig og fra Nul forskellig Konvergensradius ρ. Funktionen $f(z)$ er da regulær analytisk i hele det Indre af Konvergenscirklen $|z| = \rho$, og Rækken kan her differentieres et vilkaarligt Antal Gange ledvis, d. v. s. der gælder for ethvert $\nu = 1, 2, 3, \cdots$ Ligningen

$$f^{(\nu)}(z) = \sum_{n=\nu}^{\infty} n(n-1)\cdots(n-\nu+1)\, a_n z^{n-\nu} \qquad (|z| < \rho). \tag{2}$$

Derimod findes der altid paa selve Konvergenscirklen $|z| = \rho$ mindst et Punkt, hvori Funktionen $f(z)$ er singulær. For at afgøre, hvorvidt et vilkaarligt givet Punkt $a = \rho \cdot e^{i\theta_0}$ paa Konvergenscirklen $|z| = \rho$ er et regulært eller et singulært Punkt for $f(z)$, kan man f. Eks. betragte den *Taylor*'ske Rækkeudvikling for $f(z)$ udfra Punktet $\frac{a}{2}$, altsaa Potensrækkeudviklingen i den Variable $\left(z - \frac{a}{2}\right)$

$$f(z) = \sum_{\nu=0}^{\infty} \frac{f^{(\nu)}\left(\frac{a}{2}\right)}{\nu!} \left(z - \frac{a}{2}\right)^{\nu}, \tag{3}$$

der jo som bekendt i hvert Fald er konvergent indenfor den Cirkel med Centrum i Punktet $\frac{a}{2}$, der berører den oprindelige Konvergenscirkel $|z| = \rho$ i Punktet a; det gælder da, at dersom denne Potensrække (3) i den Variable $z - \frac{a}{2}$ har sin Konvergensradius ρ_1 netop lig med $\left|\frac{a}{2}\right|$, vil Punktet a være et singulært Punkt for $f(z)$, medens a vil være et regulært Punkt for $f(z)$, hvis ρ_1 er større end $\left|\frac{a}{2}\right|$.

2. Ved Hjælp af denne Metode til Afgørelse af, hvorvidt et paa en Potensrækkes Konvergenscirkel beliggende Punkt er regulært eller singulært for den ved Rækken fremstillede analytiske Funktion, kan man, som bemærket af *Landau* *), umiddelbart bevise følgende vigtige Sætning af *Vivanti.*

Sætning I. Er i en Potensrække (1) med Konvergensradius ρ alle Koefficienterne a_n reelle Tal $\geqq 0$, da er det Punkt ρ paa Konvergenscirklen, hvori denne skæres af den positive Halvakse, altid et singulært Punkt for Funktionen $f(z)$.

Bevis. Vi fører Beviset indirekte og antager altsaa, at Punktet ρ var et regulært Punkt for $f(z)$; da vilde Potensrækken i den Variable $\left(z-\frac{\rho}{2}\right)$

$$f(z)=\sum_{\nu=0}^{\infty}\frac{f^{(\nu)}\left(\frac{\rho}{2}\right)}{\nu!}\left(z-\frac{\rho}{2}\right)^{\nu}$$

have sin Konvergensradius $>\frac{\rho}{2}$ og altsaa for et tilstrækkeligt lille $\varepsilon>0$ være konvergent i Punktet $z=\rho+\varepsilon$, d. v. s. Rækken

$$\sum_{\nu=0}^{\infty}\frac{f^{(\nu)}\left(\frac{\rho}{2}\right)}{\nu!}\left(\frac{\rho}{2}+\varepsilon\right)^{\nu}$$

vilde være konvergent. Nu er imidlertid i Følge Formel (2)

$$\frac{f^{(\nu)}\left(\frac{\rho}{2}\right)}{\nu!}=\sum_{n=\nu}^{\infty}\binom{n}{\nu}a_n\left(\frac{\rho}{2}\right)^{n-\nu},$$

og Dobbeltrækken

$$\sum_{\nu=0}^{\infty}\sum_{n=\nu}^{\infty}\binom{n}{\nu}a_n\left(\frac{\rho}{2}\right)^{n-\nu}\left(\frac{\rho}{2}+\varepsilon\right)^{\nu}$$

(hvor Summationen først skal udføres efter n, derefter efter ν) vilde altsaa være konvergent. I denne Dobbeltrække er imid-

*) Angaaende Litteraturen henvises Læseren til den nylig udkomne lille Bog af *E. Landau:* Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie. Berlin, 1916.

lertid alle Leddene $\geqq 0$, og vi kan derfor anvende en kendt Sætning om Dobbeltrækker med positive Led, der udsiger, at dersom alle Størrelserne $\alpha_{n,\nu}$ er $\geqq 0$ og Dobbeltrækken

$$\sum_{\nu=0}^{\infty}\sum_{n=0}^{\infty} \alpha_{n,\nu} \tag{4}$$

er konvergent (med Summen A), da vil den nye Dobbeltrække

$$\sum_{n=0}^{\infty}\sum_{\nu=0}^{\infty} \alpha_{n,\nu},$$

der fremkommer ved Ombytning af Summationsordenen, paany være konvergent (og ligeledes have Summen A). Vi finder altsaa, at Dobbeltrækken

$$\sum_{n=0}^{\infty}\sum_{\nu=0}^{n} \binom{n}{\nu} a_n \left(\frac{\rho}{2}\right)^{n-\nu} \left(\frac{\rho}{2}+\varepsilon\right)^{\nu}$$

$$= \sum_{n=0}^{\infty} a_n \left(\frac{\rho}{2}\right)^n \sum_{\nu=0}^{n} \binom{n}{\nu} \left(\frac{\frac{\rho}{2}+\varepsilon}{\frac{\rho}{2}}\right)^{\nu}$$

vilde være konvergent, d. v. s. at Rækken

$$\sum_{n=0}^{\infty} a_n \left(\frac{\rho}{2}\right)^n \left(1 + \frac{\frac{\rho}{2}+\varepsilon}{\frac{\rho}{2}}\right)^n = \sum_{n=0}^{\infty} a_n (\rho+\varepsilon)^n$$

vilde være konvergent; men dette Resultat er øjensynlig i Strid med, at den givne Potensrække (1) har Konvergensradius ρ, idet det jo udsiger, at Rækken (1) er konvergent i Punktet $(\rho+\varepsilon)$ udenfor Konvergenscirklen.

Vi tilføjer til Orientering, at den Omstændighed, at man ved de benyttede Omskrivninger (Summationsordenens Ombytning) rent formelt kommer til Potensrækken i Punktet $\rho+\varepsilon$, naturligvis ikke er andet, end hvad man paa Forhaand maatte vente; Bevisets Idé ligger alene deri, at disse formelle Regninger under den givne Forudsætning $a_n \geqq 0$ faar reel Gyldighed.

Den ovenfor benyttede Hjælpesætning om, at Summationsordenen i en Dobbeltrække (4) er ligegyldig, bevarer som bekendt sin Gyldighed, selvom Forudsætningen om, at alle Tallene $a_{n,\nu}$ skal være positive, erstattes af den (mindre) Forudsætning, at disse Tal, opfattet som Vektorer i en kompleks Plan, skal være blot »væsentlig ensrettede«, d. v. s. at alle Punkter (Tal) $a_{n,\nu}$ skal ligge i et fast Vinkelrum $Re^{i\Theta}$, karakteriseret ved Ulighederne $R \geqq 0$, $V < \Theta < W$, hvor $W - V < \pi$; heraf følger umiddelbart (som et Blik paa det ovenstaaende Bevis viser), at Sætning I kan generaliseres til følgende Sætning, den saakaldte *Vivanti-Dienes*'ske Sætning.

Sætning II. Er i en Potensrække (1) med Konvergensradius ρ alle Koefficienterne a_n beliggende i et Vinkelrum $Re^{i\Theta}$, $R \geqq 0$, $V < \Theta < W$, hvor $W - V < \pi$, da vil Punktet $z = \rho$ være et singulært Punkt for Funktionen $f(z)$.

Heraf følger atter (ved Transformationen $z = z_1 \cdot e^{i\theta_0}$), at hvis θ_0 er et saadant reelt Tal, at alle Tallene $a_n e^{in\theta_0}$ ligger i et fast Vinkelrum, hvis Vinkelaabning er $< \pi$, da vil Punktet $\rho e^{i\theta_0}$ være et singulært Punkt for $f(z)$, eller i lidt anden Formulering:

Sætning III. Er z_0 et saadant Punkt paa Konvergenscirklen $|z| = \rho$, at i dette Punkt alle Leddene $a_n z_0^n$ — eller, hvad der i denne Sammenhæng naturligvis kommer ud paa det samme, alle Leddene $a_n z_0^n$ fra et vist Trin af, d. v. s. for $n > N$ — er beliggende i et Vinkelrum $Re^{i\Theta}$, $R \geqq 0$, $V < \Theta < W$, hvor $W - V < \pi$, da vil Punktet z_0 være et singulært Punkt for Funktionen $f(z)$.

3. Den ved en Potensrække fremstillede analytiske Funktion siges at have Konvergenscirklen $|z| = \rho$ til naturlig Grænse, dersom alle Punkter paa Konvergenscirklen er singulære Punkter for $f(z)$. Af stor Interesse er her en Sætning af *Hadamard*, der har faaet Navnet den Hadamard'ske »Hulsætning«, fordi den handler om Potensrækker, i hvis Koefficientfølge $a_0, a_1, a_2, \cdots$ der er store »Huller«, d. v. s. hvor talrige Koefficienter har Værdien Nul. Idet vi kun op skriver de Led $a_n z^n$, for hvilke $a_n \neq 0$, kan denne Sætning formuleres saaledes:

Hadamard's Sætning. For enhver Potensrække

$$F(z) = \sum_{p=1}^{\infty} a_{m_p} z^{m_p} \qquad (0 \leqq m_1 < m_2 < \cdots), \tag{5}$$

der opfylder Betingelsen

$$\frac{m_{p+1}}{m_p} > k \qquad (p = 1, 2, 3, \cdots),$$

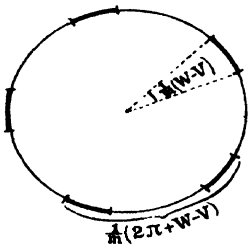
hvor k er en Konstant > 1, er Konvergenscirklen $|z| = \rho$ den naturlige Grænse for Funktionen $F(z)$.

Der er af forskellige Matematikere givet Beviser for denne Sætning, hvilke Beviser imidlertid alle er meget sammensatte og derved ret vanskelige at gennemskue. Det er maaske derfor af nogen Interesse at vise, hvorledes den *Hadamard*'ske Sætning i et Specialtilfælde, nemlig naar Konstanten k er > 3, uden Vanskelighed kan bevises ved Hjælp af den *Vivanti-Dienes*'ske Sætning III, hvorved man, som det vil fremgaa af det Følgende, faar et overmaade anskueligt Bevis for Sætningen i dette Tilfælde.

Bevis for Hadamard's Sætning i Tilfældet $k > 3$. For at bevise, at Konvergenscirklen $|z| = \rho$ er den naturlige Grænse for den ved Potensrækken (5) fremstillede Funktion $F(z)$, altsaa at alle Punkter paa denne Cirkel er singulære, er det tilstrækkeligt at vise, at de singulære Punkter ligger overalt tæt paa Cirklen $|z| = \rho$, d. v. s. at der paa enhver (selv nok saa lille) Bue B af Konvergenscirklen findes et singulært Punkt; thi heraf vil følge, at ethvert Punkt af Cirklen er singulært, idet jo et Fortætningspunkt for singulære Punkter selv er et singulært Punkt.

Vi betragter et vilkaarligt Led $a_m z^m$ i Potensrækken (5); lader vi z gennemløbe Konvergenscirklen $|z| = \rho$ een Gang, i positiv Omløbsretning og med konstant Vinkelhastighed, vil Størrelsen $Z = a_m z^m$ aabenbart gennemløbe Cirklen $|Z| = |a_m| \cdot \rho^m$ netop m Gange i positiv Omløbsretning og med en Vinkelhastighed, der er m Gange saa stor som den, hvormed z gennemløber sin Cirkel $|z| = \rho$. Heraf følger, at de Punkter z paa Konvergenscirklen $|z| = \rho$, for hvilke det gælder, at Punktet $Z = a_m z^m$ ligger i et givet Vinkelrum $Re^{i\Theta}$, $R \geqq 0$, $V < \Theta < W$, hvor $W - V < \pi$, vil udfylde m lige store og over Cirklen $|z| = \rho$ jævnt fordelte Buer, der hver har Vinkelstørrelsen

$\frac{1}{m}(W - V)$ (de stærkt optrukne Buer paa Figuren, der svarer til $m = 5$). Vi slutter heraf (se Figuren), at enhver Bue paa Cirklen $|z| = \rho$, hvis Vinkelstørrelse er større end



$$\frac{2\pi + W - V}{m}$$

i sit Indre helt indeholder mindst en af de nævnte m Buer, i hvis Punkter det gælder, at $a_m z^m$ ligger i det givne Vinkelrum $V < \Theta < W$.

Vi kan nu uden Vanskelighed (ved Hjælp af Sætning III) bevise, at der paa den vilkaarligt givne Bue B af Cirklen $|z| = \rho$ virkelig ligger et singulært Punkt z_0 for Funktionen $F(z)$, idet vi kan vise, at der paa denne Bue findes et saadant Punkt z_0, at alle Leddene $a_{m_p} z_0^{m_p}$ fra et vist Trin af ligger i et fast Vinkelrum $V < \Theta < W$, hvor Tallene V og W er valgt saadan, at Differensen $W - V$ vel er mindre end π, men er saa lidt mindre end π, at

$$\frac{2\pi + W - V}{W - V} < k,$$

hvad der er muligt, fordi Konstanten k i Følge Forudsætning er > 3.

Efter at dette Vinkelrum $V < \Theta < W$ er fastlagt, bestemmer vi det hele Tal P saa stort, at

$$\frac{2\pi + W - V}{m_P} < \beta,$$

hvor β betegner Vinkelstørrelsen af den paa Konvergenscirklen $|z| = \rho$ givne Bue B; da vil i Følge en Bemærkning ovenfor Buen B helt indeholde en Bue B_P af Vinkelstørrelsen $\frac{1}{m_P}(W - V)$, i hvis Punkter z det gælder, at Størrelsen $a_{m_P} z^{m_P}$ ligger i det fastlagte Vinkelrum $V < \Theta < W$. Nu er imidlertid

$$\frac{m_{P+1}}{m_P} > k \quad \text{og} \quad \frac{2\pi + W - V}{W - V} < k,$$

altsaa

$$\frac{W-V}{m_P} > \frac{2\pi + W - V}{m_{P+1}},$$

og vi kan derfor paa Buen B_P af Vinkelstørrelsen $\frac{1}{m_P}(W-V)$ bestemme en Delbue B_{P+1} af Vinkelstørrelsen $\frac{1}{m_{P+1}}(W-V)$, i hvis Punkter z det gælder, at det næste Led $a_{m_{P+1}} z^{m_{P+1}}$ ligeledes ligger i Vinkelrummet $V < \Theta < W$. Dernæst bestemmer vi paa denne Bue B_{P+1} en Delbue B_{P+2}, i hvis Punkter det gælder, at Leddet $a_{m_{P+2}} z^{m_{P+2}}$ ligger i Vinkelrummet $V < \Theta < W$, o. s. v.; paa denne Maade kan vi fortsætte ubegrænset, fordi der hele Tiden (d. v. s. for ethvert $p = P,\ P+1,\ P+2, \cdots$) gælder Uligheden

$$\frac{W-V}{m_p} > \frac{2\pi + W - V}{m_{p+1}}.$$

Den saaledes bestemte Følge af Cirkelbuer

$$B_P,\ B_{P+1},\ B_{P+2}, \cdots B_p, \cdots,$$

af hvilke enhver er helt indeholdt i den foregaaende, og hvis Længder $\frac{1}{m_p}(W-V)$ gaar mod Nul for $p \rightarrow \infty$, vil for $p \rightarrow \infty$ trække sig sammen om et bestemt Punkt z_0 paa den givne Bue B, og idet dette Punkt z_0 ligger paa alle Buerne $B_p\,(p \geqq P)$, vil alle Leddene $a_{m_p} z_0^{m_p}\,(p \geqq P)$ ligge i Vinkelrummet $V < \Theta < W$, hvormed er bevist, at Punktet z_0 er et singulært Punkt for Funktionen $F(z)$.

Et Eksempel paa en Bevismetode.

Af Harald Bohr.

Ved en analytisk-talteoretisk Undersøgelse stødte jeg paa en Type af Opgaver, paa hvilke den følgende er et særlig simpelt Eksempel.

Indenfor en Cirkel C med Radius 1 — som vi for at faa simple Betegnelser vil tænke os beliggende i en kompleks z-Plan med sit Centrum i Punktet 0 — er der givet et vilkaarligt Antal Punkter $z_1, \ldots z_n$, fordelt paa vilkaarlig Maade. Idet z er et variabelt Punkt indenfor C, betragter vi Middeltallet af de reciproke Afstande fra z til $z_1, \ldots z_n$, altsaa Størrelsen

$$F(z) = \frac{1}{n}\left\{\frac{1}{|z - z_1|} + \cdots + \frac{1}{|z - z_n|}\right\},$$

og skal bevise, at der findes en numerisk Konstant K (iøvrigt f. Eks. Konstanten 4), saaledes at vi i alle Tilfælde (d. v. s. hvor stor n end er, og hvorledes de n Punkter $z_1, \ldots z_n$ end er fordelt) kan finde et Punkt z_0 indenfor C, hvori det omtalte Middeltal $F(z_0)$ er $< K$.

Denne lille Sætning bevises maaske lettest ved følgende Fremgangsmaade, der kan benyttes overfor talrige Opgaver af lignende Art.

Funktionen $F(z)$ er defineret indenfor hele Cirklen C med Undtagelse af Punkterne $z_1, \ldots z_n$. For $z \rightarrow z_r$ vil $F(z)$ vokse i det Uendelige, men saaledes at $F(z)$ forbliver $< k : |z - z_r|$, og vi kan derfor tale om Fladeintegralet af $F(z)$ udstrakt over hele Cirklen C, altsaa om Integralet

$$I = \iint\limits_C F(z)\,dxdy = \frac{1}{n}\iint\limits_C \sum_{r=1}^{n} \frac{1}{|z - z_r|}\,dxdy = \frac{1}{n}\sum_{r=1}^{n}\iint\limits_C \frac{dxdy}{|z - z_r|}.$$

Nu er imidlertid Integralet under Summationstegnet mindre end en numerisk Konstant, nemlig (som man ser ved at parallelforskyde Begyndelsespunktet hen i Punktet z_r) f. Eks. mindre end Fladeintegralet af $1 : |z|$ over en Cirkel med Centrum i 0 og Radius 2, hvilket sidste Integral iøvrigt har Værdien 4π. Integralet I er altsaa $< \frac{1}{n} \cdot n \cdot 4\pi = 4\pi$. Men idet $F(z)$ er positiv indenfor hele C, og Integralet af $F(z)$ over Cirklen C (med Arealet π) er $< 4\pi$, maa Integranden $F(z)$ nødvendigvis i mindst et Punkt z_0 være $< 4\pi : \pi = 4$.

q. e. d.

Über einen Satz von Edmund Landau.

Von

Harald Bohr, Kopenhagen.

---

1. Ein wichtiger Satz in der Theorie der konformen Abbildung besagt die Existenz einer positiven absoluten Konstanten K mit folgender Eigenschaft: Für jede für $|z| \leqq 1$ reguläre Funktion

$$w = f(z) = a_0 + a_1 z + a_2 z^2 + \dots$$

mit $a_0 = 0$, $a_1 = 1$, welche die Kreisscheibe $|z| \leqq 1$ schlicht auf ein Gebiet der w-Ebene abbildet, ist der Radius r_f des größten Kreises $|w| = r_f$, dessen Punkte w sämtlich von der Funktion $f(z)$ ($|z| \leqq 1$) angenommen werden, $\geqq K$. In einer kürzlich erschienenen sehr interessanten Note[1]) hat nun Herr LANDAU bewiesen, daß in diesem Satze die Forderung, daß $f(z)$ das Gebiet $|z| \leqq 1$ schlicht abbilden soll, einfach gestrichen werden kann[2]).

2. Es sei $f(z) = \Sigma a_n z^n$ eine beliebige für $|z| \leqq 1$ reguläre Funktion mit $a_0 = 0$, und es sei $\underset{|z| = \frac{1}{2}}{\mathrm{Max}} |f(z)| = M_f$ gesetzt. Aus dem CAUCHY'schen Satze

---

1) E. LANDAU, Zum KOEBEschen Verzerrungssatz, Palermo Rendiconti, Bd. 46 (1922).

2) Es sei zur Orientierung bemerkt, daß es in der von LANDAU verallgemeinerten Form des Satzes — im Gegensatz zu der ursprünglichen Form, wo die Schlichtheit verlangt wurde — offenbar nicht erlaubt ist die Definition von r_f so zu ändern, daß r_f der Radius der größten Kreisscheibe $|w| \leqq r_f$ (statt des größten Kreises $|w| = r_f$), deren Punkte w sämtlich angenommen werden, bedeutet. Denn zu jedem $\varepsilon > 0$ gibt es Funktionen $f(z) = \Sigma a_n z^n$ mit $a_0 = 0$, $a_1 = 1$, die für $|z| \leqq 1$ regulär sind, aber nicht den Wert ε annehmen; eine solche Funktion ist ja z. B. die ganze Transzendente $f(z) = -\varepsilon\left(e^{-\frac{z}{\varepsilon}} - 1\right) = z - \frac{z^2}{2\varepsilon} + \dots$, die überall $\neq \varepsilon$ ist.

folgt sofort, daß $M_f \geqq k_1 |a_1|$ ist, wo k_1 eine positive absolute Konstante bedeutet; denn es ist ja

$$|a_1| = \left| \frac{1}{2\pi i} \int\limits_{|z| = \frac{1}{2}} \frac{f(z)}{z^2} dz \right| \leqq 2 M_f.$$

Wenn die betrachtete Funktion $f(z)$ speziell eine schlichte Abbildung der Kreisscheibe $|z| \leqq 1$ vermittelt, gilt ferner nach einem bekannten Satz aus der Theorie der konformen Abbildung die Ungleichung $M_f \leqq k_2 |a_1|$, wo k_2 (ebenso wie k_1) eine positive absolute Konstante bedeutet; aus den somit hier geltenden Ungleichungen $k_1 |a_1| \leqq M_f \leqq k_2 |a_1|$ ist unmittelbar ersichtlich, daß in dem eingangs genannten Satze über schlichte Abbildung, die Voraussetzung $a_1 = 1$ durch die Voraussetzung $M_f = 1$ ersetzt werden kann, d. h. aus $M_f = 1$ (und $a_0 = 0$) folgt $r_f \geqq K_1$, wo K_1 wiederum eine positive absolute Konstante bedeutet. In dieser Note soll nun bewiesen werden, daß auch dieser letzte Satz — ebenso wie der in 1. angeführte Satz — richtig bleibt, wenn die Forderung der Schlichtheit gestrichen wird. Der so entstehende Satz enthält den LANDAU'schen Satz, ist aber nicht unmittelbar aus diesem abzuleiten; denn für nicht-schlichte Abbildung gilt ja die Ungleichung $M_f \geq k_1 |a_1|$, aber nicht umgekehrt eine Ungleichung der Form $M_f \leqq k_2 |a_1|$ (es kann ja z. B. $a_1 = 0$, $M_f \neq 0$ sein).

3. Ich beweise den verallgemeinerten LANDAU'schen Satz in der folgenden Fassung, wo nicht eben der Kreis $|z| = \frac{1}{2}$, sondern ein beliebiger Kreis $|z| = \rho < 1$ betrachtet wird:

Es sei ρ eine gegebene Zahl im Intervalle $0 < \rho < 1$, und $w = f(z) = \Sigma a_n z^n$ eine beliebige für $|z| \leqq 1$ reguläre Funktion mit $a_0 = 0$ und $\underset{|z|=\rho}{\mathrm{Max}} |f(z)| = 1$. Dann ist der Radius r_f des größten Kreises $|w| = r_f$, dessen Punkte von $f(z)$ ($|z| \leqq 1$) sämtlich angenommen werden, $\geqq C$, wo $C = C(\rho)$ eine positive Zahl ist, die nur von ρ abhängt.

Der Beweis dieses Satzes verläuft im wesentlichen ganz wie der LANDAU'sche Beweis seines Satzes; nur benutze ich einen anderen Satz aus dem PICARD-LANDAU'schen Satzkreis als denjenigen, welcher von LANDAU angewendet wurde, nämlich den SCHOTTKY'schen Satz in der folgenden von LANDAU verschärften Form: Es sei $g(z)$ für $|z| \leqq 1$ regulär und $\neq 0, 1$, und $|g(0)| \leqq 1$. Dann gibt es zu jedem ρ im Intervalle $0 < \rho < 1$ eine nur von ρ abhängige positive Zahl $\Omega(\rho)$ derart, daß die Ungleichung $|g(z)| < \Omega(\rho)$ für alle $|z| \leqq \rho$ besteht. Aus diesem LANDAU-SCHOTTKY'schen Satze folgt in wenigen Worten

die Existenz einer Zahl $C = C(\rho)$ im Sinne des obigen Satzes, und zwar ergibt sich, daß z. B. die Zahl

$$C_0 = C_0(\rho) = \frac{1}{1 + 3\,\Omega(\rho)}$$

die erwähnte Eigenschaft besitzt. In der Tat, wäre dies nicht der Fall, so gäbe es eine für $|z| \leqq 1$ reguläre Funktion $f_0(z) = \Sigma a_n z^n$ mit $a_0 = 0$ und $\underset{|z|=\rho}{\mathrm{Max}} |f_0(z)| = 1$, für welche der Radius $r_{f_0} < C_0$ wäre, und die also weder sämtliche Werte auf dem Kreise $|w| = C_0$ noch sämtliche Werte auf dem Kreise $|w| = 2\,C_0$ annehme, etwa nicht die Werte $\alpha = C_0 e^{i\theta}$ und $\beta = 2\,C_0 e^{i\varphi}$. Ich setze

$$g(z) = \frac{f_0(z) - \alpha}{\beta - \alpha};$$

dann genügt $g(z)$ offenbar den sämtlichen Voraussetzungen des LANDAU-SCHOTTKY'schen Satzes, d. h. es ist $g(z)$ für $|z| \leqq 1$ regulär und $\neq 0, 1$, und ferner

$$|g(0)| = \left|\frac{f_0(0) - \alpha}{\beta - \alpha}\right| \leqq \frac{C_0}{2\,C_0 - C_0} = 1.$$

Also wäre für $|z| \leqq \rho$

$$|g(z)| < \Omega(\rho),$$

d. h. es wäre für $|z| \leqq \rho$

$$|f_0(z)| = |\alpha + (\beta - \alpha)\,g(z)| < C_0 + 3\,C_0\,\Omega(\rho) = 1$$

gegen die Voraussetzung $\underset{|z|=\rho}{\mathrm{Max}} |f_0(z)| = 1$.

4. Es braucht kaum gesagt zu werden, daß der geringfügigen Verallgemeinerung, die der in 3. bewiesene Satz gegenüber dem (neuartigen) LANDAU'schen Satze aufweist, an sich kein großes Interesse zukommt; wenn ich mir aber trotzdem erlaubt habe, ihn zum Gegenstand dieser Note zu machen, liegt es daran, daß ich auf gewisse funktionentheoretische Untersuchungen über Funktionen von mehreren Variablen gestoßen bin, bei welchen die Anwendung des Satzes eben in dieser Formulierung wesentlich bequemer scheint als die Anwendung in der etwas spezielleren LANDAU'schen Formulierung. Als ein möglichst einfaches Beispiel zur Charakterisierung solcher Anwendungen werde ich den folgenden Satz beweisen: Es sei $\varphi(y)$ eine (nicht konstante) ganze Transzendente von y, und $w_1 = P_1(x_1) = \sum_{n=0}^{\infty} a_n^{(1)} x_1^n, \ldots,$

$w_m = P_m(x_m) = \sum_{n=0}^{\infty} a_n^{(m)} x_m^n$ Funktionen je einer der m Variabeln $x_1, \ldots, x_m$, welche in den respektiven Einheitskreisen $|x_1| \leqq 1, \ldots, |x_m| \leqq 1$ alle regulär sind, und die Bedingungen $a_0^{(1)} = \ldots = a_0^{(m)} = 0$ erfüllen. Es sei die — im Gebiete $|x_1| \leqq 1, \ldots, |x_m| \leqq 1$ absolut konvergente — Potenzreihe in m Variabeln

$$Q(x_1, \ldots, x_m) = \varphi\{P_1(x_1) + \ldots + P_m(x_m)\} = \sum A_{n_1, \ldots, n_m} x_1^{n_1} \ldots x_m^{n_m}$$

gebildet, und es erfülle diese Reihe die Ungleichung $|Q(x_1, \ldots, x_m)| \leqq 1$ für $|x_1| \leqq 1, \ldots, |x_m| \leqq 1$. Dann gibt es zu jedem r des Intervalles $0 < r < 1$ eine nur von der Funktion φ und der Zahl r abhängige (d. h. von m und den m Funktionen $P_1, \ldots, P_m$ unabhängige) Zahl $L = L(\varphi, r)$ derart, daß für $|x_1| = \ldots = |x_m| = r$ die Majorantenreihe

$$\sum \left| A_{n_1, \ldots, n_m} x_1^{n_1} \ldots x_m^{n_m} \right|$$

$\leqq L$ ist. Es genügt offenbar die Existenz einer, nur von φ und r abhängigen, Zahl $L_1 = L_1(\varphi, r)$ zu beweisen derart, daß für $|x_1| = \ldots = |x_m| = r$ die Ungleichung

$$\sum_{n=0}^{\infty} \left| a_n^{(1)} x_1^n \right| + \ldots + \sum_{n=0}^{\infty} \left| a_n^{(m)} x_m^n \right| \leqq L_1$$

besteht, und um die Existenz einer solchen Zahl $L_1(\varphi, r)$ festzustellen, genügt es wiederum (infolge der CAUCHY'schen Ungleichungen) zu beweisen, daß

$$\operatorname*{Max}_{|x_1| = \rho} |P_1(x_1)| + \ldots + \operatorname*{Max}_{|x_m| = \rho} |P_m(x_m)| \leqq L_2$$

ist, wo ρ etwa die Zahl $\frac{1}{2}(1 + r)$ bedeutet und L_2 (ebenso wie L und L_1) nur von φ und r, d. h. von φ und ρ abhängt. Zu diesem Zwecke sei zunächst die Zahl $R_0 = R_0(\varphi)$ so groß gewählt, daß für $R = R_0$ (und also auch für $R > R_0$) die Ungleichung $\operatorname*{Max}_{|y| = R} |\varphi(y)| > 1$ besteht. Bezeichnen nun $r_1, \ldots, r_m$ die Radien der größten Kreise $|w_1| = r_1, \ldots, |w_m| = r_m$, deren sämtliche Punkte von den respektiven Funktionen $w_1 = P_1(x_1), \ldots, w_m = P_m(x_m)$ in ihren Einheitskreisen $|x_1| \leqq 1, \ldots, |x_m| \leqq 1$ angenommen werden, so enthält die Menge der Werte, welche die Summe

$$S(x_1, \ldots x_m) = P_1(x_1) + \ldots + P_m(x_m)$$

im Gebiete $|x_1| \leq 1, \ldots, |x_m| \leq 1$ annimmt, offenbar sämtliche Punkte auf dem Kreise $|y| = r_1 + \ldots + r_m$, und es muß daher $r_1 + \ldots + r_m < R_0$ sein. Nun ist aber nach dem Satze in 3.

$$\operatorname*{Max}_{|x_1|=\rho} |P_1(x_1)| \leq \frac{r_1}{C(\rho)}, \ldots, \operatorname*{Max}_{|x_m|=\rho} |P_m(x_m)| \leq \frac{r_m}{C(\rho)},$$

und es ist also

$$\operatorname*{Max}_{|x_1|=\rho} |P_1(x_1)| + \ldots + \operatorname*{Max}_{|x_m|=\rho} |P_m(x_m)| \leq \frac{1}{C(\rho)}(r_1 + \ldots + r_m) < \frac{1}{C(\rho)} R_0 = L_2(\varphi, \rho),$$

womit der Satz bewiesen ist.

Ich füge noch hinzu, daß Herr H. KLOOSTERMAN sich mit der Anwendung des genannten Satzes auf Potenzreihen mit unendlich vielen Variabeln beschäftigt hat, und seine Resultate — sowie ihre Beziehungen zu der Theorie der DIRICHLET'schen Reihen — bald veröffentlichen wird.

Rec. 23. XI. 22.

*

ON THE LIMIT VALUES OF ANALYTIC FUNCTIONS

H. Bohr‡.

1. Suppose that $f(z)$ is an analytic function of $z = x+iy$, regular in the half-strip S defined by $a < x < b$, $y > 0$. A theorem of Montel states that if $f(z)$ is *bounded* in S and tends to a limit l, when $y \to \infty$, for a certain fixed value x^* of x between a and b, then $f(z)$ tends to this limit l on every line $x = x_0$ in S, and indeed $f(z)$ tends to l uniformly for $a+\delta < x_0 < b-\delta$.

A particularly simple proof of Montel's theorem has been given incidentally by Hardy in a recent note in this Journal §. Hardy considers the function

$$f_\eta(z) = f(z+i\eta) \quad (\eta > 0)$$

in the rectangle R defined by

$$a < x < b, \quad 0 < y < 1.$$

Since the functions $f_\eta(z)$ $(0 < \eta < \infty)$ are uniformly bounded in R and tend to the limit l, when $\eta \to \infty$, on the segment $x = x^*$, $0 < y < 1$, inside

‡ Received 18 May, 1927; read 9 June, 1927.

§ G. H. Hardy, "A theorem concerning harmonic functions", *Journal London Math. Soc.*, 1 (1926), 130–131.

R, it follows from the well known theorem of Vitali on sequences of analytic functions that $f_\eta(z)$ tends to the limit l throughout the whole rectangle R (and even uniformly in every rectangle interior to R), which proves the theorem.

2. If $f(z)$ is still regular in S but *no longer bounded*, we cannot infer from the existence of a limit of $f(z)$ on a single line in S the existence of a limit of $f(z)$ on other lines in S. And even in the case when we assume the existence of a limit of $f(z)$ on every line in S, we cannot conclude that this limit is constant, *i.e.* does not depend on x. It may therefore, at first sight, seem surprising that the following theorem is nevertheless true.

THEOREM. *If the (regular) function $f(z)$ tends to a finite limit $l(x_0)$ on every line $x = x_0$ of the strip S, then this limit $l(x_0)$ is always constant in some interval $\alpha < x < \beta$ interior to $a < x < b$.*

The proof is very easy and follows the same lines as the proof by Hardy of the theorem of Montel given above. We have only to use, instead of Vitali's theorem, a well known theorem of Osgood, which states that, if a sequence of analytic functions converges in every point of a rectangle R, then the limit function $g(z)$ is necessarily an analytic function in some domain (rectangle) inside R. The detailed proof runs as follows. We consider, as above, the functions $f_\eta(z) = f(z+i\eta)$ $(0 < \eta < \infty)$ within the rectangle R defined by $a < x < b$, $0 < y < 1$. From our assumption about $f(z)$, the functions $f_\eta(z)$ tend to a limit, when $\eta \to \infty$, at every point $z = x+iy$ of R, namely to the limit $g(z) = l(x)$. But, on account of Osgood's theorem, the limit function $g(z)$ must be analytic in some smaller rectangle r, say in $\alpha < x < \beta$, $\gamma < y < \delta$, and as $g(z)$ is constant [equal to $l(x)$] on every vertical segment in R, the function $g(z)$ must be constant in the whole rectangle r. That is to say, the function $l(x)$ must be constant for $\alpha < x < \beta$, which proves our theorem.

3. A similar argument shows that, if an analytic function has a *mean value*

$$l(x) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(x+iy)\,dy$$

on every line of a strip, then there is certainly an interval in which $l(x)$ is constant.

THÉORIE DES FONCTIONS. — *Sur un problème de M. Borel.*
Note de M. **Harald Bohr**, présentée par M. Émile Borel.

Dans *L'Intermédiaire des Mathématiciens* (avril 1899), M. Borel a proposé la question suivante sur les fonctions entières :

Peut-on trouver une fonction dont le module ne dépasse l'unité qu'à l'intérieur d'un angle aussi petit que l'on veut (donné d'avance) ou même seulement à l'intérieur d'une parabole.

Comme on le sait, la réponse à cette question, dans la forme que lui a donnée M. Borel, est positive. Dans sa belle conférence au Congrès international à Rome (1908) Mittag-Leffler a donné un aperçu de la théorie à laquelle le problème de M. Borel a donné naissance.

Dans cette Note je donnerai la solution définitive du problème de M. Borel en précisant d'abord le problème de la manière suivante :

PROBLÈME. — *Quelle est la condition nécessaire et suffisante à laquelle doit satisfaire un ensemble* E *de points dans le plan complexe, pour qu'il existe une fonction entière, qui reste bornée, ou même tende uniformément vers zéro, dans tout l'ensemble complémentaire de* E.

Pour abréger, nous appelons un ensemble de points du plan complexe une « bande », s'il forme un environ (d'ailleurs arbitrairement étroit) d'une courbe continue tendant vers l'infini, c'est-à-dire s'il est ouvert et contient une telle courbe. Alors la solution du problème de M. Borel, généralisé de la manière susdite, peut s'énoncer comme suit :

SOLUTION. — *Pour qu'un ensemble* E *ait la propriété désirée, il faut et il suffit, qu'il contienne une bande.*

La nécessité de notre condition est presque évidente. Toute la difficulté consiste à démontrer qu'elle est aussi suffisante. Une démonstration détaillée, basée sur une idée différente de celle employée par Mittag-Leffler dans ses recherches mentionnées ci-dessus, sera donnée dans un autre Recueil.

Über ganze transzendente Funktionen von einem besonderen Typus.

(Beispiel einer allgemeinen Konstruktionsmethode.)

Von Harald Bohr
in Kopenhagen.

(Vorgelegt von Hrn. Bieberbach am 24. Oktober 1929 [s. oben S. 508].)

Bei einer Untersuchung über analytische fastperiodische Funktionen, bei welcher es sich darum handelte, die Existenz regulärer fastperiodischer Funktionen mit gewissen besonderen Eigenschaften nachzuweisen, wurde ich auf eine Konstruktionsmethode derartiger Funktionen geführt, die, allgemein gesprochen, darin besteht, daß zunächst meromorphe Funktionen (statt reguläre Funktionen) gebildet wurden und dann nachher die künstlich eingeführten Pole durch ein bekanntes von Runge erdachtes Verfahren einer »Polverschiebung« und einen danach folgenden Grenzübergang wieder weggeschafft wurden. Aber auch außerhalb der Theorie der fastperiodischen Funktionen läßt sich diese Methode zur Behandlung verschiedener Probleme erfolgreich verwenden. In den folgenden Zeilen soll sie an einem möglichst einfachen Beispiel, nämlich der Konstruktion ganzer transzendenter Funktionen vom Typus der Mittag-Lefflerschen E-Funktion, aber mit noch ausgeprägteren Eigenschaften, erörtert werden. Dadurch findet eine von Borel (Intermédiaire des mathématiciens, April 1899) herrührende Fragestellung ihre endgültige Antwort.

Die von Runge nachgewiesene Möglichkeit einer Polverschiebung haben wir nur in dem folgenden prinzipiell einfachsten Falle zu benutzen:

Es sei G eine beliebige offene Punktmenge der komplexen z-Ebene, und es seien z^* und z^{**} zwei beliebige Punkte von G, die durch eine ganz in G verlaufende stetige Kurve k miteinander verbunden werden können. Ferner sei $P^*(x)$ ein beliebiges Polynom, das wir ohne konstantes Glied annehmen wollen. Dann gibt es zu jedem vorgegebenen $\varepsilon > 0$ ein Polynom $P^{**}(x)$, ebenfalls ohne konstantes Glied, so daß die Ungleichung

$$\left| P^{**}\left(\frac{1}{z-z^{**}}\right) - P^*\left(\frac{1}{z-z^*}\right) \right| < \varepsilon$$

in der ganzen zu G komplementären Menge Γ besteht, d. h. in allen Punkten z, die nicht zu G gehören[1].

Und nun zur Betrachtung ganzer transzendenter Funktionen. Es sei $F(z)$ eine beliebige solche Funktion, die nur keine Konstante ist; dann gibt es bekanntlich (vgl. Bieberbach, Lehrbuch der Funktionentheorie, Bd. II, S. 272) stetige ins Unendliche laufende Kurven K, so daß $|F(z)|$ längs diesen Kurven gegen Unendlich strebt. Es erweckte seinerseits ein gewisses Aufsehen, als Mittag-Leffler demgegenüber die Existenz einer ganzen Transzendenten nachwies, die auf allen Geraden gegen den Wert Null strebte. Später hat Grandjot (Math. Ann. Bd. 91, S. 316) durch Weiterführung der Mittag-Lefflerschen Konstruktionsmethode die Existenz einer ganzen Transzendenten bewiesen, die sogar auf allen ins Unendliche gehenden algebraischen Kurvenästen gegen Null strebt. Grandjot konnte nämlich eine Transzendente $F(z) = F(x+iy)$ angeben, die in der ganzen z-Ebene außerhalb eines Streifens

$$\frac{k_1 x}{\log x} < y < \frac{k_2 x}{\log x}$$

um die transzendente Kurve $y = \dfrac{x}{\log x}$ gleichmäßig gegen Null strebte, und es ist klar, daß jede algebraische Kurve, die ins Unendliche geht, von einem

---

[1] Der ebenso einfache wie elegante Beweis von Runge verläuft folgendermaßen (vgl. etwa Bieberbach, Lehrbuch der Funktionentheorie, Bd. I, S. 294). Es sei d eine positive Zahl, die kleiner ist als der Abstand der Kurve k vom Rande des Gebietes G, und es seien die Punkte $z_0 = z^*$, z_1, $\cdots$, z_{N-1}, $z_N = z^{**}$ auf der Kurve k so gewählt, daß der Abstand je zweier aufeinander folgenden Punkte kleiner als d ist. Die Verschiebung des Poles von z^* nach z^{**} wird in N Schritten ausgeführt, indem nacheinander Polynome ohne konstantes Glied $Q_0(x) = P^*(x)$, $Q_1(x)$, $\cdots$, $Q_{N-1}(x)$, $Q_N(x) = P^{**}(x)$ so bestimmt werden, daß für jedes $1 \leq n \leq N$ die Ungleichung

$$\left| Q_n\left(\frac{1}{z-z_n}\right) - Q_{n-1}\left(\frac{1}{z-z_{n-1}}\right) \right| < \frac{\varepsilon}{N}$$

im Gebiete Γ besteht. Es genügt offenbar, den ersten Schritt, d. h. den Übergang von z_0 zu z_1, auszuführen. Um den Punkt z_1 wird der Kreis $|z - z_1| = d$ geschlagen. Da der Punkt z_0 im Innern dieses Kreises gelegen ist, so ist die Funktion $Q_0\left(\dfrac{1}{z-z_0}\right)$ regulär in allen Punkten außerhalb dieses Kreises, auch im Punkte $z = \infty$, wo sie den Wert 0 hat. Für $|z - z_1| > d$ läßt sich also die Funktion $Q_0\left(\dfrac{1}{z-z_0}\right)$ in eine Potenzreihe in $\dfrac{1}{z-z_1}$ (ohne konstantes Glied) entwickeln, etwa

$$Q_0\left(\frac{1}{z-z_0}\right) = \sum_{\nu=1}^{\infty} \frac{c_\nu}{(z-z_1)^\nu}.$$

Diese Potenzreihe konvergiert gleichmäßig im Gebiete Γ, weil dieses Gebiet ganz außerhalb des Kreises $|z - z_1| = d$ gelegen ist; wir können daher ein M so bestimmen, daß im ganzen Gebiete Γ

$$\left| Q_0\left(\frac{1}{z-z_0}\right) - \sum_{\nu=1}^{M} \frac{c_\nu}{(z-z_1)^\nu} \right| < \frac{\varepsilon}{N}$$

ist, und das Polynom $Q_1(x) = \sum\limits_{\nu=1}^{M} c_\nu x^\nu$ ist somit von der gewünschten Art.

gewissen Punkte an ganz außerhalb des genannten Streifens verläuft. Ich werde nun im folgenden zeigen, wie man durch die oben erwähnte Methode, d. h. indem man von einer meromorphen Funktion ausgeht und dann mit Hilfe des Rungeschen Verfahrens die Pole wieder wegräumt, in äußerst einfacher Weise zu ganzen Transzendenten gelangen kann, welche im obigen Sinne ein prinzipiell ausgeprägtestes Benehmen aufweisen.

Satz. *Es sei eine beliebige stetige ins Unendliche laufende Kurve K in der komplexen Ebene gegeben, etwa durch* $z = z(t)$ $(0 \leqq t < \infty)$, *wo* $z(t)$ *eine für* $0 \leqq t < \infty$ *stetige Funktion mit* $|z(t)| \to \infty$ *für* $t \to \infty$ *bedeutet. Ferner sei eine (beliebig enge) Umgebung G von K gegeben, d. h. eine beliebige offene Punktmenge, welche K enthält. Dann gibt es eine, nicht identisch verschwindende, ganze Transzendente* $F(z)$, *die gleichmäßig gegen Null strebt, wenn z innerhalb der zu G komplementären Menge* Γ *ins Unendliche geht*; d. h. zu jedem $\varepsilon > 0$ gibt es ein $\rho = \rho(\varepsilon) > 0$, so daß die Ungleichung

$$|F(z)| < \varepsilon$$

in jedem Punkt z mit $|z| > \rho$ gilt, welcher außerhalb der Menge G gelegen ist.

Beweis. Wir wählen eine beliebige Zahlenfolge $0 < t_1 < t_2 \cdots$ $(t_n \to \infty)$ derart, daß bei jedem $n = 1, 2, \cdots$ die Ungleichung

$$|z(t)| > n \qquad \text{für } t \geqq t_n$$

besteht, und bezeichnen den auf der Kurve K gelegenen Punkt $z(t_n)$ mit z_n. Ferner bezeichne G_n die offene Menge, welche aus allen Punkten aus G besteht, die außerhalb des Kreises $|z| = n$ liegen, und es bezeichne Γ_n die Komplementärmenge von G_n. Die Kreisscheibe $|z| \leqq n$ gehört also ganz zu Γ_n, und die beiden Punkte z_n und z_{n+1} liegen in G_n und lassen sich durch eine stetige ganz in G_n verlaufende Kurve (nämlich das Kurvenstück $z = z(t)$, $t_n \leqq t \leqq t_{n+1}$) verbinden.

Mit $P_1(x), P_2(x), \cdots, P_n(x), \cdots$ soll eine Folge von Polynomen ohne konstantes Glied bezeichnet werden, die durch das folgende sukzessive Verfahren bestimmt werden. Wir gehen von dem einfachsten solchen Polynom, etwa $P_1(x) = x$, aus und bilden die Funktion

$$P_1\left(\frac{1}{z - z_1}\right) = \frac{1}{z - z_1}.$$

Es sei z_0 ein beliebig gewählter Punkt von Γ, d. h. außerhalb G; wir bezeichnen die Zahl $\left|P_1\left(\frac{1}{z_0 - z_1}\right)\right| = \left|\frac{1}{z_0 - z_1}\right| > 0$ mit α und bestimmen eine Folge positiver Zahlen $\varepsilon_1, \varepsilon_2, \cdots$, so daß $\sum \varepsilon_n < \frac{\alpha}{2}$ ist. Nach dem Rungeschen Verfahren bestimmen wir das Polynom $P_2(x)$ so, daß die Ungleichung

$$\left|P_2\left(\frac{1}{z - z_2}\right) - P_1\left(\frac{1}{z - z_1}\right)\right| < \varepsilon_1$$

im ganzen Gebiete Γ_1 besteht. Danach bestimmen wir $P_3(x)$ so, daß in Γ_2

$$\left| P_3\left(\frac{1}{z-z_3}\right) - P_2\left(\frac{1}{z-z_2}\right) \right| < \varepsilon_2$$

ausfällt. Und allgemein bestimmen wir $P_{n+1}(x)$ derart, daß die Ungleichung

$$\left| P_{n+1}\left(\frac{1}{z-z_{n+1}}\right) - P_n\left(\frac{1}{z-z_n}\right) \right| < \varepsilon_n$$

in allen Punkten von Γ_n gilt.

Ich behaupte, daß der Grenzwert

$$F(z) = \lim_{n\to\infty} P_n\left(\frac{1}{z-z_n}\right)$$

für alle z existiert und eine ganze Transzendente der gewünschten Art liefert. Bei diesem Nachweis wird es bequem sein, die Funktion $F(z)$ statt als Grenzwert einer Folge lieber als Summe einer Reihe

$$(1) \qquad F(z) = P_1\left(\frac{1}{z-z_1}\right) + \sum_{n=1}^{\infty}\left\{ P_{n+1}\left(\frac{1}{z-z_{n+1}}\right) - P_n\left(\frac{1}{z-z_n}\right)\right\}$$

zu bestimmen.

1. Zunächst existiert $F(z)$ und ist eine ganze Transzendente. Denn bei festem (beliebig großem) ganzzahligem N liegen die Punkte z_n ($n \geqq N$) außerhalb des Kreises $|z| = N$, so daß die Funktionen $P_n\left(\frac{1}{z-z_n}\right)$ ($n \geqq N$) für $|z| < N$ alle regulär sind. Schreiben wir nun die Reihe (1) in der Form

$$(2) \qquad F(z) = P_N\left(\frac{1}{z-z_N}\right) + \sum_{n=N}^{\infty}\left\{ P_{n+1}\left(\frac{1}{z-z_{n+1}}\right) - P_n\left(\frac{1}{z-z_n}\right)\right\},$$

so ist jedes Glied regulär für $|z| < N$, und die Reihe konvergiert daselbst gleichmäßig, weil die Kreisscheibe $|z| < N$ in jedem Γ_n mit $n \geqq N$ gelegen ist und daher für jedes $n \geqq N$ die Ungleichung

$$\left| P_{n+1}\left(\frac{1}{z-z_{n+1}}\right) - P_n\left(\frac{1}{z-z_n}\right) \right| < \varepsilon_n$$

in $|z| < N$ besteht.

2. Die Funktion $F(z)$ strebt gleichmäßig gegen Null, wenn z innerhalb Γ gegen Unendlich strebt. Zu einem gegebenen ε können wir nämlich ein N so wählen, daß $\sum_{n=N}^{\infty} \varepsilon_n < \frac{\varepsilon}{2}$ ist. Dann folgt aus (2) für jedes z in Γ_N, also a fortiori für jedes z in Γ, die Ungleichung

$$|F(z)| \leqq \left| P_N\left(\frac{1}{z-z_N}\right)\right| + \sum_{n=N}^{\infty} \varepsilon_n < \left| P_N\left(\frac{1}{z-z_N}\right)\right| + \frac{\varepsilon}{2}.$$

Nun strebt aber die Funktion $P_N\left(\frac{1}{z-z_N}\right)$ gleichmäßig gegen Null, wenn z (in der ganzen Ebene) über alle Grenzen wächst, und wir können somit ein

$\rho > 0$ so groß wählen, daß die Ungleichung $\left|P_N\left(\frac{1}{z-z_N}\right)\right| < \frac{\varepsilon}{2}$ für alle z außerhalb des Kreises $|z| = \rho$ besteht. Also gilt die Ungleichung

$$|F(z)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

für alle z, die sowohl zu Γ gehören als auch numerisch größer als ρ sind.

3. Schließlich ist $F(z)$ nicht identisch Null. Denn in dem anfangs gewählten Punkte $z = z_0$, welcher in Γ und also a fortiori in allen Γ_n liegt, gilt ja nach (1) die Ungleichung

$$|F(z_0)| \geqq \left|P_1\left(\frac{1}{z_0 - z_1}\right)\right| - \sum_{n=1}^{\infty} \varepsilon_n > \alpha - \frac{\alpha}{2} > 0.$$

Nachtrag.

Angeregt durch eine Frage von Hrn. BIEBERBACH erlaube ich mir, zur Orientierung einige Bemerkungen allgemeiner Art hinzuzufügen.

1. Wenn ich den oben bewiesenen Satz als den bestmöglichen seiner Art bezeichnet habe, ist dies in dem genauen Sinne gemeint, daß er sofort zur vollständigen Beantwortung der folgenden allgemeinen Frage führt:

Wie muß eine Punktmenge D der komplexen Ebene beschaffen sein, damit es zu ihr eine (nicht identisch verschwindende) ganze Transzendente $F(z)$ gibt, welche in der zu D komplementären Menge Δ gleichmäßig für $z \to \infty$ gegen Null strebt?

Indem wir zur Abkürzung von einer offenen Punktmenge G der komplexen Ebene sagen, daß sie ein Streifen ist, wenn es eine stetige ins Unendliche gehende Kurve K gibt (und dann von selbst unendlich viele solche Kurven), von welcher G eine Umgebung ist, lautet die Antwort.

Die notwendige und hinreichende Bedingung dafür, daß D von der gewünschten Art ist, besteht darin, daß es einen Streifen G gibt, welcher ganz in D enthalten ist.

Nach unserem Satze ist diese Bedingung nämlich hinreichend, und daß sie notwendig ist, liegt auf der Hand; denn falls D eine Punktmenge der gewünschten Art ist und $F(z)$ eine im obigen Sinne zu ihr gehörige Funktion, wird ja $|F(z)|$ in der zu D komplementären Menge Δ beschränkt sein, etwa mit der oberen Grenze c, so daß alle Punkte z mit $|F(z)| > c$ zu D gehören müssen; die (offene) Menge aller dieser letzteren Punkte z bildet aber gerade einen Streifen G, weil sie gewiß eine ins Unendliche gehende stetige Kurve K enthält (nämlich sogar jede Kurve K, auf welcher $|F(z)|$ durchweg größer als c ist).

2. Aber auch die folgende anscheinend weitergehende Frage, wo nicht wie in 1. von einer Punktmenge, sondern von mehreren (endlich oder unendlich vielen) Punktmengen D_i die Rede ist, kann sofort durch unseren Satz erledigt werden.

Was ist die notwendige und hinreichende Bedingung, welche eine Gesamtheit von Punktmengen D_i erfüllen muß, damit es zu ihr eine ganze Transzendente $F(z)$ derart gibt, daß für jede der gegebenen Punktmengen D_i die Funktion $F(z)$ in der zu D_i komplementären Menge Δ_i gleichmäßig für $z \to \infty$ gegen Null strebt?

Falls die Anzahl der vorgelegten Punktmengen D_i eine endliche ist, fällt unsere Frage inhaltlich mit der unter 1. erledigten zusammen. Denn die Aussage, daß eine Funktion $F(z)$ in jeder der endlich vielen Komplementärmengen Δ_i gleichmäßig gegen Null strebt, ist ja damit gleichbedeutend, daß $F(z)$ in der Summe der Komplementärmengen Δ_i, d. h. in der Komplementärmenge Δ des Durchschnittes D der Mengen D_i, gleichmäßig gegen Null strebt, und die Antwort auf unsere Frage lautet also einfach, daß es einen Streifen G geben soll, welcher in den sämtlichen Punktmengen D_i enthalten ist.

Auch in dem Falle aber, wo die Anzahl der gegebenen Mengen D_i eine (abzählbar oder überabzählbar) unendliche ist, lautet die Antwort fast ebenso, nur muß man mit ihrer Formulierung etwas vorsichtiger sein:

Es soll einen festen Streifen G geben, von welchem in dem Sinne gesagt werden kann, daß sein »unendlich ferner Teil« in allen Punktmengen D_i enthalten ist, daß es zu jeder Menge D_i eine positive Zahl ρ_i so gibt, daß der Durchschnitt von G und $|z| > \rho_i$ ganz in D_i liegt[1].

Daß diese Bedingung hinreichend ist, ist wiederum eine unmittelbare Folge unseres Satzes (wir brauchen ja nur zu dem festen Streifen G eine Funktion $F(z)$ im Sinne dieses Satzes zu bestimmen). Sie ist aber auch notwendig. Wenn nämlich eine Gesamtheit von Mengen D_i von der gewünschten Art vorgelegt ist und $F(z)$ eine zu ihr gehörige Funktion bezeichnet, so können wir zunächst eine ins Unendliche gehende Kurve K bestimmen, längs welcher $|F(z)| \to \infty$, und danach (aus Stetigkeitsgründen) eine Umgebung G von K so eng wählen, daß die Relation $|F(z)| \to \infty$ gleichmäßig besteht, wenn z innerhalb G ins Unendliche geht. Dann muß aber der »unendlich ferne« Teil dieses Streifens G in jeder der Punktmengen D_i enthalten sein; denn falls c_i die (endliche) obere Grenze von $|F(z)|$ in der zu D_i komplementären Menge Δ_i bezeichnet und ρ_i so groß gewählt wird, daß für alle zu G gehörigen Punkte z mit $|z| > \rho_i$ die Ungleichung $|F(z)| > c_i$ besteht, muß ja der Durchschnitt von G und $|z| > \rho_i$ ganz in D_i liegen.

3. Als einfache Beispiele zur Erläuterung der Bemerkung 2 erwähnen wir:

I. Es sei K eine beliebige ins Unendliche gehende stetige Kurve, und es seien $D_1, D_2, \cdots, D_n, \cdots$ *abzählbar viele (beliebig gewählte) Umgebungen von K*. Dann *gibt* es eine Transzendente $F(z)$, die bei jedem n in der zu D_n komplementären Menge Δ_n gleichmäßig gegen Null strebt. In der Tat ergibt sich sofort durch eine einfache topologische Überlegung, daß die Gesamtheit dieser Mengen D_i der obigen Bedingung genügt. Wir können nämlich annehmen, daß bei jedem n die Menge D_{n+1} in der Menge D_n enthalten ist (sonst ersetze man nur für jedes n die Menge D_n durch den Durchschnitt der n ersten Mengen $D_1, D_2 \cdots D_n$);

[1] Hieraus folgt natürlich nicht, daß der Durchschnitt aller Mengen D_i einen Streifen enthält; der Durchschnitt kann sogar leer sein.

bezeichnet dann E_n den Durchschnitt von D_n etwa mit dem Kreisringe $n-1 < |z| < n+1$, wird die offene Menge $G = E_1 + E_2 + \cdots + E_n + \cdots$, welche die ganze Kurve K enthält, offenbar ein Streifen G von der verlangten Art sein, weil der Durchschnitt von G und $|z| > n$ ja ganz in D_n liegt.

II. Dagegen gibt es *keine* Transzendente $F(z)$, die in der Komplementärmenge *jeder festen Umgebung* D_ι *einer ins Unendliche gehenden Kurve* K gleichmäßig gegen Null strebt (abgesehen natürlich von dem uninteressanten Fall, wo K von der Art einer Peanokurve ist, in dem Sinne, daß die Punktmenge, welche aus der Kurve allein gebildet wird, einen Streifen enthält). Denn wie auch ein fester Streifen G gegeben wird, kann man immer eine solche Umgebung D_ι unserer Kurve K bestimmen, daß der Durchschnitt von G und $|z| > \rho$ bei keinem noch so großem ρ ganz in D_ι liegt. Wir brauchen ja nur innerhalb G eine Folge von Punkten z_n mit $z_n \to \infty$ zu wählen, welche nicht auf K gelegen sind, und danach (was aus Stetigkeitsgründen möglich ist) kleine abgeschlossene Kreisscheiben $|z - z_n| \leq \delta_n (< 1)$ so zu bestimmen, daß diese ebenfalls in G liegen und mit K keinen gemeinsamen Punkt haben; die Komplementärmenge der Summe aller dieser Kreisscheiben wird ja alsdann eine Umgebung D_ι von K von der gewünschten Art sein.

Ausgegeben am 25. November.

Berlin gedruckt in der Reichsdruckerei.

ÜBER GANZE TRANSZENDENTE FUNKTIONEN, DIE AUF JEDER DURCH DEN NULLPUNKT GEHENDEN GERADEN BESCHRÄNKT SIND

Von HARALD BOHR

EINLEITUNG.

In der vorliegenden kleinen Abhandlung soll die Struktur von ganzen transzendenten Funktionen einer gewissen Klasse — wir wollen sie zur Abkürzung die Klasse M benennen — untersucht werden.

Definition. *Von einer ganzen Funktion* $F(z) = F(re^{iv})$ *soll gesagt werden, dass sie zur Klasse* M *gehört, falls sie auf jedem vom Nullpunkte ausgehenden Halbstrahl* $v = v_0$ *beschränkt bleibt.*

Dass es überhaupt, abgesehen von dem trivialen Fall $F(z) =$ constans, ganze Funktionen der Klasse M gibt, ist bekanntlich zuerst von Mittag-Leffler bewiesen.

Der Bequemlichkeit halber wollen wir einen Halbstrahl $v = v_0$ durch Angabe des Punktes P_0 des Einheitskreises $|z| = 1$ charakterisieren, in welchem dieser Kreis von dem betreffenden Halbstrahl geschnitten wird. Bei einer Funktion $F(z)$ der Klasse M gehört also zu jedem Punkte P des Einheitskreises eine (reelle, nicht negative) Konstante $L(P)$, nämlich die obere Grenze $L(P)$ der Funktion $|F(z)|$ auf dem betreffenden Halbstrahl.

Unter einem Intervall i des Einheitskreises soll eine offene zusammenhängende Punktmenge dieses Kreises verstanden werden. Wir haben im Folgenden drei verschiedene Arten von Intervallen zu unterscheiden: 1) den ganzen Kreis, also ein Intervall ohne Randpunkte; wir wollen dieses spezielle Intervall mit I bezeichnen, 2) ein Intervall mit genau einem Randpunkte A, also den ganzen Kreis mit Ausnahme des einzigen Punktes A; dies Intervall wollen wir mit I_A bezeichnen, 3) den allgemeinen Fall, d. h. ein Intervall mit zwei verschiedenen Randpunkten A und B; wir bezeichnen dieses Intervall mit I_{AB}, wobei

die Reihenfolge der beiden Randpunkte so zu wählen sei, dass das betreffende Intervall in positiver Umlaufsrichtung durchlaufen wird, wenn ein Punkt des Einheitskreises sich von A aus nach B bewegt.

Wir sagen von einer ganzen Funktion $F(z)$ der Klasse M, dass sie in einem Intervall i des Einheitskreises beschränkt ist, falls sie auf allen Halbstrahlen, welche den Punkten des Intervalles i entsprechen, gleichartig beschränkt ist, falls es also eine Konstante C_i derart gibt, dass für alle Punkte P des Intervalles i die Ungleichung

$$L(P) < C_i$$

besteht.

Von zwei Intervallen i_1 und i_2 des Einheitskreises soll gesagt werden, dass i_1 ein Teilintervall von i_2 ist, falls nicht nur die sämtlichen Punkte von i_1 zu i_2 gehören, sondern auch jeder (etwaige) Randpunkt von i_1 im Intervalle i_2 gelegen ist. Für die folgenden Untersuchungen spielt der Begriff eines »vollen Beschränktheitsintervalles» eine wichtige Rolle. Wir sagen von einem Intervall i des Einheitskreises, dass es ein volles Beschränktheitsintervall einer gegebenen Funktion $F(z)$ der Klasse M darstellt, falls die Funktion $F(z)$ 1) in jedem Teilintervall von i beschränkt ist, dagegen 2) in keinem Intervall, welches einen Randpunkt von i enthält, beschränkt ist. Es ist klar, dass zwei volle Beschränktheitsintervalle i_1 und i_2 einer Funktion $F(z)$ der Klasse M entweder gar keinen Punkt gemeinsam haben, oder mit einander identisch sind. Ferner ist klar, dass falls $F(z)$ in einem Intervall i beschränkt ist, es ein (und nur ein) volles Beschränkheitsintervall i^* von $F(z)$ gibt, welches die sämtlichen Punkte von i enthält.

Unsere Untersuchung der Funktionen der Klasse M zerfällt in zwei Paragraphen. In § 1 beweisen wir, unter Heranziehung eines bekannten Satzes von Osgood aus der Theorie der stetigen Funktionen, den folgenden Satz.

Satz 1. *Zu jeder ganzen Funktion $F(z)$ der Klasse M gibt es Intervalle, in welchen sie beschränkt ist, und die somit vorhandenen (endlich oder abzählbar vielen) vollen Beschränktheitsintervalle i_1, i_2, ... liegen auf dem Einheitskreise überall dicht*; d. h. jedes (beliebig kleine) Intervall des Einheitskreises hat mit mindestens einem der vollen Beschränktheitsintervalle Punkte gemeinsam.

Es entsteht nun von selbst die Frage, ob es möglich ist, über die Menge der vollen Beschränktheitsintervalle einer beliebigen Funktion der Klasse M etwas näheres auszusagen als in dem Satze 1 geschehen ist (z. B. ob ihre Gesamtlänge immer gleich der ganzen Länge 2π des Einheitskreises sein sollte, oder dergleichen). Diese Frage wird in § 2 erledigt. Wir beweisen, dass sie verneinend zu beantworten ist,

dass also der Satz 1 in seiner Art das bestmöglichste Resultat liefert.

Satz 2. *Auf dem Einheitskreise sei eine beliebige (endliche oder abzählbare) Menge von Intervallen $i_1, i_2, \ldots$ gegeben, von denen keine zwei einen gemeinsamen Punkt besitzen, und die auf dem Einheitskreise überall dicht liegen. Dann lässt sich eine ganze Funktion $F(z)$ der Klasse M angeben, deren volle Beschränktheitsintervalle gerade die vorgegebenen Intervalle $i_1, i_2, \ldots$ sind.*

Beim Beweise des Satzes 2 gehen wir so vor, dass wir zunächst den Fall betrachten, wo die gegebene Intervallmenge aus nur einem einzigen Intervall I_A besteht, und einen besonders einfachen Typus einer Funktion der Klasse M aufstellen, welche gerade dieses Intervall als (einziges) volles Beschränktheitsintervall hat. Von solchen speziellen Funktionen ausgehend gelingt es danach in einfacher Weise, eine Funktion $F(z)$ der Klasse M aufzubauen, deren volle Beschränktheitsintervalle im Sinne des Satzes 2 ganz beliebig gegeben sind.

§ 1.

Den in der Einleitung erwähnten Satz von Osgood, spezialisiert für unseren Zweck (d. h. indem wir ein spezielles Gebiet betrachten) sprechen wir in der folgenden Formulierung aus, in welcher von einer kontinuierlichen Menge von Funktionen, statt wie üblich von einer abzählbaren Folge von Funktionen, die Rede ist:

Es sei $z = re^{iv}$ eine komplexe Veränderliche, und in dem Kreisringsektor

$$R: \qquad 1 < r < 2, \quad a < v < b \qquad (a < b < a + 2\pi)$$

sei eine Menge $f_\varkappa(z)$ $(0 \leqq \varkappa < \infty)$ von stetigen Funktionen gegeben, welche in jedem festen Punkte z_0 von R beschränkt ist (also $|f_\varkappa(z_0)| < k(z_0)$ für $0 \leqq \varkappa < \infty$). Dann gibt es innerhalb R einen kleineren (mit R konzentrischen) Kreisringsektor R^*, so dass die Funktionenmenge $f_\varkappa(z)$ innerhalb R^* gleichartig beschränkt ist, d. h. $|f_\varkappa(z)| < c$ für alle z in R^* und alle $0 \leqq \varkappa < \infty$ [1].

[1] Der Beweis verläuft bekanntlich folgendermassen. Wenn die Menge $f_\varkappa(z)$ im ganzen Kreisringsektor R gleichartig beschränkt ist, haben wir nichts zu beweisen. Falls dies nicht der Fall ist, bestimmen wir einen Punkt z_1 innerhalb R und eine Funktion $f_{\varkappa_1}(z)$, so dass $|f_{\varkappa_1}(z_1)| > 1$; wegen der Stetigkeit von $f_{\varkappa_1}(z)$ können wir dann innerhalb R einen Kreisringsektor R_1 um den Punkt z_1 so klein wählen, dass die Ungleichung $|f_{\varkappa_1}(z)| > 1$ im ganzen R_1 besteht. Wir wiederholen nun die obige Betrachtung: Falls die Menge $f_\varkappa(z)$ in R_1 gleichartig beschränkt ist, sind wir zum Ziele gelangt. Wenn nicht, können wir eine Funktion $f_{\varkappa_2}(z)$ und einen Punkt z_2 innerhalb R_1 so bestimmen, dass $|f_{\varkappa_2}(z_2)| > 2$ ist, und danach einen kleinen

Mit Hülfe dieses OSGOOD'schen Satzes können wir den in der Einleitung aufgestellten Satz 1 folgendermassen beweisen:

Es sei $F(z)$ eine beliebige ganze Funktion der Klasse M. Ferner sei I_{AB} ($A = e^{ia}$, $B = e^{ib}$) ein beliebiges (d. h. beliebig kleines) Intervall des Einheitskreises. Die Richtigkeit unseres Satzes in seinem vollen Umfange wird offenbar dargetan sein, wenn wir die Existenz eines Teilintervalles von I_{AB} nachweisen können, in welchem $F(z)$ beschränkt ist. Zu diesem Zwecke betrachten wir in dem Kreisringsektor $R: 1 < r < 2$, $a < v < b$ die Menge von stetigen Funktionen

$$f_s(z) = F(s \cdot z) \quad (0 \leqq s < \infty).$$

In jedem festen Punkte z ist unsere Funktionsmenge $f_s(z)$ beschränkt; denn bei festem $z\,(\neq 0)$ durchläuft ja der Punkt sz für $0 \leqq s < \infty$ gerade einen Halbstrahl vom Nullpunkte aus, und auf jedem solchen Halbstrahl ist die Funktion $F(z)$ nach Voraussetzung beschränkt. Nach dem OSGOOD'schen Satze gibt es also innerhalb R einen kleineren Kreisringsektor R^*, also a fortiori einen kleinen Kreisbogen $r = r^*$, $\alpha < v < \beta$, so dass die Funktionenmenge $f_s(z) = F(sz)$ daselbst gleichartig beschränkt ist. Dies bedeutet aber gerade, dass unsere Funktion $F(z)$ in dem ganzen Sektor $\alpha < v < \beta$, $0 \leqq r < \infty$ beschränkt bleibt, womit der Satz bewiesen ist.

§ 2.

Wir gehen nunmehr zum Beweise des Satzes 2 über, und werden dabei mehrere Falle zu unterscheiden haben.

I. Für den Fall, wo die gegebene Intervallmenge nur aus dem einzigen Intervalle I (also dem ganzen Einheitskreis) besteht, ist der Satz evident. Er wird ja hier von einer Funktion $F(z)$, die gleich einer Konstanten ist (und nur von einer solchen Funktion) befriedigt.

II. Wir betrachten danach den Fall, wo die gegebene Intervallmenge aus einem einzigen Intervalle I_A besteht, also aus dem ganzen Einheitskreis mit Ausnahme eines einzigen Punkte $A = e^{ia}$ Wir dürfen offenbar $a = 0$ (also $A = 1$) annehmen, so dass der »kri-

Kreisringsektor R_2 innerhalb R_1 so bestimmen, dass $|f_{\varkappa_2}(z)| > 2$ innerhalb R_2 gilt, und somit in R_2 die beiden Ungleichungen $|f_{\varkappa_1}(z)| > 1$ und $|f_{\varkappa_2}(z)| > 2$ bestehen In dieser Weise setzen wir fort, behaupten aber, dass der angedeutete Prozess nach endlich vielen Schritten abbrechen muss, dadurch dass wir zu einem kleinen Kreisringsektor R_n gelangen, innerhalb dessen unsere Funktionenmenge $f_\varkappa(z)$ gleichartig beschränkt ist; sonst gäbe es nämlich einen Punkt z_0 (nämlich einen gemeinsamen Punkt der Folge von ineinander geschachtelten Kreisringsektoren R_1, R_2, $R_3, \ldots$), in welchem die Funktionenmenge $f_\varkappa(z)$ gegen die Voraussetzung nicht beschränkt wäre.

tische» Halbstrahl gerade die positiv-reelle Achse ist (sonst ersetze man nur z durch ze^{-ia}). Dass der Satz in diesem Falle richtig ist, dass es also eine (nicht konstante) ganze Funktion existiert, welche auf jedem Halbstrahl vom Nullpunkte aus beschränkt ist, und welche überdies, für jedes noch so kleine ε, innerhalb des Winkelraumes $\varepsilon < v < 2\pi - \varepsilon$ gleichartig beschränkt ist, ist bekanntlich von MITTAG-LEFFLER bei seinen interessanten Untersuchungen über ganze Funktionen vom Typus seiner E-Funktion bewiesen.

Für das Folgende benötigen wir aber ein etwas weitergehendes Resultat, nämlich, dass es (nicht konstante) ganze Funktionen der Klasse M gibt — wir wollen eine (beliebig gewählte) von diesen Funktionen herausgreifen und mit $G(z)$ bezeichnen — welche, bei jedem ε, innerhalb des abgeschlossenen Winkelraums $\varepsilon \leqq v \leqq 2\pi$ beschränkt sind; das Entscheidende ist, dass hier 2π statt $2\pi - \varepsilon$ steht. Da es uns nur auf die Existenz einer solchen Funktion $G(z)$ ankommt, wollen wir, statt auf die in der früheren Literatur betrachteten speziellen Funktionen näher einzugehen, uns damit begnügen, einen allgemeinen Existenssatz heranzuziehen, welcher neulich (Sitzungsber. d. preuss. Akad., phys.-math. Klasse, 1929, XXVI) vom Verfasser bewiesen worden ist, und welcher besagt, dass es zu einer beliebig engen Umgebung U einer beliebigen ins Unendliche gehenden stetigen Kurve K immer eine (nicht identisch verschwindende) ganze Funktion gibt, welche gleichmässig gegen Null strebt, wenn z innerhalb der zu U komplementären Menge ins Unendliche geht; wenden wir diesen Satz z. B. auf die Umgebung $x > 0$, $0 < y < 1$ der Kurve $x \geqq 1$, $y = \frac{1}{2}$ an, erhalten wir ja sofort eine Funktion $G(z)$ der gewünschten Art. Neben der soeben eingeführten Funktion $G(z)$ brauchen wir auch eine Funktion $H(z)$ der Klasse M, welche gleichartig beschränkt bleibt, wenn der Punkt z sich von oben (statt wie bei $G(z)$ von unten) dem »kritischen» Halbstrahl $v = 0$ nähert, d. h. eine Funktion $H(z)$ der Klasse M mit der Eigenschaft, dass sie in jedem Winkelraum $0 \leqq v \leqq 2\pi - \varepsilon$ beschränkt bleibt; eine solche Funktion $H(z)$ können wir natürlich sofort aus $G(z)$ durch Spiegelung erhalten, nämlich

$$H(z) = \overline{G}(\bar{z}),$$

wo der Strich den Übergang zu konjugiert komplexem bezeichnet.

Bevor wir mit dem Beweise des Satzes 2 weitergehen, schalten wir noch eine kleine vorbereitende Betrachtung ein, indem wir zu einem beliebigen Intervall i (mit zwei Randpunkten) eine gewisse Hilfsfunktion $F_i(z)$ einführen. Es seien $A = e^{ia}$ und $B = e^{ib}$ $(a < b < 2\pi + a)$

zwei beliebig gewählte Punkte des Einheitskreises, und es bezeichne i das Intervall I_{AB}. Wir bilden die Funktion

$$F_i(z) = \varepsilon_{AB}\{G(ze^{-ia}) + H(ze^{-ib})\},$$

wo ε_{AB} eine positive Konstante $< \frac{1}{2}$ bezeichnet, die so klein gewählt ist, dass die Funktion $F_i(z)$ im ganzen abgeschlossenen Winkelraum $b \leqq v \leqq a + 2\pi$ numerisch kleiner als 1 ist; eine solche Wahl von ε_{AB} ist natürlich möglich, da ja sowohl $G(ze^{-ia})$ als $H(ze^{-ib})$ in dem genannten Winkelraum beschränkt sind. Aus den Eigenschaften von $G(z)$ und $H(z)$ ergibt sich sofort, dass $F_i(z)$ eine ganze Funktion der Klasse M ist, welche in jedem Teilintervall von $i = I_{AB}$ beschränkt ist, dagegen in keinem der beiden Intervalle I_{AC} und I_{CB}, wo C einen Punkt des Intervalles i bedeutet, beschränkt ist. Ferner wissen wir, dass $|F_i(z)| < 1$ in dem ganzen abgeschlossenen Winkelraum, welcher dem zu $i = I_{AB}$ komplementären (abgeschlossenen) Intervalle des Einheitskreises entspricht. Wir bemerken noch, um gewisse spätere Konvergenzbetrachtungen nicht unnötig zu komplizieren, dass die Funktion $F_i(z)$, wie auch das Intervall $i = I_{AB}$ gewählt wird, durch die feste Funktion $G(z)$ im folgenden Sinne »majorisiert» wird. Bezeichnet $M(r_0)$ das Maximum von $|G(z)|$ (und also auch von $|H(z)|$) für $|z| \leqq r_0$, so gilt, wegen $\varepsilon_{AB} < \frac{1}{2}$, bei jedem r_0 die Ungleichung

$$|F_i(z)| \leqq M(r_0) \text{ für } |z| \leqq r_0.$$

Nachdem wir diese Funktion $F_i(z)$ zur Verfügung haben, können wir nunmehr leicht den Beweis des Satzes 2 zu Ende führen.

III. Besteht die gegebene Intervallmenge aus einer endlichen Anzahl $N \geqq 2$ von (an einander stossenden) Intervallen i_1, $i_2, \ldots, i_N$, genügt offenbar die Funktion

$$F(z) = \sum_{n=1}^{N} F_{i_n}(z)$$

den Bedingungen unseres Satzes. Denn als Summe von endlich vielen ganzen Transzendenten der Klasse M ist $F(z)$ selbst eine ganze Funktion der Klasse M. Und ihre vollen Beschränktheitsintervalle sind gerade die Intervalle $i_1, \ldots, i_N$; um einzusehen, dass z. B. $i_1 = I_{A_1B_1}$ ein volles Beschränktheitsintervall von $F(z)$ darstellt, brauchen wir ja nur zu bemerken, dass $F(z)$ einerseits in jedem Teilintervall von i_1 beschränkt ist, weil die N Funktionen $F_{i_n}(z)$ daselbst alle beschränkt sind, während $F(z)$ andererseits in keinem der beiden Intervalle $I_{A_1C_1}$ und $I_{C_1B_1}$, wo C_1 einen Punkt des Intervalles i_1 bedeutet, beschränkt ist,

weil die $N-1$ Funktionen $F_{i_n}(z)$ $(n=2,\ldots,N)$ daselbst beschränkt sind, die Funktion $F_{i_1}(z)$ dagegen nicht.

IV. Schliesslich betrachten wir den (allgemeinen) Fall, wo die gegebene Intervallmenge aus unendlich vielen Intervallen $i_1, i_2, \ldots$ besteht. Es sei $\varepsilon_1, \varepsilon_2, \ldots$ eine beliebige Folge von positiven Zahlen, so dass $\sum \varepsilon_n$ konvergiert. Wir bilden die unendliche Reihe

$$(1) \qquad \sum_{n=1}^{\infty} \varepsilon_n F_{i_n}(z)$$

und behaupten, dass sie für alle z konvergiert und eine ganze Transzendente $F(z)$ der Klasse M darstellt, welche gerade die gegebenen Intervalle i_n zu Beschranktheitsintervallen hat.

Erstens ist die Reihe (1) bei jedem festen r_0 im Kreise $|z| < r_0$ gleichmässig konvergent; sie hat ja daselbst die Majorantenreihe $\sum \varepsilon_n M(r_0) = M(r_0) \sum \varepsilon_n$. Also konvergiert die Reihe (1) für alle z und stellt eine ganze Transzendente $F(z)$ dar.

Zweitens ist $F(z)$ von der Klasse M, d. h. sie ist auf jedem vom Nullpunkte ausgehenden Halbstrahl h beschrankt. Denn entweder schneidet der Halbstrahl h keines der Intervalle i_n, und es gilt alsdann auf h die Ungleichung $|F(z)| < \sum\limits_1^{\infty} \varepsilon_n$, oder er schneidet genau eines dieser Intervalle, etwa i_N, und dann gilt auf h die Ungleichung

$$|F(z)| < \varepsilon_N c_N + \sum_{n \neq N} \varepsilon_n,$$

wo c_N die obere Grenze von $|F_N(z)|$ auf h bezeichnet.

Drittens ist bei jedem N das Intervall $i_N = I_{A_N B_N}$ tatsachlich ein volles Beschränktheitsintervall. Denn schreiben wir $F(z)$ in der Form

$$F(z) = \varepsilon_N F_{i_N}(z) + \sum_{n \neq N} \varepsilon_n F_{i_n}(z) = \varepsilon_N F_{i_N}(z) + R(z),$$

gilt in dem ganzen Winkelraum, welcher dem Intervalle i_N entspricht, die Ungleichung

$$|R(z)| < \sum_{n \neq N} \varepsilon_n;$$

hieraus folgt aber sofort, dass die Funktion $F(z)$ in jedem Teilintervall von $i_N = I_{A_N B_N}$ beschränkt ist, dagegen in keinem der beiden Intervalle $I_{A_N C_N}$ und $I_{C_N B_N}$ (wo C_N einen Punkt aus i_N bedeutet) beschränkt bleibt, weil ja diese beiden Aussagen für die Funktion $F_{i_N}(z)$ zutreffen.

SCHLUSSBEMERKUNG.

Zur Orientierung füge ich hinzu, dass die (in sich abgerundete) Gesamtaussage der beiden Sätze 1 und 2 unverändert bestehen bleibt, wenn überall die Worte »eine ganze Funktion $F(z)$ der Klasse M» durch die Worte »eine in der ganzen Ebene stetige Funktion $F(z)$, welche auf jedem vom Nullpunkte ausgehenden Halbstrahl beschränkt ist» ersetzt werden. Dass man im Satze 2 die Forderung der Analytizität durch die Forderung der Stetigkeit ersetzen kann, ist ohne weiteres klar; der Satz 2 in der neuen Formulierung besagt ja viel weniger als der ursprüngliche Satz 2 (und der Beweis ist auch viel leichter zu führen; vor allem ist die Heranziehung des allgemeinen Existenzsatzes auf Seite **43** gar nicht nötig). Dagegen besagt der Satz 1 natürlich mehr, wenn er für stetige Funktionen, als wenn er nur für analytische Funktionen ausgesprochen wird; dass er tatsächlich in der weitergehenden Fassung gültig ist, lehrt ein Blick auf dem in § 1 dargestellten Beweis, weil die Anwendung des Osgood'schen Satzes ja gar nicht die Analytizität sondern nur die Stetigkeit der Funktionen $f_k(z)$ verlangt.

Über einen Satz von J. Pál.

Von HARALD BOHR in Kopenhagen.

In seiner Abhandlung: Sur des transformations de fonctions qui font converger leurs séries de Fourier (*Comptes rendus Paris*, **158** (1914), p. 101) hat J. PÁL den folgenden sehr interessanten Satz bewiesen:

Es sei $f(x)$ eine im Intervalle $0 \leqq x \leqq 2\pi$ stetige, reelle Funktion mit $f(0) = f(2\pi)$. Dann gibt es immer eine im Intervalle $0 \leqq u \leqq 2\pi$ stetige, monoton wachsende Funktion $g(u)$ mit $g(0) = 0$, $g(2\pi) = 2\pi$, derart, daß die Fourierreihe $\sum_0^\infty (a_n \cos nu + b_n \sin nu)$ der Funktion $h(u) = f(g(u))$ im ganzen Intervalle $(0, 2\pi)$ konvergiert, und zwar *gleichmäßig in jedem Teilintervall* $0 < \delta < u < 2\pi - \delta$.

Der Beweis von PÁL beruhte auf dem folgenden bekannten Satz von FEJÉR[1]): Es sei $F(z) = F(x + iy)$ eine, im offenen Einheitskreise $|z| < 1$ analytische, im abgeschlossenen Kreise $|z| \leqq 1$ stetige Funktion, deren Potenzreihe für $|z| < 1$ mit $\sum_0^\infty \alpha_n z^n$ bezeichnet wird. Ferner sei angenommen, daß die Funktion $w = F(z)$ den Kreis $|z| < 1$ auf ein schlichtes Gebiet der w-Ebene abbildet. Dann konvergiert die Potenzreihe auch auf dem Rande des Einheitskreises, und zwar gleichmäßig im ganzen abgeschlossenen Bereiche $|z| \leqq 1$.

Bemerkung: Für einen späteren Zweck erinnern wir daran, daß im Fejérschen Beweis die Annahme der Schlichtheit der Abbildung nur benutzt wird, um schließen zu können, daß das In-

---

[1]) L. FEJÉR, La convergence sur son cercle de convergence d'une série de puissance effectuant une représentation conforme du cercle sur le plan simple, *Comptes rendus Paris*, **156** (1913), p. 46.

tegral $\iint\limits_{|z|<1} |F'(z)|^2 dx\,dy < \infty$ ist. Der Fejérsche Satz gilt also auch für jede in $|z|<1$ analytische, in $|z|\leqq 1$ stetige Funktion $F(z)$, für welche das Integral $\iint\limits_{|z|<1} |F'(z)|^2 dx\,dy$ einen endlichen Wert besitzt. Natürlich genügt es hierzu z. B. $\iint\limits_{\frac{1}{2}<|z|<1} |F'(z)|^2 dx\,dy < \infty$ zu wissen.

Der Pálsche Beweis verlief nun, kurz skizziert, folgendermaßen: Die Kurve $y=f(x)$, $0\leqq x\leqq 2\pi$, wurde zu einer Jordankurve J erweitert, etwa in der in Fig. 1ᵃ angegebenen Weise. Das

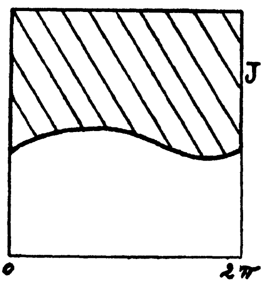


Fig. 1ᵃ

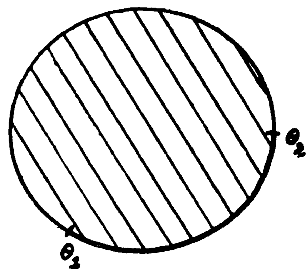


Fig. 1ᵇ

Innere dieser Jordankurve wurde danach konform auf den Einheitskreis $|\zeta|<1$ der $\zeta=re^{i\theta}$-Ebene abgebildet (Fig. 1ᵇ), etwa durch die Funktion $z=F(\zeta)=\sum\limits_{0}^{\infty} a_n \zeta^n$. Hierbei geht die Randkurve J bekanntlich stetig in den Einheitskreis $|\zeta|=1$ über, so daß die Funktion $F(\zeta)$ den Bedingungen des Fejérschen Satzes genügt. Nach diesem Satz konvergiert also die Potenzreihe $\sum\limits_{0}^{\infty} a_n \zeta^n$ gleichmäßig für $|\zeta|\leqq 1$. Bei der Randabbildung möge der aus der ursprünglich gegebenen Kurve $y=f(x)$, $0\leqq x\leqq 2\pi$, bestehende Teil von J in den abgeschlossenen Kreisbogen $r=1$, $\theta_1\leqq\theta\leqq\theta_2$ übergehen. Schließlich wurde nun das Intervall $\theta_1\leqq\theta\leqq\theta_2$ linear auf das (größere) Intervall $0\leqq u\leqq 2\pi$, etwa durch die Funktion $\theta=c+du$, abgebildet, und die entsprechende Funktion $F(e^{i(c+du)})=G(u)$, $0\leqq u\leqq 2\pi$, gebildet. Es wurde nun nachgewiesen, daß durch diese lineare Transformation der unabhängigen Veränderlichen die gleichmäßige Konvergenz der zugehörigen Fourierreihe nicht „wesentlich“ gestört wird, das heißt genau

gesprochen, daß die Fourierreihe $\sum_{0}^{\infty} \gamma_n e^{inu}$ der komplexen Funktion $G(u)$ bei jedem $\delta > 0$ für $\delta < u < 2\pi - \delta$ gleichmäßig konvergiert. Trennen wir nun die Darstellung $z = x + if(x) = \sum_{0}^{\infty} \gamma_n e^{inu} = = \sum_{0}^{\infty} (\gamma'_n + i\gamma''_n)(\cos nu + i \sin nu)$ in ihre reelle und rein imaginäre Komponente, also

$$x = \sum_{0}^{\infty} (\gamma'_n \cos nu - \gamma''_n \sin nu) = g(u) \cdot$$

$$f(x) = \sum_{0}^{\infty} (\gamma'_n \sin nu + \gamma''_n \cos nu) = h(u) = f(g(u)),$$

so haben wir offenbar in der monotonen Transformation $x = g(u)$ eine Transformation im Sinne des Pálschen Satzes.

Der Satz von PÁL hat den kleinen Schönheitsfehler, daß die Fourierreihe der transformierten Funktion $h(u)$ *nicht im ganzen Intervall* $0 \leqq u \leqq 2\pi$ *gleichmäßig konvergiert, sondern nur in jedem Teilintervall* $0 < \delta < u < 2\pi - \delta$, und dies hängt auf das engste damit zusammen, daß bei der konformen Abbildung, welche ja der entscheidene Punkt beim Beweise war, Hilfslinien etwas zufälliger Natur verwendet werden mußten, um die gegebene Kurve $y = f(x)$, $0 \leqq x \leqq 2\pi$ zu einer geschlossenen Jordankurve zu ergänzen.

Der Zweck der vorliegende Note ist nun zu zeigen, wie man durch eine kleine Abänderung des Pálschen Beweises diese kleine Unvollkommenheit in natürlicher Weise beseitigen kann, indem man, statt die gegebene Kurve $y = f(x)$, $0 \leqq x \leqq 2\pi$ zu einer im Endlichen liegenden Jordankurve zu ergänzen, die Kurve einfach *periodisch fortsetzt* und danach das eine von den beiden unbeschränkten Gebieten, welche von der so entstandenen Kurve begrenzt werden, konform auf *eine Halbebene* abbildet.

Für diese Abbildung gilt der folgende, mit Hilfe des obigen Fejérschen Satzes sofort zu beweisende Hilfssatz, wobei wir der Bequemlichkeit halber die gegebene Funktion $y = f(x)$ als überall positiv annehmen können (sonst addiere man zu $f(x)$ eine geeignete positive Konstante):

Es sei $y = f(x)$, $-\infty < x < \infty$, eine positive, stetige periodische Funktion mit der Periode 2π. In der $z = x + iy$-Ebene betrachten wir den abgeschlossenen Bereich $-\infty < x < \infty$, $y \geqq f(x)$, der mit

G bezeichnet sei (Fig. 2ᵃ). Dieser Bereich werde auf die abgeschlossene Halbebene $v \geqq 0$ der $w = u + iv$-Ebene abgebildet (Fig. 2ᵇ), und zwar konform im Innern, und daher stetig auf dem Rande; weiter soll die Abbildung so normiert sein, daß die drei Rand-

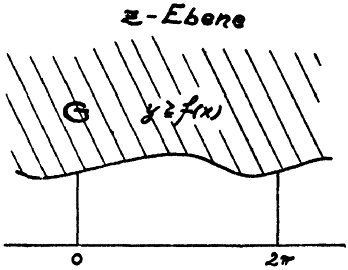


Fig. 2ᵃ

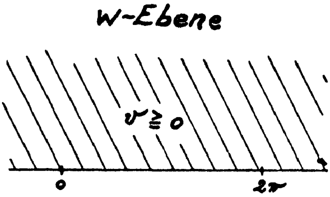


Fig. 2ᵇ

punkte $0 + if(0)$, $2\pi + if(2\pi)$, ∞ der z-Ebene in die drei Randpunkte 0, 2π, ∞ der w-Ebene übergehen. Dann wird die Abbildung durch eine Funktion der Form

$$z = w + \psi(w)$$

vermittelt, wo $\psi(w)$ periodisch mit der Periode 2π ist und eine Entwicklung der Form

$$\psi(w) = \sum_0^\infty \beta_n e^{inw}$$

zulässt, die gleichmäßig in der ganzen abgeschlossenen Halbebene $v \geqq 0$ konvergiert.

Bevor wir den einfachen Beweis dieses Satzes erbringen, bemerken wir, daß aus ihm der Pálsche Satz in der erwähnten verschärften Form unmittelbar gefolgert werden kann. Wir haben ja nur in der Darstellung

$$x + if(x) = u + \sum_0^\infty \beta_n e^{inu} = u + \sum_0^\infty (\beta_n' + i\beta_n'')\, e^{inu}$$

Reelles und Imaginäres zu trennen, also

$$x = u + \sum_0^\infty (\beta_n' \cos nu - \beta_n'' \sin nu) = g(u),$$

$$f(x) = \sum_0^\infty (\beta_n' \sin nu + \beta_n'' \cos nu) = h(u) = f(g(u))$$

zu setzen. Die monotone Transformation $x = g(u)$ liefert somit das Gewünschte.

Um schließlich den oben formulierten Hilfssatz zu beweisen, führen wir die genannte Abbildung von G auf die Halbebene $v \geqq 0$ in den folgenden Schritten aus.

Zuerst bilden wir die ganze Halbebene $y \geqq 0$ mit Hilfe der Funktion

$$s = e^{iz}$$

auf den unendlich-blättrigen Einheitskreis $|s| \leqq 1$ mit dem Windungspunkt $s = 0$ ab.

Hierbei geht die periodische Kurve $y = f(x)$ in die unendlich oft durchlaufene Jordankurve j (Fig. 3ᵃ)

$$s = e^{i(x+if(x))} = e^{-f(x)}\, e^{ix}$$

über, die übrigens von jedem von $s = 0$ ausgehenden Halbstrahl in genau einem Punkt getroffen wird.

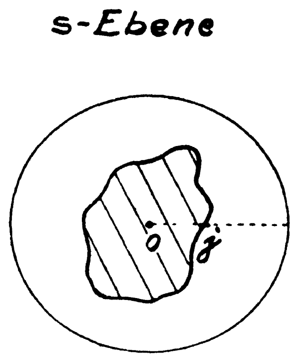


Fig. 3ᵃ

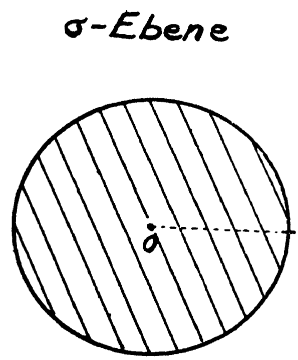


Fig. 3ᵇ

Danach wird der abgeschlossene schlichte Bereich, welcher von dieser Kurve j begrenzt wird, auf den Einheitskreis $|\sigma| \leqq 1$ der $\sigma = \sigma_1 + i\sigma_2$ Ebene abgebildet (Fig. 3ᵇ), wiederum konform im Inneren und stetig auf dem Rande, und zwar so, daß dem Punkte $s = 0$ der Punkt $\sigma = 0$ entspricht und der auf der positiven reellen Achse gelegene Punkt von j in den Punkt $\sigma = 1$ übergeht. Diese Abbildung möge durch

$$s = \Omega(\sigma) = \sigma \cdot \omega(\sigma) = \sigma \sum_{0}^{\infty} \delta_n \sigma^n$$

dargestellt sein. Hierbei ist, wegen der Schlichtheit der Abbildung, $\omega(\sigma) \neq 0$ für $|\sigma| \leqq 1$ und das Integral

$$\iint_{|\sigma|<1} |\Omega'(\sigma)|^2 \, d\sigma_1 \, d\sigma_2$$

endlich.

Schließlich wird der nunmehr als unendlichblättrig aufgefaßte Einheitskreis $|\sigma| \leqq 1$ auf die Halbebene $v \geqq 0$ der w-Ebene durch die Funktion

$$\sigma = e^{iw}$$

abgebildet.

Durch Zusammensetzung der drei besprochenen Abbildungen

$$z = \frac{1}{i} \log s, \quad s = \Omega(\sigma), \quad \sigma = e^{iw}$$

erhalten wir offenbar die gewünschte Abbildung der Halbebene $v \geqq 0$ der w-Ebene auf den Bereich G der z-Ebene, da ja (nach passender Normierung von $\log s$) die Punkte $0, 2\pi, \infty$ in $if(0)$, $2\pi + if(2\pi), \infty$ übergehen.

Die Funktion $z = \lambda(w)$, welche diese Abbildung liefert, wird somit durch

$$z = \frac{1}{i} \log \Omega(\sigma) = \frac{1}{i} \log \sigma + \frac{1}{i} \log \omega(\sigma), \quad \sigma = e^{iw}$$

gegeben.

Indem wir die für $|\sigma| < 1$ analytische, für $|\sigma| \leqq 1$ stetige Funktion $\frac{1}{i} \log \omega(\sigma)$ mit $L(\sigma)$ und ihre Potenzreihe mit $\sum_0^\infty \beta_n \sigma^n$ bezeichnen, erhalten wir also schließlich

$$z = w + L(e^{iw}) = w + \psi(w) = w + \sum_0^\infty \beta_n e^{inw}.$$

Vorläufig wissen wir aber nur, daß die Entwicklung $\sum_0^\infty \beta_n \sigma^n$ der Funktion $L(\sigma)$ im Inneren des Einheitskreises gilt; um den Beweis zu Ende zu führen, haben wir noch darzutun, daß die Potenzreihe im *abgeschlossenen* Kreise $|\sigma| \leqq 1$ gleichmäßig konvergiert. Hierzu genügt es nach der anfangs gemachten Bemerkung zum Fejérschen Satz zu zeigen, daß das Integral

$$\iint_{\frac{1}{2} < |\sigma| < 1} |L'(\sigma)|^2 \, d\sigma_1 \, d\sigma_2 = \iint_{\frac{1}{2} < |\sigma| < 1} \left| \frac{\omega'(\sigma)}{\omega(\sigma)} \right|^2 d\sigma_1 \, d\sigma_2 < \infty$$

ist.

Dies folgt aber sofort aus der Tatsache, daß

$$\iint_{|\sigma| < 1} |\Omega'(\sigma)|^2 \, d\sigma_1 \, d\sigma_2 < \infty$$

ist. Bezeichnen wir nämlich Min $|\omega(\sigma)|$ für $|\sigma| \leqq 1$ mit m (>0), so folgt aus

$$\Omega'(\sigma) = \sigma \,.\, \omega'(\sigma) + \omega(\sigma)$$

für $\frac{1}{2} < |\sigma| < 1$

$$\left|\frac{\omega'(\sigma)}{\omega(\sigma)}\right| \leqq \frac{1}{|\sigma|}\left\{\frac{|\Omega'(\sigma)|}{|\omega(\sigma)|}+1\right\} \leqq 2\left\{\frac{|\Omega'(\sigma)|}{m}+1\right\},$$

also

$$\left|\frac{\omega'(\sigma)}{\omega(\sigma)}\right|^2 \leqq 8\left\{\frac{|\Omega'(\sigma)|^2}{m^2}+1\right\}$$

und somit

$$\iint\limits_{\frac{1}{2}<|\sigma|<1} \left|\frac{\omega'(\sigma)}{\omega(\sigma)}\right|^2 d\sigma_1\, d\sigma_2 \leqq \frac{8}{m^2} \iint\limits_{|\sigma|<1} |\Omega'(\sigma)|^2\, d\sigma_1\, d\sigma_2 + 8\pi < \infty.$$

(Eingegangen am 4. April 1935.)

ZUM PICARDSCHEN SATZ.

Von

Harald Bohr in Kopenhagen.

Zu den wichtigsten Errungenschaften der neueren Theorie der analytischen Funktionen gehören die beiden folgenden Sätze von Picard:

1. Picardscher Satz. *Jede ganze transzendente Funktion nimmt sämtliche Werte mit höchstens einer einzigen Ausnahme an.*

2. Picardscher Satz. *In der Umgebung einer isolierten wesentlich singulären Stelle nimmt eine analytische Funktion sämtliche Werte mit höchstens einer einzigen Ausnahme an.*

Ausführlicher und für das folgende bequemer formuliert lässt sich der 2. Satz auch folgendermassen aussprechen, wobei wir uns die betrachtete singuläre Stelle im Punkte $z = \infty$ denken:

Es sei die Funktion $f(z)$ in der Umgebung $R < |z| < \infty$ des Punktes $z = \infty$ eindeutig, regulär und verschieden von a und b (wobei die Zahlen a und b natürlich von einander verschieden sind). Dann ist die Funktion $f(z)$ im Punkte $z = \infty$ regulär oder besitzt dort einen Pol, oder, um die beiden Fälle zusammenzufassen, es gibt eine positive ganze Zahl N_0, so dass $f(z) z^{-N_0}$ im Punkte $z = \infty$ regulär ist; oder, was auf dasselbe hinausläuft: es gibt eine positive ganze Zahl N, so dass bei dem Grenzübergange $z \to \infty$ eine Ungleichung der Form

$$|f(z)| < |z|^N$$

stattfindet.

Die Picardschen Beweise dieser beiden Sätze wurden bekanntlich durch Heranziehung der Modulfunktion geführt. Während der Beweis des 1. Satzes äusserst einfach verlief, forderte der Beweis des zweiten, den ersten umfassenden Satzes kompliziertere Ueberlegungen. Es war daher in methodischer Hinsicht ein wesentlicher Fortschritt, als es Lindelöf durch eine sinnreiche Ueberlegung zu zeigen gelang, dass dieser 2. Picardsche Satz fast unmittelbar aus einem bekannten Schottkyschen Satze abgeleitet werden kann — welcher seinerseits in naher Beziehung zu der berühmten Landauschen Verallgemeinerung des 1. Picardschen Satzes steht und wie dieser sehr einfach mittels der Modulfunktion bewiesen werden kann. Dieser Satz lautet:

Schottkyscher Satz. *Es sei $f(z)$ im Einheitskreise $|z| < 1$ regulär und verschieden von a und b, und ϑ eine positive Zahl zwischen 0 und 1. Dann gibt es eine nur von den vier Zahlen $f(0)$, a, b und ϑ abhängige positive Konstante Ω derart, dass*

$$|f(z)| < \Omega \quad \textit{für} \quad |z| \leqq \vartheta.$$

E 14. *Сборник Посвященный Памяти Д. А. Граве* 1940.
Also in *Mat. Tidsskr. B* 1940.

Bei der Lindelöfschen Verwendung dieses Satzes zum Beweis des 2. Picardschen Satzes war übrigens eine völlig triviale Abschätzung der Konstanten Ω nötig, nämlich dass Ω unabhängig von der Grösse $f(0)$ gewählt werden kann, falls $f(0)$ in „sicherer Entfernung" von a, b und ∞ bleibt.

In der Abhandlung „Zur Theorie der fastperiodischen Funktionen III" (Acta Mathematica, Bd. 47), in welcher die Theorie der analytischen fastperiodischen Funktionen einer komplexen Veränderlichen $s = \sigma + it$ entwickelt wurde, zeigte der Verfasser, dass der 2. Picardsche Satz auf fastperiodische Singularititen verallgemeinert werden kann. Hierzu mussten aber ziemlich weitgehende, von Iversen geschaffene Hilfsmittel aus der Theorie der allgemeinen analytischen Funktionen herangezogen werden, weil die oben erwähnte Lindelöfsche Methode nicht auf den fastperiodischen Fall übertragen werden konnte.

Neuerdings hat aber der Verfasser eine Variante des Lindelöfschen Beweises gefunden, die — im Gegensatz zu dem ursprünglichen Lindelöfschen Beweise — nicht nur auf fastperiodische Funktionen angewendet werden kann, sondern zugleich eine Verallgemeinerung des 2. Picardschen Satzes mehr prinzipieller Art liefert. Diese Verallgemeinerung darzustellen, ist das Ziel der vorliegenden kleinen Abhandlung; ihre Beziehung zur Theorie der fastperiodischen Funktionen werde ich an anderer Stelle erörtern.

Für die neue Formulierung des Lindelöfschen Beweises des 2. Picardschen Satzes — oder vielmehr der erwähnten Verallgemeinerung des letzteren — reicht allerdings der Schottkysche Satz in der obigen Formulierung nicht aus, sondern es muss eine recht genaue Abschätzung der darin vorkommenden Konstanten $\Omega = \Omega(a, b, f(0), \vartheta)$ herangezogen werden. Eine solche Abschätzung ist aber schon bekannt und übrigens ebenfalls mit Hilfe der Modulfunktionen sehr leicht abzuleiten. In einer gemeinsamen Arbeit von Landau und dem Verfasser „Ueber das Verhalten von $\zeta(s)$ und $\zeta_\varkappa(s)$ in der Nähe der Geraden $\sigma = 1$" (Göttinger Nachrichten 1910) findet sich nämlich in § 6 (welcher Paragraph an frühere Untersuchungen von Landau über den Picardschen Satz anknüpft und, wie in der Einleitung erwähnt, von ihm allein herrührt), der folgende Satz:

Schottky-Landauscher Satz. *Es seien a und b zwei unter einander verschiedene komplexe Zahlen. Dann gibt es eine Konstante D, die nur von a und b abhängt, mit der folgenden Eigenschaft: für jede im Einheitskreise $|z| < 1$ reguläre und von a und b verschiedene Funktion $f(z)$ gilt in dem kleineren Kreise $|z| \leqslant \vartheta$ $(0 < \vartheta < 1)$ die Abschätzung*

$$|f(z)| < e^{\frac{D \ln (|f(0)| + 2)}{1-\vartheta}}. \tag{1}$$

Bei der Anwendung dieser Ungleichung (1) auf die ζ-Funktion in der oben zitierten Arbeit war die Art der Abhängigkeit der rechten Seite von der Grösse $f(0)$ von besonderer Bedeutung, während die Art der Abhängigkeit von ϑ für unseren damaligen Zweck unwesentlich war, weil bei der Anwendung von (1) ein fester Wert von ϑ $\left(\text{z. B. } \vartheta = \frac{1}{2}\right)$ zugrunde gelegt wurde. Für den jetzigen Zweck (den Beweis der Verallgemeinerung des 2. Picardschen Satzes) liegt die Sache gerade umgekehrt, d. h. es ist die Art ausschlaggebend, in welcher die Grösse ϑ in der rechten Seite von (1)

auftritt, während die Abhängigkeit von $f(0)$ nur grob ausgenutzt wird. Wir betonen dies, indem wir das folgende Corollar des Schottky-Landauschen Satzes, welches gerade soviel von diesem Satze enthält, wie wir für das folgende gebrauchen, explicite als ein Lemma formulieren:

Lemma. *Es seien a und b zwei unter einander verschiedene komplexe Zahlen und k eine positive Grösse. Dann gibt es eine positive Konstante $K = K(a, b, k)$ derart, dass jede im Kreise $|z| < 1$ reguläre und von a und b verschiedene sowie die Ungleichung $|f(0)| < k$ befriedigende Funktion $f(z)$ im ganzen Kreise $|z| \leq \vartheta$ $(0 < \vartheta < 1)$ der Ungleichung*

$$|f(z)| < e^{\frac{K}{1-\vartheta}} \tag{2}$$

genügt.

Aus diesem Lemma leiten wir zunächst mittels einer einfachen konformen Abbildung den folgenden Satz ab, welcher insofern als der Hauptsatz dieser kleinen Abhandlung betrachtet werden kann, als sich aus ihm die erwähnte Verallgemeinerung des 2. Picardschen Satzes in wenigen Worten ergeben wird:

Satz. *Es sei $F(s) = F(\sigma + it)$ eine in der Halbebene $\sigma > 0$ reguläre und von a und b verschiedene Funktion, welche auf der ganzen Geraden $\sigma = 1$ dem Betrage nach kleiner als k ist. Dann gilt in der ganzen Halbebene $\sigma > 1$ die Ungleichung*

$$|F(s)| < e^{K\sigma}, \tag{3}$$

wo K eine nur von a, b, k abhängige positive Konstante bezeichnet, als welche übrigens jede im Sinne des vorhergehenden Lemmas verwendbare Konstante K benutzt werden kann.

Bemerkung. Uebrigens gilt im Streifen $0 < \sigma < 1$ die Ungleichung $|f(s)| < e^{K/\sigma}$. Diese Ungleichung ist völlig analog der für $\sigma > 1$ gültigen Ungleichung (3) abzuleiten, wird aber für unseren vorliegenden Zweck nicht benötigt.

Beweis. Es genügt natürlich zu zeigen, dass für jede für $\sigma > 0$ reguläre und von a und b verschiedene Funktion $G(s)$, welche im Punkte $s = 1$ der Ungleichung $|G(1)| < k$ genügt, in jedem Punkte $s = \sigma$ der reellen Strecke $1 < \sigma < \infty$ die Ungleichung

$$|G(\sigma)| < e^{K\sigma} \tag{4}$$

gilt; denn die gewünschte Ungleichung (3) des Satzes, d. h. die Ungleichung

$$|F(\sigma + it_0)| < e^{K\sigma}$$

für einen beliebigen festen Punkt $\sigma + it_0$ der Halbebene $\sigma > 1$ folgt ja sofort aus dieser Ungleichung (4), wenn sie auf die Funktion $G(s) = F(s + it_0)$ angewendet wird.

Um die Ungleichung (4) darzutun, bilden wir die Halbebene $\sigma > 0$ der s-Ebene auf den Einheitskreis $|z| < 1$ der z-Ebene konform ab und zwar so, dass der Punkt $s = 1$ dem Punkte $z = 0$ und der Randpunkt $s = \infty$ dem Randpunkte $z = 1$ entspricht, d. h. wir setzen

$$z = \frac{s-1}{s+1}, \quad s = \frac{z+1}{1-z}.$$

Hierdurch geht $G(s)$ in die für $|z|<1$ reguläre und von a und b verschiedene Funktion

$$f(z)=G\left(\frac{z+1}{1-z}\right)$$

über, für welche $f(0)=G(1)$, also $|f(0)|<k$ ist. Dem reellen Punkte $s=\sigma>1$ entspricht bei dieser Abbildung der Punkt $z=\vartheta=\frac{\sigma-1}{\sigma+1}$ der Strecke $0<z<1$. Wegen

$$1-\vartheta=1-\frac{\sigma-1}{\sigma+1}=\frac{2}{\sigma+1}>\frac{2}{2\sigma}=\frac{1}{\sigma}$$

folgt aus dem obigen Lemma — indem dieses speziell auf den positiven Randpunkt $z=\vartheta$ des Kreises $|z|\leqslant\vartheta$ angewendet wird — die Ungleichung

$$|G(\sigma)|=|f(\vartheta)|<e^{\frac{K}{1/\sigma}}=e^{K\sigma},$$

womit der Satz bewiesen ist.

Nunmehr formuliere ich die genannte

Verallgemeinerung des 2. Picardschen Satzes. *Es sei S die Riemannsche Fläche des Logarithmus, d. h. die unendlichblättrige Fläche der Veränderlichen $z=re^{iv}$; $0<r<\infty$, $-\infty<v<\infty$. In einer unendlichblättrigen Umgebung $R<|z|<\infty$ des Windungspunktes $z=\infty$ sei die Funktion $f(z)$ regulär und verschieden von a und b. Ferner gebe es ein $\rho>R$, so dass $f(z)$ auf der (unendlich oft durchlaufenen) Kreislinie $r=\rho$, $-\infty<v<\infty$ beschränkt bleibt, etwa*

$$|f(z)|<k.$$

Dann gilt gleichmässig in allen Blättern bei dem Grenzübergang $z\to\infty$ eine Ungleichung der Form

$$|f(z)|<|z|^K,$$

wo K eine von R, ρ, a, b und k abhängige Konstante ist.

Bemerkung. Es ist klar, dass hierin der 2. Picardsche Satz enthalten ist, nämlich dem speziellen Falle entsprechend, wo $f(z)$ in der schlichten Umgebung $R<|z|<\infty$ des Punktes $z=\infty$ regulär ist, oder vielmehr, wo $f(z)$, auf der Riemannschen Fläche S betrachtet, periodisch mit der Periode 2π in Bezug auf die Amplitude v ist. Denn in diesem Falle ist ja von selbst bei jedem $\rho>R$ die Funktion $f(z)$ für $|z|=\rho$ beschränkt.

Beweis. Ohne Beschränkung der Allgemeinheit darf offenbar beim Beweise $R=1$, $\rho=e$ angenommen werden; sonst ersetze man nur z durch $Rz^{\ln\frac{\rho}{R}}$. Wir setzen $z=e^s$ und betrachten die Funktion

$$F(s)=f(e^s),$$

welche in der Halbebene $\sigma>0$ regulär und verschieden von a und b ist und auf der ganzen Geraden $\sigma=1$ (welche dem unendlich oft durchlaufenen Kreise $|z|=e$ entspricht) der Ungleichung

$$|F(s)|<k$$

genügt. Dann gilt nach dem obigen Satze in der ganzen Halbebene $\sigma > 1$ eine Ungleichung der Form

$$|F(s)| < e^{K\sigma},$$

d. h., wenn wir zur z-Ebene zurückkehren: es gilt im unendlichblättrigen Gebiete $|z| > e$ die Ungleichung

$$|f(z)| < |z|^K,$$

womit der Satz bewiesen ist.

Om Potensrækker med Huller.

En Pseudo-Kontinuitetsegenskab.

Af Harald Bohr.

1. Lad $\sum a_n z^n$ være en Potensrække i den Variable $z = re^{i\Theta}$ med en endelig og fra Nul forskellig Konvergensradius ϱ; uden at indskrænke Betragtningernes Almindelighed kan vi antage $\varrho = 1$. Den ved Rækken indenfor Enhedscirklen $|z| < 1$ fremstillede analytiske Funktion vil vi betegne med $f(z)$. Man taler om, at der er „Huller" i Potensrækken, hvis talrige af dens Led $a_n z^n$ mangler, derved at de tilsvarende Koefficienter a_n er Nul. Ved Undersøgelser over Potensrækker med Huller er det ofte bekvemt kun at opskrive de Led $a_n z^n$, for hvilke $a_n \neq 0$; vi skriver da (idet vi af Bekvemmelighedsgrunde vil tænke os det konstante Led $a_0 = 0$)

$$f(z) = \sum_{n=1}^{\infty} b_n z^{m_n} \quad (0 < m_1 < m_2 \cdots).$$

2. Klassiske velkendte Eksempler paa Potensrækker $\sum b_n z^{m_n}$ med Huller er Rækker som

$$\sum_{n=1}^{\infty} z^{q^n} \ (q \text{ hel} > 1) \quad \text{og} \quad \sum_{n=1}^{\infty} z^{n!}.$$

Enhver af disse har, som man næsten umiddelbart ser, Konvergenscirklen $|z| = 1$ til naturlig Grænse, d. v. s. alle Punkter paa $|z| = 1$ er singulære Punkter for den ved Rækken fremstillede Funktion $f(z)$. For at indse dette er det nok at vise, at der ligger singulære Punkter overalt tæt paa $|z| = 1$. Og for den sidste Række er aabenbart alle Punkter af Formen $e^{2\pi i r}$ (r rational), for den første alle Punkter af Formen $e^{\frac{2\pi i p}{q^m}}$ singulære, idet det jo i ethvert af de nævnte Punkter z_0 gælder, at alle Potensrækkens Led fra et vist Trin af vil være positive og $= 1$, og $|f(z)|$ derfor vil $\to \infty$ for $z \to z_0$ langs Radiusvektor.

Ogsaa Rækker som

$$\sum_{n=1}^{\infty} \frac{1}{n^2} z^{n!} \quad \text{og} \quad \sum_{n=1}^{\infty} \frac{1}{n^2} z^{q^n}$$

vil, skønt de er absolut konvergente paa Konvergenscirklen $|z| = 1$ (og derfor fremstiller Funktioner, der er kontinuerte i den lukkede Cirkelskive $|z| \leqq 1$), have denne Cirkel til naturlig Grænse. Thi ogsaa her ses de ovenfor nævnte Punkter $e^{2\pi i r}$ henholdsvis $e^{\frac{2\pi i p}{q^m}}$ alle at være singulære, i Kraft af en simpel og vigtig Sætning af *Vivanti* om, at et Punkt z_0 paa Konvergenscirklen for en Potensrække $f(z) = \sum a_n z^n$ altid (hvad enten $\sum a_n z_0{}^n$ er divergent eller konvergent) er et singulært Punkt for $f(z)$, naar alle Leddene $a_n z_0{}^n$ fra et vist Trin af er positive.

Som opdaget af *Hadamard* drejer det sig imidlertid ikke her om en særlig Egenskab ved de anførte og lignende Rækker — hvor Forholdene ligger særlig simpelt, fordi hver Eksponent er et Multiplum af den foregaaende — men om en almen Egenskab ved alle Potensrækker med tilstrækkelig store Huller. Den „*Hadamard*'ske Hulsætning" udsiger:

Er der Huller mellem to hvilkesomhelst optrædende Led i Potensrækken $f(z) = \sum b_n z^{m_n}$, og er disse Huller saa store, at

$$\frac{m_{n+1}}{m_n} > k > 1 \qquad \text{for alle } n,$$

da har Funktionen $f(z)$ altid Konvergenscirklen $|z| = 1$ til naturlig Grænse.

I det Tilfælde, hvor Hullerne antages saa store, at

$$\frac{m_{n+1}}{m_n} > k > 3 \qquad \text{for alle } n,$$

kan Beviset for den *Hadamard*ske Hulsætning, som jeg for en Række Aar siden har vist her i Tidsskriftet (Aarg. 1919), føres paa en særlig anskuelig Maade — i nær Tilslutning til det ovenfor anførte Bevis for de specielle Rækker med Eksponenter q^n og $n!$ — idet man baserer det paa en af *Dienes* given (næsten umiddelbar) Generalisation af *Vivanti*'s Sætning, som udsiger, at hvis det om en Potensrække med Konvergensradius 1 gælder, at i et Punkt z_0 paa Konvergenscirklen er alle Leddene $a_n z_0{}^n$ fra et vist Trin af (omend ikke helt ensrettede, saa dog) beliggende i et fast Vinkelrum $< \pi$ med Toppunkt i 0, da vil dette Punkt z_0 være et singulært Punkt for $f(z)$. Man kan nemlig

umiddelbart vise, at hvis $\frac{m_{n+1}}{m_n} > k > 3$, vil den Vinkelhastighed, hvormed de enkelte Potenser z^{m_n} bevæger sig paa Enhedscirklen, naar z selv bevæger sig paa $|z| = 1$ med konstant Hastighed, vokse saa hurtigt med n, at der vil eksistere en paa Enhedscirklen overalt tæt liggende Punktmængde, i hvis Punkter z_0 det vil gælde, at alle Leddene $b_n z_0^{m_n}$ fra et vist Trin af vil ligge i et fast Vinkelrum $< \pi$ med Toppunkt i 0; disse Punkter vil derfor alle være singulære Punkter for $f(z)$, og Cirklen $|z| = 1$ altsaa naturlig Grænse for $f(z)$.

3. Ved en Undersøgelse af næstenperiodiske analytiske Funktioners Forhold i den umiddelbare Nærhed af Grænsen for Næstenperiodicitetens Ophør førtes jeg til et Problem vedrørende Potensrækker $\Sigma a_n z^n$ med Konvergensradius 1, som fremstiller Funktioner $f(z)$, der ikke er begrænsede i $|z| < 1$. Det drejer sig om et Spørgsmaal vedrørende den Ændring, Funktionen $f(z)$ undergaar, naar z-Planen drejes en Vinkel τ, altsaa vedrørende Differensen

$$f(z e^{i\tau}) - f(z) = \sum_0^\infty a_n z^n (e^{in\tau} - 1).$$

Herved vil jeg, for i det følgende at kunne benytte en bekvem Formulering, betegne et Tal τ i Intervallet $0 < \theta < 2\pi$ som et Drejningstal for $f(z)$, hvis Differensen $f(ze^{i\tau}) - f(z)$ er begrænset i hele Enhedscirklen $|z| < 1$; er

$$|f(ze^{i\tau}) - f(z)| \leqq k \qquad \text{for } |z| < 1,$$

vil jeg, nærmere præciseret, betegne τ som et til k hørende Drejningstal for $f(z)$. Vi bemærker, at samtidig med τ er naturligvis ogsaa $2\pi - \tau$ et til k hørende Drejningstal for $f(z)$.

Eksempler. Kvotientrækken $\frac{1}{1-z} = \sum_0^\infty z^n$ med den ene Pol $z = 1$ besidder øjensynlig ikke noget Drejningstal. En Potensrække som

$$\frac{1}{1-z} + \frac{1}{1+z} + z = z + 2\sum_0^\infty z^{2n}$$

med de to Poler $z = 1$ og $z = -1$ har det ene Drejningstal

$\tau = \pi$, forøvrigt hørende til $k = 2$ (og dermed ogsaa til ethvert Tal $k > 2$). Som et sidste Eksempel nævner vi, at en i $|z| < 1$ ubegrænset analytisk Funktion $f(z) = \sum a_n z^n$ specielt kan have Drejningstal τ hørende til $k = 0$, saakaldte „Drejningsperioder" τ, for hvilke $f(ze^{i\tau}) = f(z)$ i hele $|z| < 1$. En saadan Drejningsperiode τ maa dog, som man let overbeviser sig om, altid være et rationelt Multiplum af 2π, og er $\frac{p}{q} \cdot 2\pi$ (p primisk med q) en Drejningsperiode, vil ogsaa $\frac{1}{q} \cdot 2\pi$ være det, idet vi jo kan finde et Multiplum af p, som ved Division med q giver Resten 1. Vi tilføjer, at hvis $f(z)$ har en saadan Drejningsperiode $\tau = \frac{2\pi}{q}$, giver denne sig „aabenlyst" til Kende derved, at Potensrækken kun indeholder Led $a_n z^n$, hvis Eksponenter alle er Multipla af q. En Funktion $f(z)$ kan derfor kun have et endeligt Antal Drejningsperioder, og disse er alle Multipla af en af dem.

4. Det omhandlede Problem er:

Findes der en Potensrække $\sum a_n z^n$ med Konvergensradius 1 og fremstillende en for $|z| < 1$ ubegrænset Funktion $f(z)$, som for ethvert $\varepsilon > 0$ besidder vilkaarlig smaa Drejningstal hørende til ε?

Spørgsmaalet kan aabenbart ogsaa formuleres saaledes: Findes der en i $|z| < 1$ analytisk ubegrænset Funktion $f(z)$, som i den Forstand er pseudo kontinuert (eller udførligere „pseudo ligelig kontinuert ved Drejning"), at der eksisterer en mod Nul aftagende Følge af positive Tal τ_n saaledes, at der ligelig i hele Cirklen $|z| < 1$ gælder Grænseligningen

$$\lim_{n \to \infty} f(ze^{i\tau_n}) = f(z).$$

Bemærkning. Vi anfører straks til Orientering, at dersom $f(z)$ overhovedet har Drejningstal hørende til vilkaarlig smaa ε, maa disse Tal τ — naar vi ser bort fra det Tilfælde, hvor $f(z)$ besidder Drejningsperioder — nødvendigvis af sig selv ligge i den umiddelbare Nærhed af 0 (og 2π). Thi ellers kunde vi jo bestemme en Følge af positive Tal $\varepsilon_1, \varepsilon_2, \cdots$ med $\varepsilon_n \to 0$ og en tilsvarende Følge af Drejningstal $\tau_1, \tau_2, \cdots$, hvor τ_n hører til ε_n, saaledes at τ_n for $n \to \infty$ nærmede sig til en Grænseværdi $\tau^* \neq 0, 2\pi$; af den for ethvert z i $|z| < 1$ gældende Ulighed

$$|f(ze^{i\tau_n})-f(z)| \leqq \varepsilon_n$$

vilde da ved Grænseovergangen $n \to \infty$ følge, at $f(ze^{i\tau^*}) = f(z)$ i hele $|z| < 1$, i Strid med, at $f(z)$ ikke har nogen Drejningsperiode.

5. Det er klart, at de sædvanlige skikkelige Potensrækker, der fremstiller i $|z| < 1$ ubegrænsede Funktioner, ikke har den nævnte Egenskab. Og man indser da ogsaa umiddelbart, at det kun er Potensrækker, der har Konvergenscirklen $|z| = 1$ til naturlig Grænse, der har Mulighed for at besidde den omhandlede Pseudo-Kontinuitet. Thi er $f(z)$ ubegrænset indenfor $|z| < 1$, maa den have mindst et Randpunkt z_0 til „Uendelighedspunkt" i den Forstand, at $f(z)$ indenfor enhver nok saa lille Omegn af z_0 indenfor $|z| < 1$ antager numerisk vilkaarlig store Værdier, og har Funktionen vilkaarlig smaa Drejningstal τ (og dermed ogsaa Drejningstal, der ligger overalt tæt i Intervallet $0 < \theta < 2\pi$), maa den have Uendelighedspunkter overalt tæt paa $|z| = 1$ (hvoraf iøvrigt atter følger, at ethvert Punkt paa $|z| = 1$ er Uendelighedspunkt), idet en Drejning paa τ jo maa føre et Uendelighedspunkt over i et Uendelighedspunkt igen. Vi ledes derved naturligt til at betragte Potensrækker som f. Eks. netop de to tidligere anførte $\sum z^{q^n}$ og $\sum z^{n!}$, som har Konvergenscirklen $|z| = 1$ til naturlig Grænse og fremstiller for $|z| < 1$ ubegrænsede Funktioner. Som vi skal se, har den første af disse Rækker ikke (for noget q) den omhandlede Egenskab; derimod gælder dette for den anden Række, hvoraf altsaa specielt følger, at det opstillede Spørgsmaal maa besvares bekræftende.

Beviset for, at den førstnævnte Række

$$f(z) = \sum_{n=1}^{\infty} z^{q^n}$$

(der forøvrigt har Drejningsperioden $\frac{2\pi}{q}$ og, som man let ser, ethvert Tal τ af Formen $2\pi \cdot \frac{p}{q^m}$ til Drejningstal) ikke besidder den omhandlede Pseudokontinuitet, fører vi ved at vise, at der i Intervallet $0 < \theta < \frac{2\pi}{q+1}$ ikke findes noget Drejningstal

for $f(z)$ hørende til $\frac{4}{q+1}$. Hertil bemærker vi først, at hvis en Funktion $f(z) = \Sigma a_n z^n$ har Drejningstallet τ hørende til k, altsaa

$$|f(ze^{i\tau}) - f(z)| = |\Sigma a_n(e^{in\tau} - 1)z^n| \leqq k \quad \text{i} \quad |z| < 1,$$

maa i Følge *Cauchy*'s Uligheder Koefficienterne $a_n(e^{in\tau} - 1)$ $(n = 1, 2, \cdots)$ alle være numerisk $\leqq k$. For vor Række $f(z) = \Sigma z^{q^n}$ gælder det derfor, at en nødvendig Betingelse for, at τ er et Drejningstal hørende til $\frac{4}{q+1}$, er, at

$$|e^{i\tau q^n} - 1| \leqq \frac{4}{q+1} \quad \text{for } n = 1, 2, \cdots.$$

Disse uendelig mange Uligheder kan imidlertid ikke samtidig være opfyldt for noget τ i $0 < \theta < \frac{2\pi}{q+1}$. Thi skal Tallet $|e^{it} - 1|$, d. v. s. Korden, der forbinder Punkterne 1 og e^{it} paa Enhedscirklen, være $\leqq \frac{4}{q+1} (< 2)$, maa jo, $\left(\text{idet } \frac{\text{Bue}}{\text{Korde}} < \frac{\pi}{2}\right)$

$$|t| < \frac{2\pi}{q+1} \quad (\text{mod. } 2\pi),$$

hvorved menes, at t maa afvige mindre end $\frac{2\pi}{q+1}$ fra et helt Multiplum af 2π. Dette kan imidlertid umuligt være Tilfældet for samtlige Tal $t = \tau q, \tau q^2, \tau q^3, \cdots$. Thi idet

$$q \cdot \frac{2\pi}{q+1} = 2\pi - \frac{2\pi}{q+1},$$

og τ ligger i Intervallet $0 < \theta < \frac{2\pi}{q+1}$, er $q \cdot \tau < 2\pi - \frac{2\pi}{q+1}$, og skal $|q\tau| < \frac{2\pi}{q+1}$ (mod 2π), maa $q\tau$ derfor selv være mindre end $\frac{2\pi}{q+1}$ d. v. s. ligesom τ ligge i Intervallet $0 < \theta < \frac{2\pi}{q+1}$. Paa ordret samme Maade indses dernæst, at $q^2\tau$, $q^3\tau$, o. s. v. alle maa være $< \frac{2\pi}{q+1}$, i Strid med, at $q^n\tau \to \infty$ for $n \to \infty$.

Derimod har Rækken

$$f(z) = \sum_{n=1}^{\infty} z^{n!}$$

den ønskede Pseudokontinuitet. Vi kan nemlig vise, at $\tau_n = \frac{2\pi}{n!}$ er et Drejningstal for $f(z)$ hørende til $\frac{4\pi}{n}$. Idet $\frac{\nu!}{n!}$ er hel for $\nu \geqq n$, gælder jo i hele $|z| < 1$

$$f(ze^{i\tau_n}) - f(z) = \sum_{\nu=1}^{\infty} z^{\nu!}(e^{2\pi i \frac{\nu!}{n!}} - 1) = \sum_{\nu=1}^{n-1} z^{\nu!}(e^{2\pi i \frac{\nu!}{n!}} - 1).$$

Benytter vi her den for vilkaarligt reelt t gældende trivielle Ulighed $|e^{it} - 1| \leqq |t|$ (Korde < Bue), finder vi

$$|f(ze^{i\tau_n}) - f(z)| \leqq 2\pi \sum_{\nu=1}^{n-1} \frac{\nu!}{n!},$$

hvoraf, idet

$$\sum_{\nu=1}^{n-1} \frac{\nu!}{n!} = \frac{1}{n} + \frac{1}{n(n-1)} + \cdots + \frac{1}{n(n-1)\cdots 2} < \frac{1}{n}\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-2}}\right) < \frac{2}{n},$$

den ønskede Ulighed

$$|f(ze^{i\tau_n}) - f(z)| \leqq \frac{4\pi}{n}.$$

6. Det viste sig nu — og at paavise dette, er det egentlige Formaal i den foreliggende Afhandling — at den Tankegang, der laa til Grund for mit ovenfor antydede Bevis for den *Hadamard*'ske Hulsætning, kunde anvendes til at bevise, at endog enhver Potensrække $\sum b_n z^{m_n}$ med tilstrækkelig store Huller, d. v. s. for hvilken Eksponentfølgen $m_1, m_2, \cdots$ vokser tilstrækkelig hurtigt, besidder den omhandlede Pseudokontinuitet. Hertil maa vi dog forlange „meget store“ Huller (Eksponentfølgen q^n frembød jo ikke tilstrækkelig store Huller, og naar Beviset fungerede for Eksponentfølgen $n!$, skyldtes det i Virkeligheden kun, at Eksponenterne her besad den tidligere fremhævede særlige Egenskab, at hver af dem er et Multiplum af den foregaaende).

Jeg skal dog ikke præcisere, eller bevise, nogen herom gældende helt almindelig Sætning, men skal for at faa det væsentlige frem — idet det mere er Eksponenterne end Koefficienterne,

der interesserer os — nøjes med at bevise følgende Sætning om Potensrækker, der er Delrækker i Kvotientrækken $\sum z^n$, d. v. s. hvis Koefficienter a_n alle er 1 eller 0; vi bemærker straks, at enhver uendelig Delrække i Kvotientrækken naturligvis har Konvergensradius 1, samt at den fremstiller en i $|z| < 1$ ubegrænset Funktion $f(z)$, idet jo $f(x) \to \infty$ for $x \to 1$ fra venstre.

Sætning. Vokser Følgen af positive hele Tal $m_1 < m_2 < m_3 \cdots$ saa hurtigt, at

$$\sum_{n=1}^{\infty} \frac{m_n}{m_{n+1}}$$

er konvergent [Eks. $m_n = (n!)^2$], da vil det om den ved Potensrækken

$$\sum_{n=1}^{\infty} z^{m_n}$$

for $|z| < 1$ fremstillede ubegrænsede Funktion $f(z)$ gælde, at der til ethvert $\varepsilon > 0$ paa enhver selv nok saa lille Bue $0 < \theta < d$ af Enhedscirklen findes en Argumentværdi τ, saaledes at τ er et Drejningstal for $f(z)$ hørende til ε, altsaa saaledes at

$$|f(ze^{i\tau}) - f(z)| \leqq \varepsilon \qquad \text{i hele } |z| < 1.$$

7. Inden vi paabegynder selve Beviset, vil vi forudskikke nogle simple forberedende Bemærkninger.

Bemærkning 1. Det vil ved Beviset være bekvemt at antage, at $m_1 > 3$, samt at Uligheden

$$\frac{m_n}{m_{n+1}} < \frac{1}{2},$$

der jo (fordi $\frac{m_n}{m_{n+1}}$ som det n-te Led i en konvergent Række $\to 0$ for $n \to \infty$) sikkert gælder for alle n fra et vist Trin af, gælder for alle $n = 1, 2, 3, \cdots$. Disse Antagelser kan vi naturligvis gøre, idet vi ellers blot bortkaster et endeligt Antal Begyndelsesled i den forelagte Række $\sum z^{m_n}$, hvorved $f(z)$ jo kun ændres med en for $|z| \leqq 1$ kontinuert Funktion.

Vi tilføjer, at af $\frac{m_n}{m_{n+1}} < \frac{1}{2}$ for alle n følger, at

$$\sum_{\nu=1}^{n} m_\nu < 2m_n \qquad \text{for alle } n;$$

vi har jo

$$\sum_{\nu=1}^{n} m_\nu = m_n + m_n \frac{m_{n-1}}{m_n} + m_n \frac{m_{n-1}}{m_n} \cdot \frac{m_{n-2}}{m_{n-1}} + \cdots + m_n \frac{m_{n-1}}{m_n} \cdots \frac{m_1}{m_2}$$

$$< m_n\left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}\right) < 2\,m_n.$$

Bemærkning 2. For ethvert reelt t gælder, som allerede tidligere nævnt, Uligheden

$$|e^{it} - 1| \leqq |t|.$$

Denne Ulighed vil man dog kun finde paa at anvende, naar $|t| \leqq \pi$. Gør vi ingen saadan indskrænkende Forudsætning om t, vil det være naturligt at anvende følgende Ulighed, der umiddelbart fremgaar af den anførte (idet venstre Side jo er periodisk med Perioden 2π): Er $0 < \delta < \pi$, og er

$$|t| \leqq \delta \pmod{2\pi},$$

da er

$$|e^{it} - 1| \leqq \delta.$$

Bemærkning 3. For et vilkaarligt positivt helt Tal m og et vilkaarligt Tal δ i $0 < \delta < \pi$ vil de Punkter $z = e^{i\Theta}$ paa Enhedscirklen, for hvilke

$$|m\,\Theta| \leqq \delta \pmod{2\pi},$$

udfylde m lige store Buer af Længde $\frac{2\delta}{m}$, som ved en Drejning paa $\frac{2\pi}{m}$ gaar over i hinanden. Specielt gælder det, at enhver Bue af en Længde $> \frac{2\pi}{m} + \frac{2\delta}{m}$, altsaa yderligere enhver Bue af Længden $\frac{4\pi}{m}$, i sit Indre helt vil indeholde en af de nævnte m Buer af Længden $\frac{2\delta}{m}$.

Vi tilføjer, at i ethvert Punkt $z = e^{i\Theta}$ paa en af disse m Buer vil der, i Følge Bem. 2, gælde Uligheden

$$|e^{im\Theta} - 1| \leqq \delta.$$

8. Og nu til Beviset for den opstillede Sætning. Vi sætter til Afkortning

$$\delta_n = 2\pi \frac{m_n}{m_{n+1}} \qquad (n = 1, 2, \cdots);$$

herved er $\sum \delta_n$ konvergent, og hvert δ_n er $< \pi$ (idet jo $\frac{m_n}{m_{n+1}} < \frac{1}{2}$ i Følge Bem. 1).

Med I_n betegner vi ethvert af de m_n paa Enhedscirklen jævnt fordelte Intervaller af Længden $\frac{2\delta_n}{m_n}$, hvori $|m_n \theta| \leqq \delta_n \pmod{2\pi}$ og dermed tillige

$$|e^{i m_n \theta} - 1| \leqq \delta_n.$$

Idet $\frac{2\delta_n}{m_n} = \frac{4\pi}{m_{n+1}}$ vil (Bem. 3) ethvert Interval I_n i sit Indre indeholde et Interval I_{n+1}.

Endvidere vil vi til Afkortning sætte

$$\eta_n = \frac{4\pi}{m_{n+1}} \qquad (n = 1, 2, \cdots);$$

herved er $\eta_n < \pi$ for alle n (idet jo $m_1 > 3$). I Følge Bem. 3 vil Bueintervallet $0 < \theta < \eta_n$ i sit Indre indeholde et Interval I_{n+1}. I Følge Bem. 1 gælder Uligheden

$$\eta_n \cdot \sum_{\nu=1}^{n} m_\nu < 2\eta_n m_n = 8\pi \frac{m_n}{m_{n+1}} = 4\delta_n,$$

hvorved det afgørende er, at der paa højre Side staar et Tal, der gaar mod 0 (og ikke, at dette Tal netop er $4\delta_n$), altsaa at η_n gaar saa hurtigt mod 0 for $n \to \infty$, at endog $\eta_n \cdot \sum_{\nu=1}^{n} m_\nu$ vil $\to 0$.

Opgaven er at vise, at der til det vilkaarligt opgivne positive $d < \pi$ og det vilkaarligt givne ε findes et Tal τ i Intervallet $0 < \theta < d$, som opfylder Betingelsen

$$|f(ze^{i\tau}) - f(z)| \leqq \varepsilon \qquad \text{i hele } |z| < 1.$$

Vi vælger hertil n saa stor, at $\eta_n < d$, og at

$$4\delta_n + \sum_{\nu=n+1}^{\infty} \delta_\nu < \varepsilon.$$

I det Indre af Intervallet $0 < \theta < \eta_n$ bestemmer vi et Interval I_{n+1}, deri atter et Interval I_{n+2}, deri atter et Interval I_{n+3}, o. s. v. Lad τ $(0 < \tau < \eta_n)$ betegne det Punkt, hvorom disse indeni hinanden liggende Intervaller trækker sig sammen. Dette Tal τ, som ligger i det givne Interval $0 < \theta < d$, vil da, som vi skal se, opfylde vor Betingelse. Vi har

$$|f(ze^{i\tau}) - f(z)| = \left|\sum_{\nu=1}^{\infty} z^{m_\nu}(e^{im_\nu\tau} - 1)\right| \leqq \sum_{\nu=1}^{\infty} |e^{im_\nu\tau} - 1|$$
$$= \sum_{\nu=1}^{n} |e^{im_\nu\tau} - 1| + \sum_{\nu=n+1}^{\infty} |e^{im_\nu\tau} - 1|.$$

Paa hvert Led i den første Sum anvender vi simpelthen Uligheden $|e^{it} - 1| < t$ for $t > 0$, altsaa

$$|e^{im_\nu\tau} - 1| < m_\nu\tau \qquad (\nu = 1, 2, \cdots, n),$$

og for hvert Led i den anden Sum gælder det, at

$$|e^{im_\nu\tau} - 1| < \delta_\nu \qquad (\nu = n+1,\ n+2, \cdots),$$

fordi τ jo for ethvert $\nu > n$ ligger i et Interval I_ν. Vi finder derved sluttelig

$$|f(ze^{i\tau}) - f(z)| < \tau \cdot \sum_{\nu=1}^{n} m_\nu + \sum_{\nu=n+1}^{\infty} \delta_\nu < \eta_n \cdot \sum_{\nu=1}^{n} m_\nu + \sum_{\nu=n+1}^{\infty} \delta_\nu$$
$$< 4\,\delta_n + \sum_{\nu=n+1}^{\infty} \delta_\nu < \varepsilon.$$

Hermed er Sætningen bevist.

ADDITION OF CONVEX CURVES

OM ADDITION AF UENDELIG MANGE KONVEKSE KURVER

AF

HARALD BOHR

Indledning.

Ved en Undersøgelse over en vis almindelig Klasse af uendelige Rækker[1] førtes jeg til følgende Opgave: Lad der være givet uendelig mange positive Tal $\varsigma_1, \varsigma_2, \ldots, \varsigma_n, \ldots$ saaledes, at $\sum_{n=1}^{\infty} \varsigma_n$ er konvergent. Hvilke Værdier antager da Funktionen $F(\varphi_1, \varphi_2, \ldots, \varphi_n, \ldots) = \sum_{n=1}^{\infty} \varsigma_n e^{i\varphi_n}$, naar de reelle Tal $\varphi_1, \varphi_2, \ldots, \varphi_n, \ldots$ uafhængig af hinanden gennemløber alle Værdier fra $-\infty$ til $+\infty$, eller, hvad der øjensynlig kommer ud paa det samme, alle Værdier fra 0 (incl.) til 2π (excl.)?

Som man meget let viser, gælder her følgende Sætning:

1) Hvis der i Talfølgen $\varsigma_1, \varsigma_2, \ldots, \varsigma_n, \ldots$ ikke eksisterer noget Element ς_n, der er større end Summen af alle de andre, antager Funktionen F alle Værdier z, for hvilke $|z| \leq \sum_{n=1}^{\infty} \varsigma_n$, og ingen andre; altsaa enhver kompleks Værdi, hvis Billede i den komplekse Plan er beliggende indenfor eller paa Randen af en Cirkel med Centrum i Begyndelsespunktet og Radius $\sum_{n=1}^{\infty} \varsigma_n$.

2) Hvis der derimod i Talfølgen $\varsigma_1, \varsigma_2, \cdots, \varsigma_n, \ldots$ findes et Element ς_N, der er større end Summen af alle de andre, antager

[1] Lösung des absoluten Konvergenzproblems einer allgemeinen Klasse Dirichletscher Reihen (som snart vil fremkomme i Acta Matematica).

Funktionen F alle Værdier z, for hvilke $\varsigma_N - \sum_{n \neq N} \varsigma_n \leq |z| \leq \sum_{n=1}^{\infty} \varsigma_n$, og ingen andre; altsaa enhver kompleks Værdi, hvis Billede i den komplekse Plan er beliggende indenfor eller paa Randen af en Cirkelring med Centrum i Begyndelsespunktet, og hvis ydre og indre Radius er henholdsvis $\sum_{n=1}^{\infty} \varsigma_n$ og $\varsigma_N - \sum_{n \neq N} \varsigma_n$.

Idet det til Tallet $\varsigma_n e^{i\varphi_n}$ svarende Punkt i den komplekse Plan, naar φ_n varierer fra 0 til 2π, gennemløber en Cirkelperiferi med Centrum i Begyndelsespunktet og Radius ς_n, kan denne Sætning øjensynlig ogsaa (i ikke helt præcis Formulering) udsiges saaledes:

Den Punktmængde M i den komplekse Plan, der fremkommer ved Addition af uendelig mange Cirkler C_n ($n = 1, 2, 3, \ldots$) med Centrum i Begyndelsespunktet, og for hvilke Summen af Radierne danner en konvergent Række, (idet herved forstaas den Punktmængde M, der svarer til Mængden af alle komplekse Tal $z = \sum_{n=1}^{\infty} z_n$, hvor z_n er det til et vikaarligt Punkt paa den n^{te} Cirkel svarende komplekse Tal) danner enten det Indre af en Cirkel (incl. Rand) med Centrum i Begyndelsespunktet, eller det Indre af en Cirkelring (incl. Rand) omkring Begyndelsespunktet.

Ved en Undersøgelse over visse i den analytiske Primtalteori forekommende Funktioner førtes jeg til følgende almindeligere Problem, nemlig til at undersøge, hvilken Punktmængde der fremkommer ved en Addition som den ovenfor betragtede, ikke længer specielt af uendelig mange Cirkler med Centrum i Begyndelsespunktet, men almindeligere af uendelig mange vilkaarlige lukkede konvekse Kurver.

Svarende til den ovenfor omtalte specielle Sætning om Addition af uendelig mange Cirkler, beviste jeg i Almindelighed, at den Punktmængde i den komplekse Plan, der fremkommer ved Addition af uendelig mange lukkede konvekse Kurver, enten er det Indre (incl. Rand) af en lukket konveks

2

Kurve. eller et Omraade (incl. Rand) begrændset af to indenfor hinanden liggende lukkede konvekse Kurver.

Det er denne Sætning i nøjagtig Formulering og Beviset derfor, som jeg skal tillade mig at meddele i den foreliggende Afhandling.

Idet Sætningen i Virkeligheden er en ren geometrisk Sætning, har jeg foretrukket at føre Beviset i sin naturlige geometriske Form, fremfor gennem Indførelsen af komplekse Tal at iklæde Sætningen og Beviset en arithmetisk Skikkelse.

I § 1 meddeles nogle almindelige orienterende Bemærkninger om Addition af uendelig mange Punktmængder; § 2 omhandler Addition af to lukkede konvekse Kurver; endelig behandles i § 3 Addition af uendelig mange lukkede konvekse Kurver.

I en senere Afhandling skal jeg anvende Resultaterne af den foreliggende Undersøgelse paa de i den analytiske Primtaltheori forekommende Funktioner, specielt paa den Riemannske Zetafunktion.

§ 1. Nogle almindelige Bemærkninger om Addition af uendelig mange Punktmængder.

Lad i en Plan, med Begyndelsespunkt O, p_1 og p_2 være to givne Punkter; jeg vil da paa sædvanlig Maade ved Summen $p_1 + p_2$ forstaa det Punkt i Planen, der bestemmes som modstaaende Vinkelspids til O i det Parallelogram, hvor p_1 og p_2 er de to andre Vinkelspidser. Begrebet Sum af to Punkter udvides umiddelbart til Begrebet Sum af et vilkaarligt endeligt Antal Punkter. Lad $p_1, p_2, \ldots, p_n, \ldots$ være en given uendelig Følge af Punkter i Planen; den „uendelige Række“ $\sum_{n=1}^{\infty} p_n$ skal da siges at være konvergent og dens Sum at være Punktet p, hvis Punktet $q_N = \sum_{n=1}^{N} p_n$, naar N vokser ud over alle Grændser, nærmer sig til det endelige og bestemte Grændsepunkt p, d. v. s. hvis $\lim_{N=\infty} q_N = p$.

Lad M_1 med Elementer (Punkter) m_1 og M_2 med Elementer m_2 være to givne Punktmængder i Planen; jeg vil da ved Summen $M_1 + M_2$ betegne den Punktmængde, hvis Elementer er alle Punkter af Formen $m_1 + m_2$, med andre Ord: Punktmængden $M_1 + M_2$ indeholder ethvert Punkt, der kan dannes som Sum af et Element i M_1 og et Element i M_2, og ingen andre. Bestaar f. Ex. M_1 af alle Punkter paa det rette Liniestykke OA (Endepunkterne medregnede), og M_2 af alle Punkter paa det rette Liniestykke OB (Endepunkterne medregnede), bestaar $M_1 + M_2$ af alle Punkter indenfor og paa Begrændsningen af det Parallelogram, der har A og B til to modstaaende Vinkelspidser og en tredie Vinkelspids faldende i O. Bestaar f. Ex. M_1 af alle Punkter paa en Cirkelperiferi med Centrum i O og Radius ϱ_1, og M_2 af alle Punkter paa en Cirkelperiferi med Centrum i O og Radius $\varrho_2 \leq \varrho_1$, bestaar $M_1 + M_2$, som man meget let ser, af alle Punkter indenfor og paa Randen af en Cirkelring med Centrum i O, og hvis ydre og indre Radius er henholdsvis $\varrho_1 + \varrho_2$ og $\varrho_1 - \varrho_2$. o. s. fr.

Begrebet Sum af to Punktmængder udvides umiddelbart til Begrebet Sum af et vilkaarligt endeligt Antal Punktmængder; er $M_1, M_2, \ldots M_n$ Punktmængder, hvis Elementer henholdsvis betegnes med $m_1, m_2, \ldots, m_n$, bestaar Punktmængden $M_1 + M_2 + \ldots + M_n$ af alle Punkter $m_1 + m_2 + \ldots + m_n$ (hvert Punkt kun medregnet én Gang), og ingen andre.

Lad der være givet en uendelig Følge af Punktmængder, M_1 med Elementer m_1, M_2 med Elementer m_2, ..., M_n med Elementer m_n, ... Den uendelige Række $\sum_{n=1}^{\infty} M_n$ skal da siges at være konvergent, hvis enhver uendelig Række $\sum_{n=1}^{\infty} m_n$ (hvor m_n er et vilkaarligt Punkt i M_n) er konvergent, og $\sum_{n=1}^{\infty} M_n$ skal, hvis den er konvergent, siges at fremstille (have Summen) M, hvor M er den Punktmængde, hvis Elementer m er alle Punkter af Formen $\sum_{n=1}^{\infty} m_n$ (hvert Punkt kun medregnet én Gang), og ingen andre.

4

Er $\sum_{n=1}^{\infty} M_n$ konvergent, er det muligt til ethvert $\varepsilon > 0$ at finde et helt positivt Tal $N = N(\varepsilon)$, saaledes at Punktet $\sum_{n=N_1}^{N_1+p} m_n$, for $N_1 \geq N$ og $p \geq 0$, stedse (d. v. s. hvordan end m_n er udvalgt blandt Elementerne i M_n) ligger indenfor en Cirkel med Begyndelsespunktet O som Centrum og Radius ε (man kunde udtrykke dette ved at sige, at den uendelige Række $\sum_{n=1}^{\infty} m_n$, hvor m_n gennemløber Punktmængden M_n, er ligelig konvergent); thi i modsat Fald maatte der eksistere et bestemt Tal $e > 0$ med følgende Egenskab: der eksisterer en Følge af hele positive Tal $N_1 \leq N_1 + p_1 < N_2 \leq N_2 + p_2 < \ldots < N_r \leq N_r + p_r < \ldots$ og en dertil svarende Punktfølge $m'_1, m'_2, \ldots, m'_n, \ldots$ (hvor m'_n er Element i M_n) saaledes, at for alle $r = 1, 2, \ldots$ Punktet $\sum_{n=N_r}^{N_r+p_r} m'_n$ ligger udenfor eller paa Randen af Cirklen med O som Centrum og Radius e; men heraf vilde umiddelbart følge, at den uendelige Række $\sum_{n=1}^{\infty} m'_n$ ikke var konvergent, i Modstrid med Antagelsen om, at enhver uendelig Række $\sum_{n=1}^{\infty} m_n$ er konvergent.

Af denne Bemærkning følger specielt, at der, hvis $\sum_{n=1}^{\infty} M_n$ er konvergent, til ethvert $\varepsilon > 0$ eksisterer et helt Tal $N = N(\varepsilon)$ saaledes, at for $n \geq N$ Punktmængden M_n er beliggende helt indenfor en Cirkel med O som Centrum og Radius ε.

Endvidere følger let, at hvis $\sum_{n=1}^{\infty} M_n$ er konvergent, og enhver af Punktmængderne M_n ($n = 1, 2, \ldots$) er beliggende helt i det Endelige, d. v. s. indenfor en Cirkel med O som Centrum og Radius r_n (i Følge den foregaaende Bemærkning vil da alle Punktmængderne M_n ogsaa ligge indenfor en Cirkel med O som Centrum og fast (d. v. s. af n uafhængig) Radius) vil Punktmængden $M = \sum_{n=1}^{\infty} M_n$ ligeledes være beliggende helt i det Endelige.

Endelig kan bemærkes, at hvis $\sum_{n=1}^{\infty} M_n$ er konvergent og

5

M_n $(n = 1, 2, \ldots)$ beliggende helt i det Endelige, samt hvis endvidere enhver af Punktmængderne M_n $(n = 1, 2, \ldots)$ er afsluttet (d. v. s. indeholder sine Grændsepunkter), vil Punktmængden $M = \sum_{n=1}^{\infty} M_n$ ligeledes være en afsluttet Punktmængde. Lad nemlig p være et Grændsepunkt for Punktmængden M; da er p Grændsepunkt for en Punktfølge $m^{(q)} = \sum_{n=1}^{\infty} m_n^{(q)}$ $(q = 1, 2, \ldots)$, hvor $m_n^{(q)}$ er Element i M_n. Idet imidlertid alle Punkter $m_n^{(q)}$ er beliggende indenfor en Cirkel med Centrum i O og fast Radius, vil der i Følge en bekendt Sætning om Dobbeltfølger eksistere en saadan Følge af hele positive Tal $q_1, q_2, \ldots, q_r, \ldots$, at, for ethvert fast $n = 1, 2, \ldots$, $\lim_{r=\infty} m_n^{(q_r)}$ eksisterer og følgelig er lig et Punkt m_n i den afsluttede Punktmængde M_n; og her vil øjensynlig Punktet $m = \sum_{n=1}^{\infty} m_n$ (paa Grund af den ligelige Konvergens af $\sum_{n=1}^{\infty} m_n^{(q)}$) være Grændsepunkt for Punktfølgen $m^{(q_r)}$ $(r = 1, 2, \ldots)$, d. v. s. $\lim_{r=\infty} m^{(q_r)} = m$. Følgelig vil Punktet p være lig Punktet m og altsaa være Element i M. Hermed er den ovenstaaende Paastand bevist. Lader jeg enhver af Punktmængderne M_n, for hvilken $n > N$, indeholde kun det ene Punkt O, følger specielt, at Summen $\sum_{n=1}^{N} M_n$ af et endeligt Antal afsluttede og helt i det Endelige beliggende Punktmængder $M_1, M_2, \ldots, M_N$ vil være en afsluttet Punktmængde. [Fordringen om at Punktmængderne M_n skal være helt i det Endelige beliggende er væsentlig, d. v. s. Sætningen vil ikke bevare sin Gyldighed. hvis denne Betingelse udelades. Saaledes vil f. Ex., hvis a_1 og a_2 er to positive Tal, for hvilke $a_1 : a_2$ er et irrationalt Tal, og hvis M_1 og M_2 er to Punktmængder, hvis Elementer alle er beliggende paa en ret Linie L gennem Begyndelsespunktet O, saaledes at M_1 indeholder alle de Punkter m_1 paa L, for hvilke Afstanden Om_1 (regnet med Fortegn i Overensstemmelse med en valgt positiv Retning paa L) er lig na_1, hvor n er et

6

vilkaarligt helt Tal $\gtreqless 0$, medens M_2 indeholder alle de Punkter m_2 paa L, for hvilke Afstanden Om_2 (regnet med Fortegn) er lig na_2, hvor n er et helt Tal $\gtreqless 0$, enhver af Punktmængderne M_1 og M_2 være en afsluttet Punktmængde, medens $M_1 + M_2$ vil være en Punktmængde, der ligger overalt tæt paa Linien L uden at indeholde noget Kontinuum, altsaa en ikke afsluttet Punktmængde].

Lad $M_1, M_2, \ldots, M_n, \ldots$ være en uendelig Følge af Punktmængder, saaledes at $\sum_{n=1}^{\infty} M_n$ er konvergent; jeg vil da sammen med Punktmængden $M = \sum_{n=1}^{\infty} M_n$ betragte den anden Punktmængde M^*, hvis Elementer er alle saadanne Punkter m^*, for hvilke der til ethvert $\varepsilon > 0$ svarer et helt Tal n_1 saaledes, at enhver af Punktmængderne $L_N = \sum_{n=1}^{N} M_n$ $(N \geq n_1)$ indeholder mindst ét Punkt l_N, hvis Afstand fra m^* er mindre end ε (eller, hvad der øjensynlig er ensbetydende hermed, alle saadanne Punkter m^*, for hvilke det er muligt at udtage et Punkt l_r i Punktmængden $L_r = \sum_{n=1}^{r} M_n$ saaledes, at $\lim_{r=\infty} l_r = m^*$).

Man indser meget let udfra denne Definition, at Punktmængden M^* maa være en afsluttet Punktmængde.

Endvidere indses umiddelbart, at ethvert Element m i Punktmængden $M = \sum_{n=1}^{\infty} M_n$ maa tilhøre Punktmængden M^*; thi idet $m = \sum_{n=1}^{\infty} m_n$, er m Grændsepunkt for Punktfølgen $l_r = \sum_{n=1}^{r} m_n$, d. v. s. $m = \lim_{r=\infty} l_r$. Derimod behøver ikke omvendt alle Punkter i M^* at være Elementer i M; dette følger allerede deraf, at M (i Modsætning til M^*) ikke behøver at være en afsluttet Punktmængde. (Er f. Ex. M_1 en ikke afsluttet Punktmængde, medens M_n for $n \geq 2$ kun indeholder det ene Punkt O, er M øjensynlig lig M_1 og følgelig ikke afsluttet). Derimod vises let, at ethvert Punkt m^* i M^* enten tilhører M eller er Grændsepunkt for M, eller, hvad der udtrykker

det samme, at der, hvis m^* er et vilkaarligt Element i M^* og $\varepsilon > 0$ et vilkaarligt lille Tal, findes mindst ét Punkt i Punktmængden M, hvis Afstand fra m^* er mindre end ε. Rigtigheden af denne sidste Paastand indses saaledes: Lad $m'_1, m'_2, \ldots, m'_n, \ldots$ være en fast valgt Punktfølge, hvor m'_n er Element i M_n; idet $\sum_{n=1}^{\infty} m'_n$ er konvergent, kan vi vælge et Tal N saa stort, at for $N_1 \geqq N$ er Punktet $\sum_{n=N_1+1}^{\infty} m'_n$ beliggende indenfor en Cirkel med O som Centrum og Radius $\frac{\varepsilon}{2}$. Idet m^* er Element i M^* kan vi, i Følge Definitionen for M^*, efter at N er fastlagt, vælge et fast Tal $N_1 \geqq N$ og dertil svarende Elementer m''_1 i M_1, m''_2 i $M_2, \ldots, m''_{N_1}$ i M_{N_1} saaledes, at Punktet $l_{N_1} = \sum_{n=1}^{N_1} m''_n$'s Afstand fra Punktet m^* er mindre end $\frac{\varepsilon}{2}$; da vil øjensynlig Punktet $\sum_{n=1}^{N_1} m''_n + \sum_{n=N_1+1}^{\infty} m'_n$ være et Punkt i M, hvis Afstand fra m^* er mindre end ε, q. e. d.

Af det ovenstaaende følger umiddelbart, at hvis $M = \sum_{1}^{\infty} M_n$ er en afsluttet Punktmængde (altsaa specielt, hvis de enkelte Punktmængder M_n i den konvergente Række $\sum_{1}^{\infty} M_n$ er afsluttede og helt i det Endelige beliggende Punktmængder), er $M = M^*$.

Den uendelige Række $\sum_{n=1}^{\infty} M_n$, hvor de enkelte Punktmængder M_n antages beliggende helt i det Endelige, kaldes **ubetinget** konvergent, hvis det er muligt, for ethvert $n = 1, 2, \ldots$, at indeslutte Punktmængden M_n indenfor en Cirkel med Centrum i O og Radius r_n saaledes, at $\sum_{n=1}^{\infty} r_n$ er konvergent. En ubetinget konvergent Række $\sum_{1}^{\infty} M_n$ er øjensynlig ogsaa konvergent. [Et Exempel paa en konvergent men ikke ubetinget konvergent Række $\sum_{n=1}^{\infty} M_n$ er f. Ex. den Række $\sum_{n=1}^{\infty} M_n$, hvor Punktmængden M_n $(n = 1, 2, \ldots)$ kun indeholder et enkelt Punkt m_n og hvor alle Punkter m_n $(n = 1, 2, \ldots)$ er beliggende paa en ret Linie

8

gennem Begyndelsespunktet med positiv Retning L, saaledes at Afstanden fra O til m_n regnet med Fortegn er lig $\frac{(-1)^n}{n}$].
I en ubetinget konvergent Række er øjensynlig Leddenes Orden ligegyldig, d. v. s. Rækken vedbliver at være ubetinget konvergent og fremstille den samme Punktmængde efter en vilkaarlig Omordning af dens Led.

Enhver uendelig konvergent Række $\sum_{n=1}^{\infty} M_n$, hvis enkelte Led M_n er beliggende helt i det Endelige, og hvor Begyndelsespunktet O tilhører enhver af Punktmængderne M_n $(n = 1, 2, \ldots)$ er ubetinget konvergent.

Lad ς_n betegne Maximum af Afstanden $\overline{Om_n}$, hvor m_n gennemløber Punktmængden M_n; den ovenstaaende Paastand er da øjensynlig identisk med Paastanden: $\sum_{n=1}^{\infty} \varsigma_n$ er konvergent. Konvergensen af $\sum_{n=1}^{\infty} \varsigma_n$ bevises saaledes: Lad os antage $\sum_{n=1}^{\infty} \varsigma_n$ divergent. Gennem Punktet O trækkes da tre Halvlinier L_1, L_2 og L_3 saaledes, at (idet der fastlægges en positiv Omløbsretning i Planen) $\angle L_1 L_2 = \angle L_2 L_3 = \angle L_3 L_1 = 120°$.

Lad henholdsvis $\varsigma_{n,1}$, $\varsigma_{n,2}$ og $\varsigma_{n,3}$ betegne Maksimum af Afstanden $\overline{Om_n}$, idet m_n gennemløber den Punktmængde (indeholdende mindst det ene Punkt O), der bestaar af alle de Punkter tilhørende Punktmængden M_n, der er beliggende indenfor eller paa Randen henholdsvis af Vinkelaabningen OL_1L_2, OL_2L_3 og OL_3L_1. Da i det mindste et af Tallene $\varsigma_{n,1}$, $\varsigma_{n,2}$ $\varsigma_{n,3}$ er lig Tallet ς_n (altsaa $\varsigma_n \leq \varsigma_{n,1} + \varsigma_{n,2} + \varsigma_{n,3}$) og idet $\sum_{n=1}^{\infty} \varsigma_n$ er antaget divergent, kan ikke alle tre uendelige Rækker $\sum_{n=1}^{\infty} \varsigma_{n,1}$, $\sum_{n=1}^{\infty} \varsigma_{n,2}$ og $\sum_{n=1}^{\infty} \varsigma_{n,3}$ være konvergente; lad os f. Ex. antage, at $\sum_{n=1}^{\infty} \varsigma_{n,1}$ er divergent. Lad m'_n være et saadant (sikkert eksisterende) Punkt tilhørende Punktmængden M_n og beliggende indenfor eller paa Begrændsningen af Vinkelrummet OL_1L_2, at Afstanden $\overline{Om'_n} \geq \frac{1}{2}\varsigma_{n,1}$; lad L betegne den Halvlinie ud fra O,

der halverer $\angle L_1 L_2$, og lad l_n være Projektionen af Punktet m'_n paa Halvlinien L. Da er Afstanden $\overline{Ol_n} \geqq \overline{Om'_n} \cdot \cos(60^\circ) \geqq \frac{1}{4}\varsigma_{n,1}$. Følgelig er, idet Punkterne l_n $(n = 1, 2, \ldots)$ alle er beliggende paa Halvlinien L, $\sum_{n=1}^{\infty} l_n$ divergent. Paa den anden Side følger imidlertid umiddelbart, idet l_n er Projektion af m'_n paa den faste Halvlinie L, og idet $\sum_{n=1}^{\infty} m'_n$ i Følge Forudsætning er konvergent, at $\sum_{n=1}^{\infty} l_n$ maa være konvergent. Vor Antagelse $\sum_{n=1}^{\infty} \varsigma_n$ divergent har altsaa ført os til en Modstrid; følgelig er $\sum_{n=1}^{\infty} \varsigma_n$ konvergent, d. v. s. $\sum_{n=1}^{\infty} M_n$ er ubetinget konvergent, q. e. d.

Ved Hjælp af det foregaaende Resultat kan vi nu umiddelbart bevise Rigtigheden af følgende Sætning: Lad M_n $(n = 1, 2, \ldots)$ være en helt i det Endelige beliggende Punktmængde og lad $\sum_{n=1}^{\infty} M_n$ være konvergent. Der eksisterer da en Punktfølge $c_1, c_2, \ldots, c_n, \ldots$, hvor $\sum_{n=1}^{\infty} c_n$ er konvergent, og en dertil svarende Følge af positive Tal $r_1, r_2, \ldots r_n, \ldots$, hvor $\sum_{n=1}^{\infty} r_n$ er konvergent; med følgende Egenskab: Punktmængden M_n $(n = 1, 2, \ldots)$ er helt beliggende indenfor en Cirkel med c_n som Centrum og Radius r_n.

Bevis: Lad c_n være et vilkaarligt Punkt i M_n; da er $\sum_{n=1}^{\infty} c_n$ konvergent. Lad M'_n være den Punktmængde, der fremkommer ved at parallelforskyde Punktmængden M_n, saaledes at Punktet c_n falder i Begyndelsespunktet O. Idet $\sum_{n=1}^{\infty} c_n$ er konvergent, vil da, som man umiddelbart indser, den uendelige Række $\sum_{n=1}^{\infty} M'_n$ være konvergent. Da endvidere enhver af Punktmængderne M'_n $(n = 1, 2, \ldots)$ indeholder Punktet O, vil $\sum_{n=1}^{\infty} M'_n$ være ubetinget konvergent. Følgelig vil Punktmængden M'_n være helt indeholdt i en Cirkel med Centrum i Begyndelses-

10

punktet og Radius r_n, hvor $\sum_{n=1}^{\infty} r_n$ er konvergent; men heraf følger umiddelbart, at Punktmængden M_n maa være indeholdt i en Cirkel med c_n som Centrum og Radius r_n. Hermed er den ovenstaaende Sætning bevist.

Jeg skal slutte denne Paragraf med nogle faa Bemærkninger om den uendelige Række $\sum_{n=1}^{\infty} M_n$, hvor enhver af Punktmængderne er en JORDAN'sk Kurve[1]. Da M_n i dette Tilfælde er afsluttet og beliggende helt i det Endelige, vil $\sum_{n=1}^{\infty} M_n$, hvis den er konvergent, ligeledes være afsluttet og helt i det Endelige beliggende, og Punktmængden $M = \sum_{n=1}^{\infty} M_n$ vil være identisk med Punktmængden M^*.

Lad $\sum_{n=1}^{\infty} M_n$ være konvergent; der eksisterer da, som ovenfor bevist, en Punktfølge c_n, hvor $\sum_{n=1}^{\infty} c_n$ er konvergent, og en positiv Talfølge r_n, hvor $\sum_{n=1}^{\infty} r_n$ er konvergent, saaledes at Kurven M_n er helt beliggende indenfor en Cirkel C_n med c_n som Centrum og Radius r_n. Heraf drages følgende Slutninger: 1) Hvis Begyndelsespunktet O er beliggende indenfor enhver af Kurverne M_n $(n = 1, 2, \ldots)$, vil $\sum_{n=1}^{\infty} M_n$ være ubetinget konvergent; thi idet Punktet O er beliggende indenfor Kurven M_n og følgelig ogsaa indenfor Cirklen C_n, vil denne Cirkel og følgelig yderligere Kurven M_n være beliggende helt indenfor en Cirkel med Centrum i O og Radius $2r_n$. 2) Lad p_n være et vilkaarligt Punkt indenfor M_n; da vil p_n ligeledes være beliggende indenfor Cirklen C_n; følgelig vil, som

[1] Ved en JORDAN'sk Kurve forstaas som bekendt en lukket kontinuert Kurve uden Dobbeltpunkter, eller præcisere: en Punktmængde, der kan afbildes éenentydig og kontinuert paa en Cirkelperiferi. En JORDAN'sk Kurve deler som bekendt Mængden af alle de af Planens Punkter, der ikke er beliggende paa selve Kurven, i to Omraader, et indenfor Kurven og et udenfor Kurven beliggende Omraade. Kurven selv danner Begrændsningen mellem Omraadet af indre og Omraadet af ydre Punkter.

11

man umiddelbart indser, $\sum_{n=1}^{\infty} p_n$ være konvergent. Lad M_n' være den Punktmængde (JORDAN'ske Kurve), der fremkommer ved at parallelforskyde Kurven M_n saaledes, at Punktet p_n (der tænkes i fast Forbindelse med M_n) falder i Begyndelsespunktet O (der tænkes fast). Da vil den uendelige Række $\sum_{n=1}^{\infty} M_n'$ være konvergent, og Punktmængden $M = \sum_{n=1}^{\infty} M_n$ vil umiddelbart fremkomme af Punktmængden $M' = \sum_{n=1}^{\infty} M_n'$ ved at parallelforskyde denne sidste Punktmængde saaledes, at Punktet O (der nu tænkes i fast Forbindelse med M') falder i Punktet $p = \sum_{n=1}^{\infty} p_n$ (idet dette sidste Punkt tænkes fastliggende). Af denne sidste Bemærkning følger aabenbart, at man ved en Undersøgelse over Addition af uendelig mange lukkede kontinuerte Kurver, uden at indskrænke Undersøgelsens Almindelighed tør antage, at Begyndelsespunktet O er beliggende indenfor enhver af Kurverne.

§ 2. Addition af to konvekse Kurver.

Ved en lukket konveks Kurve, eller simpelthen ved en konveks Kurve, forstaas en JORDAN'sk Kurve med følgende Egenskab: Hvis en ret Linie har mer end to Punkter fælles med Kurven, vil de for Kurven og den rette Linie fælles Punkter være samtlige Punkter paa et Liniestykke AB (Endepunkterne medregnede). Heraf følger specielt, at en ret Linie, der indeholder et indenfor Kurven beliggende Punkt, vil skære Kurven i to og kun to Punkter (denne Egenskab kunde iøvrigt ogsaa være lagt til Grund for den konvekse Kurves Definition).

Idet vi ved et konvekst Omraade forstaar en saadan Punktmængde i Planen, at den, hvis den indeholder de to Punkter A og B, indeholder hele Liniestykket AB, er den ovenstaaende Definition som bekendt identisk med følgende

12

anden Definition: En konveks Kurve er Begrændsningen[1] for et konvekst Omraade, der er beliggende helt i det Endelige, uden at være beliggende helt paa en ret Linie.

Ved Arealet af en konveks Kurve vil vi i det følgende forstaa det (altid eksisterende) Areal af det indenfor Kurven beliggende Omraade.

Lad M_1 og M_2 være to konvekse Kurver, saaledes at Begyndelsespunktet O er beliggende indenfor begge Kurver, og saaledes at Arealet af $M_1 \geqq$ Arealet af M_2. Jeg vil da i denne Paragraf undersøge den Punktmængde $M_1 + M_2$, der fremkommer ved Addition af de to konvekse Kurver.

I den følgende Undersøgelse vil jeg med $\div M_2$ betegne den Punktmængde, hvis Elementer m' er karakteriserede ved følgende Egenskab: der findes svarende til m' et Element m_2 i M_2, saaledes at $m' + m_2 = O$; med andre Ord, $\div M_2$ er den Punktmængde, der fremkommer, naar M_2 drejes 180° om Begyndelsespunktet O; $\div M_2$ vil følgelig være en konveks Kurve med samme Areal som M_2 og ligeledes indeholdende Punktet O i sit Indre. Endvidere vil jeg, idet l er et vilkaarligt givet Punkt i Planen, ved $N(l)$ forstaa den Punktmængde, der fremkommer af $\div M_2$, naar denne sidste Punktmængde parallelforskydes saaledes. at Punktet O (der tænkes i fast Forbindelse med $\div M_2$) falder i Punktet l (der tænkes fastliggende). $N(l)$ er følgelig en konveks Kurve med samme Areal som M_2 og indeholdende Punktet l i sit Indre. Med denne Betegnelse vil det øjensynlig være e n n ø d v e n d i g o g t i l s t r æ k k e l i g B e t i n g e l s e f o r a t P u n k t e t l h ø r e r m e d t i l P u n k t m æ n g d e n $M_1 + M_2$ (d. v. s. for at der eksisterer et Element m_1 i M_1 og et Element m_2 i M_2 saaledes, at $m_1 + m_2 = l$), a t K u r v e r n e M_1 og $N(l)$ s k æ r e r h i n a n d e n (d. v. s. har mindst ét Punkt fælles).

[1] Et Punkt p siges at tilhøre Begrændsningen for en given Punktmængde M, hvis der indenfor enhver Cirkel med Centrum i p og vilkaarlig lille Radius findes saavel Punkter tilhørende Punktmængden M som Punkter ikke tilhørende M.

Ved Benyttelse af denne Bemærkning indses umiddelbart, at ethvert Punkt l paa Kurven M_1 maa tilhøre Punktmængden $M_1 + M_2$; thi i modsat Tilfælde vilde de to Kurver M_1 og $N(l)$ ingen Punkter have fælles; følgelig maatte $N(l)$ enten ligge helt indenfor M_1, eller ligge helt udenfor M_1, eller ogsaa maatte M_1 ligge helt indenfor $N(l)$; men intet af disse tre Tilfælde kan indtræffe, thi, idet Punktet l ligger paa Kurven M_1 men indenfor Kurven $N(l)$, er de to første Muligheder udelukkede, og idet Arealet af $M_1 \geqq$ Arealet af $M_2 =$ Arealet af $N(l)$ er ogsaa den tredje Mulighed udelukket. Hermed er Paastanden om, at ethvert Punkt paa Kurven M_1 tilhører Punktmængden $M_1 + M_2$, bevist.

For den senere Undersøgelse vil følgende videregaaende Bemærkning være af Vigtighed: Lad e_2 betegne Minimum af Afstanden $\overline{Om_2}$, idet m_2 gennemløber Kurven M_2 (da er e_2 tillige Minimum af Afstanden $\overline{ln}$, idet n gennemløber Kurven $N(l)$), og lad m_1 være et vilkaarligt Punkt paa Kurven M_1; da vil ethvert Punkt l i Planen, hvis Afstand fra m_1 er mindre end e_2, tilhøre Punktmængden $M_1 + M_2$. Rigtigheden af denne Bemærkning indses ved Benyttelse af ganske samme Slutningsmaade som ovenfor, idet det bemærkes, at Kurven $N(l)$ ogsaa her vil indeholde et Punkt m_1 paa Kurven M_1 i sit Indre.

Lad L være en vilkaarlig ret Linie i Planen; jeg vil da søge at bestemme de Punkter l paa Linien L, der tilhører Punktmængden $M_1 + M_2$, altsaa de Punkter l paa Linien L, for hvilke den tilsvarende Kurve $N(l)$ skærer den faste Kurve M_1.

Lad l være et vilkaarligt Punkt paa L. Idet den tilsvarende Kurve $N(l)$ er en konveks Kurve, eksisterer der som bekendt to indbyrdes forskellige, med L parallele, rette Linier L_1 og L_2 (Kurvens Grændselinier i den ved L bestemte Retning) med følgende Egenskaber: en vilkaarlig ret Linie parallel med L vil, hvis den ligger helt udenfor Parallelstrimlen $L_1 L_2$, ikke have noget Punkt fælles med Kurven, hvis den ligger helt indenfor Parallelstrimlen $L_1 L_2$, skære Kurven i to og kun to

Punkter, og endelig, hvis den er sammenfaldende med én af Linierne L_1 og L_2, skære Kurven $N(l)$ enten i ét og kun ét Punkt eller i et Liniestykke (Endepunkterne medregnede). Idet l ligger indenfor Kurven $N(l)$, vil Linien L øjensynlig være beliggende mellem Grændselinierne L_1 og L_2.

Betragter vi alle Punkter indenfor Parallelstrimlen $L_1 L_2$ (incl. Rand) kan disse Punkter øjensynlig inddeles i følgende fire Grupper, alt efter deres Beliggenhed i Forhold til den konvekse Kurve $N(l)$.

Gruppe 1: De paa Kurven $N(l)$ beliggende Punkter.

Gruppe 2: De indenfor Kurven $N(l)$ beliggende Punkter.

Gruppe 3: Punkter, der ligger over $N(l)$ [idet vi om et Punkt p indenfor eller paa Randen af Parallelstrimlen siger, at det ligger over Kurven $N(l)$, hvis Halvlinien (Endepunktet medregnet) draget ud fra p parallel med L og til den Side, der bestemmes ved L's positive Retning, intet Punkt har fælles med Kurven $N(l)$].

Gruppe 4: Punkter der ligger under $N(l)$ [idet vi om et Punkt q indenfor eller paa Randen af Parallelstrimlen siger, at det ligger under Kurven $N(l)$ hvis Halvlinien (Endepunktet medregnet) draget ud fra q parallel med L og til den Side, der er modsat L's positive Retning, intet Punkt har fælles med Kurven $N(l)$].

Man indser umiddelbart, at hvis en kontinuert Kurve K (idet jeg ved en kontinuert Kurve forstaar en saadan Punktmængde i Planen, der kan afbildes énéntydig og kontinuert enten paa en hel Cirkelperiferi eller paa en Cirkelbue (Endepunkterne medregnede), specielt et enkelt Punkt) har alle sine Punkter beliggende indenfor Parallelstrimlen $L_1 L_2$ (incl. Rand) uden at have noget Punkt fælles med Kurven $N(l)$, maa K enten være beliggende helt i det Indre af $N(l)$, eller helt over $N(l)$, eller helt under $N(l)$.

Lad l og l' være to Punkter paa L saaledes, at Retningen fra l til l' er overensstemmende med L's positive Retning, og

lad $N(l)$ og $N(l')$ være de tilsvarende (indbyrdes kongruente) konvekse Kurver; idet $N(l')$ umiddelbart fremkommer af $N(l)$ ved en Parallelforskydning bestemt i Størrelse og Retning ved Liniestykket ll', vil Kurverne $N(l)$ og $N(l')$ have de samme Grændselinier L_1 og L_2. Man indser nu umiddelbart ud fra Definitionen, at ethvert Punkt p, der ligger over $N(l')$ ogsaa maa ligge over $N(l)$, medens ethvert Punkt, der ligger under $N(l)$ ogsaa maa ligge under $N(l')$.

Efter disse indledende Bemærkninger skal jeg nu gaa over til den direkte Undersøgelse af, hvilke Punkter l paa Linien L der tilhører Punktmængden $M_1 + M_2$, altsaa af, hvilke Punkter l, der er saaledes beliggende paa Linien L, at Kurven $N(l)$ skærer Kurven M_1. Da $N(l)$ stedse (d. v. s. hvor end l er beliggende paa L) befinder sig i det Indre (incl. Rand) af den ved L_1 og L_2 bestemte Parallelstrimmel, behøver vi i denne Sammenhæng øjensynlig kun at betragte den Del af Kurven M_1, der er beliggende indenfor Parallelstrimlen $L_1 L_2$ (incl. Rand). Der kan nu indtræffe et af følgende tre Tilfælde:

Tilfælde 1: M_1 er helt beliggende udenfor Parallelstrimlen $L_1 L_2$ (incl. Rand). Da vil $N(l)$ aldrig, hvor end l er beliggende paa Linien L, have noget Punkt fælles med M_1. Følgelig indeholder Linien L i dette Tilfælde intet Punkt af Punktmængden $M_1 + M_2$.

Tilfælde 2: M_1 er enten helt beliggende indenfor Parallelstrimlen $L_1 L_2$ (excl. Rand), eller M_1 har mindst ét Punkt fælles med mindst én af Randene L_1 og L_2, uden at M_1 dog skærer enhver af Linierne L_1 og L_2 i to og kun to Punkter. I alle disse under Tilfælde 2 sammenfattede Tilfælde vil de Punkter af M_1, der er beliggende indenfor eller paa Randen af Parallelstrimlen $L_1 L_2$, øjensynlig danne en kontinuert Kurve K. Jeg vil bevise, at i dette Tilfælde 2 vil de Punkter l paa Linien L, der tilhører Punktmængden $M_1 + M_2$, danne et Liniestykke AB (Endepunkterne medregnede) specielt et enkelt Punkt. Da nemlig M_1 har mindst ét Punkt beliggende indenfor eller paa

16

Randen af Parallelstrimlen $L_1 L_2$, vil for det første øjensynlig mindst ét Punkt paa Linien L tilhøre Punktmængden $M_1 + M_2$, d. v. s. der findes mindst ét Punkt l paa L, for hvilket $N(l)$ indeholder et Punkt paa Kurven K. Idet endvidere enhver af Punktmængderne M_1 og M_2 og følgelig ogsaa Punktmængden $M_1 + M_2$ er en helt i det Endelige liggende afsluttet Punktmængde, vil den ovenstaaende Paastand aabenbart være bevist, naar vi har eftervist, at hvis l_1 og l_2 er to indbyrdes forskellige Punkter paa Linien L, der begge tilhører Punktmængden $M_1 + M_2$ (og hvor vi vil antage, at Retningen $l_1 l_2$ er overensstemmende med L's positive Retning), at da ethvert Punkt l_3 beliggende paa L og mellem l_1 og l_2 ligeledes vil tilhøre Punktmængden $M_1 + M_2$. Men at dette sidste vil være Tilfældet, indses umiddelbart saaledes: Idet den Del af M_1, der er beliggende indenfor Parallelstrimlen $L_1 L_2$ (incl. Rand) er en kontinuert Kurve K, maatte denne Punktmængde K, hvis den intet Punkt havde fælles med Kurven $N(l_3)$, være beliggende enten helt i det Indre af $N(l_3)$, eller helt over $N(l_3)$, eller helt under $N(l_3)$; men intet af disse tre Tilfælde kan indtræffe. For det første kan K ikke være beliggende helt i det Indre af $N(l_3)$; K har nemlig enten mindst et Punkt fælles med en af Randene af Parallelstrimlen $L_1 L_2$ (og et saadant Randpunkt kan ikke være indre Punkt for $N(l_3)$), eller ogsaa er K identisk med hele Kurven M_1 (men, idet Arealet af $M_1 >$ Arealet af M_2 = Arealet af $N(l)$, kan K heller ikke i dette Tilfælde være beliggende helt i det Indre af $N(l_3)$). Endvidere kan K ikke være beliggende helt over $N(l_3)$, da K da ogsaa maatte være beliggende helt over $N(l_1)$, hvad der ikke kan være Tilfældet, da K i Følge Antagelsen har Punkter fælles med $N(l_1)$. Endelig kan K ikke være beliggende helt under $N(l_3)$; thi da vilde K ogsaa være beliggende helt under $N(l_2)$, i Modstrid med at K har Punkter fælles med $N(l_2)$. Hermed er Paastanden om, at de Punkter paa Linien L, der i dette Tilfælde tilhører Punktmængden $M_1 + M_2$, udgør et Liniestykke

AB (incl. Endepunkter), specielt et enkelt Punkt A, fuldkommen bevist.

Tilfælde 3: Enhver af Linierne L_1 og L_2 skærer Kurven M_1 i to og kun to Punkter. I dette Tilfælde vil de indenfor Parallelstrimlen $L_1 L_2$ (Randen incl.) beliggende Punkter af Kurven M_1 danne to kontinuerte Kurver K_1 og K_2, som ikke har noget Punkt fælles, og saaledes at K_i $(i = 1, 2)$ skærer enhver ret Linie parallel med L og beliggende indenfor eller paa Randen af Parallelstrimlen $L_1 L_2$ i ét og kun ét Punkt, altsaa specielt skærer Linien L i ét og kun ét Punkt l_i. For at i dette Tilfælde et Punkt l paa Linien L skal tilhøre Punktmængden $M_1 + M_2$ er det nødvendigt og tilstrækkeligt, at Kurven $N(l)$ har et Punkt fælles med mindst én af de to kontinuerte Kurver K_1 og K_2. Lad os søge de Punkter l paa Linien L, for hvilke Kurven $N(l)$ har et Punkt fælles med den kontinuerte Kurve K_i $(i = 1, 2)$. Man beviser her ved Benyttelse af ganske den samme Slutningsmaade som den ved Tilfælde 2 anvendte, at de søgte Punkter l danner et Liniestykke $A_i B_i$ (Endepunkterne medregnede). Endvidere indses, at det Punkt l_i, hvori Linien L skærer den kontinuerte Kurve K_i, øjensynlig maa tilhøre Liniestykket $A_i B_i$, thi K_i vil øjensynlig skære Kurven $N(l_i)$, da K_i indeholder et Punkt (nemlig l_i), der tilhører det Indre af Kurven $N(l_i)$, uden at K_i er helt beliggende i denne Kurves Indre (K_i indeholder nemlig Punkter paa Parallelstrimlens Rand); iøvrigt indses paa ganske samme Maade, at ikke blot Punktet l_i men ogsaa alle Punkter paa L, hvis Afstand fra l_i er mindre end e_2 (hvor e_2 som tidligere betegner Minimum af Afstanden Om_2 idet, m_2 gennemløber M_2) maa tilhøre Liniestykket $A_i B_i$. Hermed er bevist, at i dette Tilfælde 3 vil de Punkter af $M_1 + M_2$, der er beliggende paa Linien L, enten danne et enkelt Liniestykke AB (incl. Endepunkter), nemlig hvis $A_1 B_1$ og $A_2 B_2$ har Punkter fælles, eller to adskilte Liniestykker $A_1 B_1$ og $A_2 B_2$; i dette sidste Tilfælde

vil ethvert af Liniestykkerne $A_i B_i$ $(i = 1, 2)$ indeholde et (og kun et) Punkt paa Kurven M_1 i sit Indre.

Resumerer vi Resultaterne af den forudgaaende Undersøgelse har vi bevist følgende: Punktmængden $M_1 + M_2$ er en afsluttet og helt i det Endelige beliggende Punktmængde. En vilkaarlig ret Linie L vil, hvis den overhovedet har Punkter fælles med $M_1 + M_2$, skære denne Punktmængde enten i et enkelt Liniestykke AB (Endepunkterne medregnede) specielt i et enkelt Punkt, eller i to Liniestykker $A_1 B_1$ og $A_2 B_2$; i dette sidste Tilfælde vil Linien L skære Kurven M_1 i to og kun to Punkter l_1 og l_2, hvor l_1 vil tilhøre det Indre af Liniestykket $A_1 B_1$, l_2 det Indre af Liniestykket $A_2 B_2$. Endelig vil Punktmængden $M_1 + M_2$ indeholde ethvert Punkt m_1 paa Kurven M_1, almindeligere ethvert Punkt indenfor en Cirkel med Centrum i m_1 og Radius e_2; heraf følger specielt, at intet Punkt m_1 paa Kurven M_1 kan være Begrændsningspunkt for Punktmængden $M_1 + M_2$.

Efter den forudgaaende Undersøgelse kan vi nu uden Vanskelighed bevise følgende

Sætning: Lad M_1 og M_2 være to konvekse Kurver, som begge indeholder Begyndelsespunktet O som indre Punkt, og saaledes at Arealet af $M_1 \geq$ Arealet af M_2. Da vil Punktmængden $M_1 + M_2$ enten være et Omraade (incl. Rand) begrændset af en enkelt konveks Kurve Y_2, og saaledes at Kurven M_1 ligger helt indenfor Kurven Y_2, eller ogsaa vil $M_1 + M_2$ være et Omraade (incl. Rand) begrændset af to konvekse Kurver Y_2 og I_2, hvor I_2 er beliggende helt indenfor Y_2, og saaledes at Kurven M_1 er helt indeholdt i det Indre af Kurven Y_2, men indeholder Kurven I_2 helt i sit Indre. Endvidere vil, idet ς_i $(i = 1, 2)$ henholdsvis e_i $(i = 1, 2)$ betegner Maximum henholdsvis Minimum af Afstanden Om_i, hvor m_i gennemløber Kurven M_i, Kurven Y_2 være helt inde-

19

holdt i en Cirkel (incl. Rand) med O som Centrum og Radius $\varsigma_1+\varsigma_2$ samt helt i sit Indre (incl. Rand) indeholde en Cirkel med O som Centrum og Radius e_1+e_2, ligesom Kurven I_2, i det Tilfælde, hvor M_1+M_2 er begrændset af to konvekse Kurver, vil være helt indeholdt i en Cirkel (incl. Rand) med Centrum i O og Radius ς_1-e_2. Endelig vil Arealet af det mellem M_1 og Y_2 beliggende Omraade være $>$ Arealet af M_2, ligesom, i det Tilfælde, hvor M_1+M_2 er begrænset af to konvekse Kurver, Arealet af det mellem M_1 og I_2 beliggende Omraade vil være $>$ Arealet af M_2; følgelig vil, i det Tilfælde, hvor M_1+M_2 er begrændset af en enkelt konveks Kurve, Arealet af Omraadet M_1+M_2 være $>$ Arealet af M_1 + Arealet af M_2, medens, i det Tilfælde, hvor M_1+M_2 er begrændset af to konvekse Kurver, Arealet af Omraadet M_1+M_2 vil være $>$ $2 \cdot$ Arealet af M_2.

Bevis: 1) Lad os antage, at enhver ret Linie, der overhovedet har Punkter fælles med M_1+M_2, skærer M_1+M_2 i et enkelt Liniestykke (incl. Endepunkter) specielt i et enkelt Punkt; i dette Tilfælde vil M_1+M_2 være et konvekst Omraade; da M_1+M_2 endvidere er helt i det Endelige beliggende og ikke ligger helt paa en ret Linie samt er afsluttet, vil M_1+M_2 i dette Tilfælde bestaa af det Indre (incl. Rand) af en lukket konveks Kurve Y_2. Da endelig ethvert Punkt m_1 paa M_1 tilhører M_1+M_2 uden at være Begrændsningspunkt for M_1+M_2, maa M_1 være beliggende helt indenfor Y_2.

2) Lad os antage, at der findes en ret Linie, der skærer M_1+M_2 i to adskilte Liniestykker. Jeg vil da først søge at bestemme den Punktmængde P, der til Elementer har dels alle Punkter indenfor Kurven M_1 og dels alle saadanne Punkter udenfor eller paa Kurven M_1, som tilhører Punktmængden M_1+M_2 (med andre Ord, Punktmængden P fremkommer ved til Punktmængden M_1+M_2 at føje de Punkter, der ligger

20

indenfor Kurven M_1 uden at tilhøre Punktmængden $M_1 + M_2$). Lad L være en vilkaarlig ret Linie; jeg vil da vise, at, hvis L overhovedet har Punkter fælles med Punktmængden P, danner disse Punkter et Liniestykke (Endepunkterne medregnede), specielt et enkelt Punkt. Dette indses saaledes: Hvis L ingen Punkter har fælles med $M_1 + M_2$, har L heller ingen Punkter fælles med P; thi i modsat Fald vilde L indeholde et Punkt indenfor Kurven M_1, følgelig ogsaa et Punkt paa Kurven M_1, altsaa et Punkt tilhørende $M_1 + M_2$, i Modstrid med vor Antagelse. Antages dernæst, at L skærer $M_1 + M_2$ i et enkelt Liniestykke AB (Endepunkterne medregnede) specielt i et enkelt Punkt A, vil de for L og P fælles Punkter øjensynlig være de samme, som de for L og $M_1 + M_2$ fælles Punkter; thi i modsat Fald vilde L indeholde et Punkt l udenfor Liniestykket AB, f. Ex. i AB's Forlængelse ud over A, der var beliggende indenfor Kurven M_1; men da maatte L indeholde et Punkt i Forlængelsen af Liniestykket BAl udover l, som laa paa Kurven M_1 og altsaa tilhørte Punktmængden $M_1 + M_2$, i Modstrid med vor Antagelse. Lad os endelig betragte det Tilfælde, hvor L skærer $M_1 + M_2$ i to adskilte Liniestykker A_1B_1 og A_2B_2 (Betegnelserne tænkes valgte saaledes, at Punkterne B_1 og A_2 ligger imellem Punkterne A_1 og B_2). Idet Linien L skærer M_1 i to og kun to Punkter l_1 og l_2, hvoraf l_1 tilhører Liniestykket A_1B_1 og l_2 tilhører Liniestykket A_2B_2, vil ethvert Punkt paa Linien L beliggende mellem B_1 og A_2 tilhøre det Indre af Kurven M_1, medens ethvert Punkt paa Linien L beliggende enten i Forlængelsen af A_1B_1 udover A_1 eller i Forlængelsen af A_2B_2 udover B_2 vil være beliggende udenfor Kurven M_1. Følgelig vil de for L og P fælles Punkter være samtlige Punkter paa Liniestykket A_1B_2 (Endepunkterne medregnede). Hermed er Paastanden om, at enhver ret Linie, hvis den overhovedet har Punkter fælles med P, skærer P i et Liniestykke (Endepunkterne medregnede) specielt i et enkelt Punkt, bevist. Da P endvidere

21

øjensynlig er helt i det Endelige beliggende og ikke ligger helt paa en ret Linie samt er afsluttet, vil P være det Indre (incl. Rand) af en konveks Kurve Y_2, der øjensynlig vil indeholde Kurven M_1 helt i sit Indre. For at bestemme Punktmængden M_1+M_2 har vi nu kun tilbage at bestemme den Punktmængde Q, der bestaar af alle de indenfor M_1 beliggende Punkter, som ikke tilhører Punktmængden M_1+M_2. Er Q først bestemt, faar vi umiddelbart M_1+M_2 ved blot at fradrage P de Punkter, der tilhører Q. Lad L være en vilkaarlig ret Linie; da vil L, hvis den overhovedet har Punkter fælles med Q, øjensynlig skære Q i et Liniestykke (Endepunkterne ikke medregnede); thi af den umiddelbart foregaaende Undersøgelse følger, at L, hvis den ikke skærer Punktmængden M_1+M_2, eller hvis den skærer denne Punktmængde i et Liniestykke, specielt i et enkelt Punkt, intet Punkt vil have fælles med Q, medens L, hvis den skærer M_1+M_2 i to adskilte Liniestykker A_1B_1 og A_2B_2 (hvor Betegnelserne som ovenfor tænkes valgte saaledes, at B_1 og A_2 ligger mellem A_1 og B_2) vil skære Q i Liniestykket B_1A_2 (Endepunkterne ikke medregnede). Q er følgelig et helt i det Endelige beliggende konvekst Omraade. Idet M_1+M_2 er en afsluttet Punktmængde, kan Q endvidere ikke indeholde noget Punkt af sin Begrændsning, altsaa specielt ikke være beliggende helt paa en ret Linie. Punktmængden Q er følgelig det Indre (excl. Rand) af en lukket konveks Kurve I_2. Da endelig intet Punkt paa Kurven M_1 er Begrændsningspunkt for Punktmængden M_1+M_2, maa I_2 være beliggende helt indenfor M_1. Punktmængden M_1+M_2 er følgelig i dette Tilfælde et Omraade (incl. Rand) begrændset af to lukkede konvekse Kurver Y_2 og I_2, hvor Y_2 indeholder M_1 helt i sit Indre, medens M_1 indeholder I_2 helt i sit Indre. Hermed er den første Del af den opstillede Sætning bevist.

Idet ethvert Punkt tilhørende M_1+M_2 har Formen m_1+m_2, indses umiddelbart, at „den ydre Begrændsningskurve“ Y_2 maa være helt beliggende indenfor en Cirkel (incl. Rand) med Cen-

trum i Begyndelsespunktet O og Radius $\varsigma_1+\varsigma_2$. Idet endvidere ethvert Punkt, hvis Afstand fra et vilkaarligt Punkt m_1 paa M_1 er mindre end e_2, tilhører Punktmængden M_1+M_2, indser man, at Kurven Y_2 helt i sit Indre (incl. Rand) maa indeholde en Cirkel med Centrum i O og Radius e_1+e_2; thi, idet O er beliggende indenfor M_1, vil enhver Halvlinie L ud fra O skære enhver af Kurverne M_1 og Y_2 i ét og kun ét Punkt m_1 og y_2, og her vil Afstanden $\overline{Om_1}$ være $\geqq e_1$, Afstanden $\overline{m_1y_2}$ være $\geqq e_2$, altsaa Afstanden $\overline{Oy_2}$ være $\geqq e_1+e_2$. Endelig vil i det Tilfælde, hvor M_1+M_2 er begrændset af to konvekse Kurver, Kurven I_2 (idet c betegner et vilkaarligt Punkt i Planen og ς betegner Maximum af Afstanden $\overline{cm_1}$ for m_1 gennemløbende M_1) være helt indeholdt i en Cirkel C (incl. Rand) med Centrum i c og Radius $\varsigma-e_2$; thi lad L være en vilkaarlig Halvlinie ud fra c; hvis da L intet Punkt har fælles med M_1, vil L heller intet Punkt have fælles med I_2; hvis derimod L har mindst ét Punkt fælles med M_1, og hvis jeg med m_1 betegner det af de for L og M_1 fælles Punkter, hvis Afstand fra c er størst, vil øjensynlig intet Punkt paa Halvlinien L, hvis Afstand fra c er større end Afstanden $\overline{cm_1}$, ligge paa Kurven I_2, almindeligere, intet Punkt paa L, hvis Afstand fra c er større end det største af Tallene $\overline{cm_1}-e_2$ og 0, altsaa i hvert Fald intet Punkt paa L, hvis Afstand fra c er større end $\varsigma-e_2$; hermed er bevist, at Kurven I_2 vil ligge helt indenfor Cirklen C (incl. Rand); vælges specielt Punktet c i Begyndelsespunktet følger, at I_2 vil ligge indenfor en Cirkel (incl. Rand) med Centrum i O og Radius ς_1-e_2. [Det kan i denne Sammenhæng bemærkes, at Punktet O ikke nødvendigvis er beliggende indenfor (eller paa) Kurven I_2.]

Jeg skal nu gaa over til en Behandling af de i Sætningens sidste Del fremsatte Paastande, nemlig at Arealet af det mellem M_1 og Y_2 beliggende Omraade er $>$ Arealet af M_2, samt at i det Tilfælde, hvor M_1+M_2 er begrændset af to konvekse Kurver, Arealet af det mellem M_1 og I_2 beliggende Omraade

23

er $>$ Arealet af M_2. Jeg skal her nøjes med at bevise den sidste af disse to Sætninger; Beviset for den første føres paa ganske tilsvarende Maade. Lad l være et Punkt i det Indre af Kurven I_2. Da vil Kurven $N(l)$ ingen Punkter have fælles med M_1. Idet Punktet l imidlertid er beliggende indenfor begge Kurverne $N(l)$ og M_1, kan $N(l)$ ikke ligge helt udenfor M_1; idet endvidere Arealet af $N(l)$ = Arealet af $M_2 \leqq$ Arealet af M_1, kan $N(l)$ heller ikke helt omslutte M_1; følgelig maa $N(l)$ ligge helt indenfor M_1. [Heraf følger specielt, at Arealet af $N(l)$ maa være $<$ (ikke blot $\leqq$) Arealet af M_1; altsaa er bevist, at hvis Arealet af M_1 = Arealet af M_2, maa Punktmængden $M_1 + M_2$ nødvendigvis være begrændset af kun én konveks Kurve.]

Lad os ud fra det betragtede Punkt l trække en Halvlinie L; idet l er beliggende indenfor enhver af de tre konvekse Kurver M_1, I_2 og $N(l)$, skærer L enhver af disse Kurver i ét og kun ét Punkt; lad os betegne disse Punkter med henholdsvis m_1, i_2 og n, og lad os betegne Afstandene $\overline{lm_1}$, $\overline{li_2}$ og $\overline{ln}$ henholdsvis med r_1, r_2 og r_3. Da er for det første $r_1 > r_2$ og $r_1 > r_3$. Jeg vil imidlertid bevise, at der gælder den skarpere Ulighed $r_1 \geqq r_2 + r_3$. Dette indses saaledes: Lad l' betegne det Punkt paa Halvlinien L, hvis Afstand fra l er lig $r_1 - r_3$; Punktet l' hører da med til Punktmængden $M_1 + M_2$, da Kurven $N(l')$ øjensynlig indeholder Punktet m_1, altsaa har et Punkt fælles med M_1; følgelig ligger l' udenfor eller paa Kurven I_2; altsaa er $r_2 \leqq r_1 - r_3$, d. v. s. $r_1 \geqq r_2 + r_3$, q. e. d. Dette gælder for enhver Stilling af Halvlinien L ud fra l. Lad X være en fast Halvlinie ud fra l, og lad os vælge en positiv Omløbsretning i Planen, samt bestemme Halvlinien L ved Vinklen $\varphi = \sphericalangle (XL)$ $(0 \leqq \varphi < 2\pi)$. Da er r_1, r_2 og r_3 Funktioner af φ, og der gælder for alle φ Uligheden $r_1 \geqq r_2 + r_3$. Nu er imidlertid

$$\text{Arealet af } M_1 = \frac{1}{2}\int_0^{2\pi} r_1^2 d\varphi; \quad \text{Arealet af } I_2 = \frac{1}{2}\int_0^{2\pi} r_2^2 d\varphi,$$

samt Arealet af M_2 = Arealet af $N(l) = \frac{1}{2}\int_0^{2\pi} r_3^2 d\varphi$.

Følgelig er Arealet af det mellem M_1 og I_2 beliggende Omraade lig

$$\frac{1}{2}\int_0^{2\pi}(r_1^2 - r_2^2)\,d\varphi =$$

$$= \frac{1}{2}\int_0^{2\pi}(r_1+r_2)(r_1-r_2)\,d\varphi > \frac{1}{2}\int_0^{2\pi}(r_1-r_2)^2 d\varphi \geqq \frac{1}{2}\int_0^{2\pi} r_3^2\,d\varphi =$$

= Arealet af M_2, q. e. d.

Hermed er den opstillede Sætning fuldstændig bevist.

Jeg bemærker sluttelig, at, som det umiddelbart fremgaar af Beviset, den ovenstaaende Sætning [alene med Undtagelse af den ene Bemærkning om, at Y_2 helt i sit Indre (incl. Rand) vil indeholde en Cirkel med Centrum i O og Radius $e_1 + e_2$] vil bevare sin Gyldighed uforandret, hvis i denne Sætning Forudsætningen om, at O er beliggende indenfor Kurven M_1, udelades. Denne Bemærkning vil vi faa Brug for i det følgende.

§ 3. Addition af uendelig mange konvekse Kurver.

Lad $M_1, M_2, \ldots, M_n, \ldots$ være uendelig mange konvekse Kurver saaledes, at $\sum_1^{\infty} M_n$ er konvergent. Jeg skal da i denne Paragraf undersøge Punktmængden $\sum_{n=1}^{\infty} M_n$. Som tidligere vist kan vi uden at indskrænke Undersøgelsens Almindelighed antage, at Begyndelsespunktet O er beliggende indenfor enhver af Kurverne M_n; da er $\sum_{n=1}^{\infty} M_n$ tillige ubetinget konvergent, d. v. s. den uendelige Række med positive Tal $\sum_{n=1}^{\infty} \varsigma_n$, hvor ς_n betegner Maximum af Afstanden $\overline{Om_n}$, idet m_n gennemløber Kurven M_n, er konvergent; lad denne Række have Summen R; da vil alle Punkter tilhørende Punktmængden $\sum_{n=1}^{\infty} M_n$ (ligesom for ethvert N, alle Punkter tilhørende $\sum_{n=1}^{N} M_n$) være beliggende

indenfor en Cirkel (incl. Rand) med Centrum i O og Radius R. Jeg vil antage, hvad der øjensynlig er tilladeligt, og hvad der vil simplificere Undersøgelsen i væsentlig Grad, at Arealet af M_1 er $\geqq$ Arealet af enhver af Kurverne M_n ($n = 1, 2, \ldots$).

Inden jeg gaar over til den direkte Undersøgelse af Punktmængden $\sum_{n=1}^{\infty} M_n$, vil jeg først bevise følgende Sætning om Addition af et vilkaarligt endeligt Antal konvekse Kurver:

Punktmængden $\sum_{n=1}^{N} M_n$ ($N = 2, 3, \ldots$) er et Omraade (incl. Rand) begrændset enten af en enkelt konveks Kurve Y_N (og dette Tilfælde vil (for $N \geqq 3$) altid indtræffe, hvis Punktmængden $\sum_{n=1}^{N-1} M_n$ er begrændset af en enkelt konveks Kurve) eller af to konvekse Kurver Y_N og I_N, hvor I_N er beliggende helt indenfor Y_N. Idet vi sætter $Y_1 = I_1 = M_1$, vil i begge Tilfælde Kurven Y_N ($N \geqq 2$) helt omslutte Kurven Y_{N-1}, ligesom i det Tilfælde, hvor $\sum_{n=1}^{N} M_n$ er begrændset af de to Kurver Y_N og I_N, Kurven I_N vil være helt beliggende indenfor Kurven I_{N-1}. Endvidere vil, idet ς_n og e_n betegner Maksimum henholdsvis Minimum af Afstanden $\overline{Om_n}$ for m_n gennemløbende Kurven M_n, Kurven Y_N være beliggende indenfor en Cirkel (incl. Rand) med Centrum i O og Radius $\sum_{n=1}^{N} \varsigma_n$, og helt i sit Indre (incl. Rand) indeholde en Cirkel med Centrum i O og Radius $\sum_{n=1}^{N} e_n$, ligesom, i det Tilfælde, hvor $\sum_{n=1}^{N} M_n$ er begrændset af to konvekse Kurver, Kurven I_N vil være helt indeholdt i en Cirkel (incl. Rand) med Centrum i O og Radius $\varsigma_1 - \sum_{n=2}^{N} e_n$. [Heraf følger specielt, at hvis $\varsigma_1 \leqq \sum_{n=2}^{N} e_n$, vil Punktmængden $\sum_{n=1}^{N} M_n$ nødvendigvis være begrændset af kun én konveks Kurve.]

Endelig vil Arealet af det mellem Y_N og Y_{N-1} beliggende Omraade, ligesom ogsaa, i det Tilfælde, hvor $\sum_{n=1}^{N} M_n$ er begrændset af to konvekse Kurver; Arealet af det mellem I_N og I_{N-1} beliggende Omraade være $>$ Arealet af Kurven M_N. [Heraf følger specielt, at hvis Arealet af M_1 er $\leq \sum_{n=2}^{N}$ (Arealet af M_n), vil Punktmængden $\sum_{n=1}^{N} M_n$ nødvendigvis være begrændset af kun én konveks Kurve.]

Beviset for denne Sætning føres let gennem Induktion; Sætningen er rigtig for $N = 2$. Vi antager da Sætningen rigtig, naar Antallet af konvekse Kurver er $\leq N$, og vil da herudfra bevise dens Rigtighed, naar Antallet af konvekse Kurver er $N + 1$.

Lad $N'(l)$ betegne den konvekse Kurve, der fremkommer af M_{N+1} paa ganske samme Maade som den i forrige Paragraf betragtede Kurve $N(l)$ fremkom af M_2. [$N'(l)$ er altsaa den konvekse Kurve, der fremkommer ved at parallelforskyde Kurven $\div M_{N+1}$ saaledes, at O falder i det vilkaarligt givne Punkt l.] Da er det en nødvendig og tilstrækkelig Betingelse for at l tilhører Punktmængden $\sum_{n=1}^{N+1} M_n$, at Kurven $N'(l)$ har Punkter fælles med Punktmængden $\sum_{n=1}^{N} M_n$. Jeg vil nu først bevise, at Punktmængden $\sum_{n=1}^{N+1} M_n$ indeholder ethvert Punkt, der tilhører Punktmængden $\sum_{n=1}^{N} M_n$. Dette indses umiddelbart saaledes: Lad l være et vilkaarligt Punkt i Punktmængden $\sum_{n=1}^{N} M_n$; hvis l da ikke tilhørte Punktmængden $\sum_{n=1}^{N+1} M_n$, vilde $N'(l)$ ingen Punkter have fælles med $\sum_{n=1}^{N} M_n$. Idet Punktet l imidlertid ligger indenfor $N'(l)$ og indenfor eller paa Kurven Y_N, kan $N'(l)$ ikke ligge helt udenfor Y_N; endvidere kan $N'(l)$, i det Tilfælde, hvor $\sum_{n=1}^{N} M_n$ begrændses af de to lukkede Kurver Y_N og I_N, heller

ikke ligge helt indenfor I_N, da Punktet l ligger indenfor $N'(l)$ men udenfor eller paa Kurven I_N; følgelig maatte $N'(l)$, hvis den ingen Punkter havde fælles med $\sum_{n=1}^{N} M_n$, helt omslutte Kurven Y_N, men dette er umuligt, da Arealet af $Y_N >$ Arealet af $M_1 \geqq$ Arealet af M_{N+1} = Arealet af $N'(l)$. Hermed er bevist, at ethvert Punkt l tilhørende Punktmængden $\sum_{n=1}^{N} M_n$ ogsaa maa tilhøre Punktmængden $\sum_{n=1}^{N+1} M_n$.

Jeg vender mig nu til en Betragtning af de udenfor Y_N beliggende Punkter. Lad os først betragte den Punktmængde $Y_N + M_{N+1}$, der fremkommer ved Addition af de to konvekse Kurver Y_N og M_{N+1}, der begge indeholder Punktet O som indre Punkt. Da Arealet af $Y_N >$ Arealet af M_{N+1}, vil der, hvad enten Punktmængden $Y_N + M_{N+1}$ er begrændset af én eller af to konvekse Kurver, eksistere en saadan konveks Kurve Y_{N+1}, helt indeholdende Y_N i sit Indre, at de udenfor Y_N beliggende Punkter af Punktmængden $Y_N + M_{N+1}$ er de og kun de udenfor Y_N beliggende Punkter, der ligger indenfor eller paa Kurven Y_{N+1}. Idet Punktmængden $Y_N + M_{N+1}$ imidlertid er helt indeholdt i Punktmængden $\sum_{n=1}^{N+1} M_n$ (da Y_N er indeholdt i $\sum_{n=1}^{N} M_n$), vil $\sum_{n=1}^{N+1} M_n$ følgelig indeholde ethvert udenfor Y_N beliggende Punkt, der ligger indenfor eller paa Kurven Y_{N+1}. Derimod paastaar jeg, vil $\sum_{n=1}^{N+1} M_n$ ikke indeholde noget Punkt l udenfor Y_{N+1}; thi i modsat Fald vilde Kurven $N'(l)$ have Punkter fælles med $\sum_{n=1}^{N} M_n$, følgelig, da $\sum_{n=1}^{N} M_n$ er beliggende indenfor eller paa Kurven Y_N, ogsaa med Kurven Y_N, d. v. s. l maatte tilhøre Punktmængden $Y_N + M_{N+1}$, hvad der ikke er Tilfældet.

Hermed er bevist Eksistensen af en lukket konveks Kurve Y_{N+1}, der helt omslutter Y_N, saaledes at alle de udenfor Y_N beliggende Punkter af $\sum_{n=1}^{N+1} M_n$ er samtlige de udenfor Y_N belig-

gende Punkter, der ligger indenfor eller paa Kurven Y_{N+1}. Da Y_{N+1} er bestemt som „den ydre Grændsekurve“ for $Y_N + M_{N+1}$, vil endvidere det mellem Kurverne Y_N og Y_{N+1} beliggende Areal være $>$ Arealet af M_{N+1}. Tillige vil, idet Maximum af Afstanden $\overline{Oy_N}$ for y_N gennemløbende Y_N er $\leq \sum_{n=1}^{N} \varsigma_n$, og Minimum af Afstanden $\overline{Oy_N}$ for y_N gennemløbende Y_N er $\geq \sum_{n=1}^{N} e_n$, Kurven Y_{N+1} være helt indeholdt i en Cirkel (incl. Rand) med Centrum i O og Radius $\sum_{n=1}^{N+1} \varsigma_n$ samt helt i sit Indre (incl. Rand) indeholde en Cirkel med Centrum i O og Radius $\sum_{n=1}^{N+1} e_n$.

For at bestemme Punktmængden $\sum_{n=1}^{N+1} M_n$ har vi nu kun tilbage, i det Tilfælde, hvor $\sum_{n=1}^{N} M_n$ er begrændset af de to Kurver Y_N og I_N, at undersøge, hvilke Punkter indenfor I_N der tilhører $\sum_{n=1}^{N+1} M_n$. Hvis Arealet af I_N er $<$ Arealet af M_{N+1}, vil ethvert Punkt l indenfor I_N tilhøre $\sum_{n=1}^{N+1} M_n$; thi da $N'(l)$ i dette Tilfælde øjensynlig hverken kan være beliggende helt udenfor Y_N, eller helt indenfor I_N, eller helt omslutte Y_N, maa $N'(l)$ nødvendigvis have Punkter fælles med $\sum_{n=1}^{N} M_n$. Lad os dernæst antage, at Arealet af I_N er $\geq$ Arealet af M_{N+1}. Jeg betragter da den Punktmængde $I_N + M_{N+1}$, der fremkommer ved Addition af de to konvekse Kurver I_N og M_{N+1}, af hvilke i hvert Fald M_{N+1} indeholder Begyndelsespunktet O i sit Indre. Hvis denne Punktmængde $I_N + M_{N+1}$ er begrændset af kun én konveks Kurve Y', vil, da I_N er beliggende helt indenfor Y', alle Punkter indenfor I_N tilhøre Punktmængden $I_N + M_{N+1}$, altsaa yderligere tilhøre Punktmængden $\sum_{n=1}^{N+1} M_n$. Er Punktmængden $I_N + M_{N+1}$ derimod begrændset af de to konvekse Kurver Y' og I_{N+1} (helt indenfor Y'), vil alle Punkter indenfor I_N men udenfor eller paa Kurven I_{N+1} tilhøre $I_N + M_{N+1}$, altsaa yder-

29

ligere tilhøre $\sum_{n=1}^{N+1} M_N$; derimod vil intet Punkt l indenfor I_{N+1} tilhøre $\sum_{n=1}^{N+1} M_n$, thi i modsat Fald maatte Kurven $N'(l)$ indeholde Punkter af $\sum_{n=1}^{N} M_n$, følgelig, da $\sum_{n=1}^{N} M_n$ ligger helt udenfor eller paa Kurven I_N, ogsaa Punkter paa I_N, d. v. s. l maatte tilhøre Punktmængden $I_N + M_{N+1}$, i Modstrid med at l ligger indenfor Kurven I_{N+1}. Idet I_{N+1} er bestemt som „den indre Begrændsningskurve" for $I_N + M_{N+1}$, følger endvidere, at Arealet af det mellem I_N og I_{N+1} beliggende Omraade er $>$ Arealet af M_{N+1}; samt, idet Maksimum af Afstanden $\overline{Oi_N}$, for i_N gennemløbende I_N, er $\leqq \varsigma_1 - \sum_{n=2}^{N} e_n$, at Kurven I_{N+1} er helt indeholdt i en Cirkel (incl. Rand) med Centrum i O og Radius $\varsigma_1 - \sum_{n=2}^{N+1} e_n$.

Hermed er den ovenstaaende Sætning fuldstændig bevist.

Jeg gaar nu over til den direkte Undersøgelse af Punktmængden $M = \sum_{n=1}^{\infty} M_n$. Som tidligere bevist er denne Punktmængde identisk med den tilsvarende Punktmængde M^*, hvis Elementer m^* er karakteriserede gennem følgende Egenskab: svarende til et vilkaarligt Tal $\varepsilon > 0$ eksisterer der et helt Tal $N_1 = N_1(\varepsilon)$, saaledes at enhver af Punktmængderne $\sum_{n=1}^{N} M_n$ $(N \geqq N_1)$ indeholder mindst ét Punkt, hvis Afstand fra m^* er mindre end ε. Idet Punktmængden $\sum_{n=1}^{N} M_n$ er helt indeholdt i Punktmængden $\sum_{n=1}^{N+1} M_n$, indses umiddelbart, at ethvert Punkt tilhørende en af Punktmængderne $\sum_{n=1}^{N} M_n$ maa tilhøre Punktmængden M^*, altsaa tilhøre Punktmængden M.

Sammen med Punktmængden $\sum_{n=1}^{N} M_n$ vil jeg betragte den Punktmængde P_N, der bestaar af samtlige Punkter indenfor og paa den konvekse Kurve Y_N (med andre Ord, hvis $\sum_{n=1}^{N} M_n$ er begrændset af kun én konveks Kurve, er P_n identisk med $\sum_{n=1}^{N} M_n$; hvis $\sum_{n=1}^{N} M_n$ derimod er begrændset af to konvekse

Kurver, er P_N den Punktmængde, der fremkommer ved til $\sum_{n=1}^{N} M_n$ at føje alle de indenfor M_1 beliggende Punkter, der ikke tilhører $\sum_{n=1}^{N} M_n$). Da er P_N et konvekst Omraade. Lad P betegne den Punktmængde, hvis Elementer p er karakteriserede derigennem, at der til ethvert $\varepsilon > 0$ svarer et helt Tal N_1, saaledes at enhver af Punktmængderne P_N ($N \geqq N_1$) indeholder et Punkt, hvis Afstand fra p er mindre end ε. Da vil et Punkt udenfor M_1 øjensynlig da og kun da tilhøre $M = M^*$, naar det tilhører P. Jeg vil bevise, at P er et konvekst Omraade; dette indses umiddelbart saaledes: Hvis den rette Linie L har de to Punkter A og B fælles med P, vil, for alle tilstrækkelig store N, Punktmængden P_N indeholde to Punkter A_N og B_N saaledes, at Afstandene $\overline{AA_N}$ og $\overline{BB_N}$ begge er mindre end ε; idet P_N imidlertid er et konvekst Omraade, vil P_N indeholde alle Punkter paa Liniestykket $A_N B_N$; lad C være et vilkaarligt Punkt paa L mellem A og B; der vil da findes et Punkt C_N paa Liniestykket $A_N B_N$ (altsaa et Punkt C_N tilhørende P_N) saaledes, at Afstanden $\overline{CC_N}$ er mindre end ε; heraf følger imidlertid, at C tilhører Punktmængden P. Hermed er bevist, at Punktmængden P, hvis den indeholder de to Punkter A og B, indeholder hele Liniestykket AB, d. v. s. at P er et konvekst Omraade. Da P endvidere er afsluttet og helt i det Endelige beliggende samt ikke ligger helt paa en ret Linie, maa P være et Omraade (incl. Rand) begrændset af en konveks Kurve Y; og det er klart, at enhver af Kurverne Y_N maa ligge helt indenfor Kurven Y. Alle de udenfor M_1 beliggende Punkter, som tilhører Punktmængden $M = \sum_{n=1}^{\infty} M_n$, er altsaa alle de udenfor M_1 beliggende Punkter, der ligger indenfor eller paa en konveks Kurve Y, der helt omslutter M_1. Idet Y helt omslutter Y_N ($N = 1, 2, \ldots$) vil endvidere det mellem M_1 og Y beliggende Areal være $> \sum_{n=2}^{N}$ (Arealet af M_n), altsaa $\geqq \sum_{n=2}^{\infty}$ (Arealet af M_n). [At denne sidste uendelige Række er

31

konvergent, følger af Udledelsen, men er trivielt, da Arealet af $M_n \leq \pi \varsigma_n^2$, og $\sum_{n=1}^{\infty} \varsigma_n^2$ umiddelbart ses at være konvergent, da $\sum_{n=1}^{\infty} \varsigma_n$ er konvergent.] Iøvrigt vil det mellem M_1 og Y beliggende Areal være $\geq$ det mellem M_1 og Y_2 beliggende Areal $+ \sum_{n=3}^{\infty}$ (Arealet af M_n), altsaa $>$ (ikke blot $\geq$) $\sum_{n=2}^{\infty}$ (Arealet af M_n). Endelig indses, at Kurven Y vil være beliggende indenfor en Cirkel (incl. Rand) med Centrum i O og Radius $\sum_{n=1}^{\infty} \varsigma_n$ samt i sit Indre (incl. Rand) vil indeholde en Cirkel med Centrum i O og Radius $\sum_{n=1}^{\infty} e_n$.

For helt at bestemme Punktmængden $M = \sum_{n=1}^{\infty} M_n$, har vi endnu tilbage, i det Tilfælde, hvor samtlige Punktmængder $\sum_{n=1}^{N} M_n$ ($N = 2, 3, \ldots$) er begrændsede af to konvekse Kurver Y_N og I_N, at bestemme de Punkter indenfor M_1, der tilhører Punktmængden M. Ved en Betragtning ganske tilsvarende til den ovenfor benyttede beviser man her, at hvis der overhovedet findes Punkter indenfor M_1, der ikke tilhører M, vil den af disse Punkter dannede Punktmængde Q være et konvekst Omraade, der ikke indeholder noget Punkt af sin Begrændsning (altsaa specielt ikke er helt beliggende paa en ret Linie), og som ligger helt indenfor enhver af Kurverne I_N. Følgelig vil der, hvis M ikke indeholder alle Punkter indenfor Kurven M_1, eksistere en saadan konveks Kurve I, at de indenfor M_1, til M hørende, Punkter er samtlige de indenfor M_1 beliggende Punkter, der ligger udenfor eller paa Kurven I. Endvidere vises paa ganske samme Maade som ovenfor, at det mellem M_1 og I beliggende Areal er $> \sum_{n=2}^{\infty}$ (Arealet af M_n), samt at Kurven I er helt beliggende indenfor en Cirkel (incl. Rand) med Centrum i O og Radius $\varsigma_1 - \sum_{n=2}^{\infty} e_n$.

Resumerer vi de fundne Resultater, har vi bevist følgende

Sætning: Lad M_n $(n=1,2,\ldots)$ være en lukket konveks Kurve, der indeholder Begyndelsespunktet O i sit Indre, og saaledes, at $\sum_{n=1}^{\infty} M_n$ er konvergent; og lad Arealet af M_1 være $\geq$ Arealet af M_n $(n=2,3,\ldots)$. Punktmængden $M=\sum_{n=1}^{\infty} M_n$ vil da være et Omraade (incl. Rand) begrændset enten af en enkelt lukket konveks Kurve Y eller af to lukkede konvekse Kurver Y og I, hvor I ligger helt indenfor Y. Endvidere vil, idet ς_n henholdsvis e_n betegner Maksimum henholdsvis Minimum af Afstanden $\overline{Om_n}$ for m_n gennemløbende M_n, Kurven Y ligge indenfor en Cirkel (incl. Rand) med Centrum i O og Radius $\sum_{n=1}^{\infty} \varsigma_n$, samt i sit Indre (incl. Rand) indeholde en Cirkel med Centrum i O og Radius $\sum_{n=1}^{\infty} e_n$, ligesom Kurven I, i det Tilfælde, hvor M begrændses af de to Kurver Y og I, vil være beliggende indenfor en Cirkel (incl. Rand) med O som Centrum og Radius $\varsigma_1 - \sum_{n=2}^{\infty} e_n$. Endelig vil Kurven Y helt omslutte Kurven M_1 og Arealet af det mellem M_1 og Y beliggende Omraade vil være $>$ $\sum_{n=2}^{\infty}$ (Arealet af M_n), ligesom Kurven I, i det Tilfælde, hvor M er begrændset af de to Kurver Y og I, vil være beliggende helt indenfor Kurven M_1 og saaledes, at Arealet af det mellem M_1 og I beliggende Omraade vil være $> \sum_{n=2}^{\infty}$ (Arealet af M_n). Følgelig vil, hvis M er begrændset af kun den ene Kurve Y, Arealet af Omraadet M være $> \sum_{n=1}^{\infty}$ (Arealet af M_n), medens Arealet af Omraadet M i det Tilfælde, hvor M er begrændset af de to lukkede konvekse Kurver

Y og I, vil være $> 2 \cdot \sum_{n=2}^{\infty}$ (Arealet af M_n). [Af Sætningen fremgaar specielt, at hvis $\varsigma_1 \leqq \sum_{n=2}^{\infty} e_n$, eller hvis Arealet af M_1 er $\leqq \sum_{n=2}^{\infty}$ (Arealet af M_n), vil M nødvendigvis være begrændset af kun én lukket konveks Kurve].

København, 11—2—1912.

SUR L'ADDITION D'UN NOMBRE INFINI DE COURBES CONVEXES.

RÉSUMÉ.

Introduction.

Au cours de recherches relatives à une classe générale de séries infinies [1] j'ai été amené à me poser le problème suivant:

Étant donnée une suite infinie de nombres positifs $\varsigma_1, \varsigma_2, \ldots, \varsigma_n, \ldots$ telle que $\sum_{n=1}^{\infty}\varsigma_n$ soit convergente, quelles seront les valeurs prises par la fonction $F(\varphi_1, \varphi_2, \ldots, \varphi_n, \ldots) = \sum_{n=1}^{\infty}\varsigma_n e^{i\varphi_n}$ si nous supposons que les nombres réels $\varphi_1, \varphi_2, \ldots, \varphi_n, \ldots$ parcourent indépendamment les uns des autres toutes les valeurs depuis $-\infty$ jusqu'à $+\infty$ ou bien, ce qui revient évidemment au même, toutes les valeurs comprises entre 0 (inclusivement) et 2π (exclusivement)?

Il est très facile de démontrer que le théorème suivant aura lieu:

1° Au cas où la suite de nombres $\varsigma_1, \varsigma_2, \ldots, \varsigma_n, \ldots$ ne comprend pas un élément, ς_n, supérieur à la somme de tous les autres, la fonction F prendra toutes les valeurs z pour lesquelles $|z| \leq \sum_{n=1}^{\infty}\varsigma_n$, et ces valeurs seulement; en d'autres termes: elle prendra toutes les valeurs complexes dont la représentation dans le plan complexe soit située à l'intérieur ou sur la circonférence d'un cercle ayant l'origine pour centre et de rayon $\sum_{n=1}^{\infty}\varsigma_n$.

2° Dans le cas où la suite $\varsigma_1, \varsigma_2, \ldots, \varsigma_n, \ldots$ comprend au contraire un élément ς_N supérieur à la somme de tous les autres, la fonction F prendra toutes les valeurs z pour lesquelles

[1] Voir mon mémoire intitulé: *Lösung des absoluten Konvergenzproblems einer allgemeinen Klasse Dirichletschen Reihen*, qui va paraître dans les Acta Mathematica.

$\varsigma_N - \sum_{n \neq N} \varsigma_n \leq |z| \leq \sum_{n=1}^{\infty} \varsigma_n$, et ces valeurs seulement; en d'autres termes: elle prendra toutes les valeurs complexes dont la représentation dans le plan complexe soit située à l'intérieur ou sur le bord d'un anneau circulaire ayant son centre à l'origine et dont les rayons extérieur et intérieur soient respectivement $\sum_{n=1}^{\infty} \varsigma_n$ et $\varsigma_N - \sum_{n \neq N} \varsigma_n$.

En tenant compte de ce fait que, φ_n variant de 0 à 2π, le point du plan complexe qui correspond au nombre $\varsigma_n e^{i\varphi_n}$ parcourra une circonférence ayant l'origine pour centre et de rayon ς_n, le même théorème peut évidemment, sous une forme un peu moins précise, il est vrai, s'énoncer comme suit: L'ensemble M de points du plan complexe qui s'obtient par l'addition d'une infinité de circonférences ayant pour centre l'origine et dont les rayons forment une série convergente (c'est-à-dire: l'ensemble M de points qui correspond à l'ensemble de tous les nombres complexes $z = \sum_{n=1}^{\infty} z_n$, où z_n est le nombre complexe qui correspond à un point arbitraire situé sur la $n^{\text{ième}}$ circonférence) formera ou l'intérieur d'un cercle (y compris la circonférence) ayant son centre à l'origine, ou bien celui d'un anneau circulaire (y compris les circonférences intérieure et extérieure) ayant également l'origine pour centre.

L'étude de certaines fonctions auxquelles on a affaire dans la théorie analytique des nombres premiers m'a conduit au problème plus général de savoir quel ensemble de points résulterait d'une addition portant non pas en particulier sur une infinité de circonférences ayant l'origine pour centre, mais sur une infinité de courbes convexes fermées arbitraires. Je démontrais alors une proposition générale, correspondant à celle que je viens d'énoncer sur l'addition d'une infinité de circonférences et qui disait: que l'ensemble de points qu'on obtient dans le plan complexe par l'addition d'une infinité de courbes convexes fermées sera ou l'intérieur (y compris la frontière) d'une courbe convexe fermée, ou bien un domaine (y compris la frontière) limité par deux courbes convexes fermées dont l'une sera intérieure à l'autre.

La présente Note contient la formulation exacte et la démonstration de cette proposition. Comme il s'agit au fond d'un

théorème de géométrie il m'a semblé plus naturel de donner la démonstration sous sa forme géométrique que d'exposer le théorème et la démonstration sous leur aspect arithmétique en introduisant des nombres complexes.

Au § 1 je fais quelques remarques générales d'orientation sur l'addition d'une infinité d'ensembles de points; le § 2 traite de l'addition de deux courbes convexes fermées; le § 3, enfin est relatif à l'addition d'une infinité de courbes convexes fermées.

Dans un mémoire subséquent je me propose d'appliquer les résultats de la présente recherche aux fonctions appartenant à la théorie analytique des nombres premiers, et notamment à la fonction zéta de Riemann.

§ 1. Quelques remarques générales sur l'addition d'une infinité d'ensembles de points.

Soient p_1 et p_2 deux points dans un plan d'origine O, et entendons, comme on le fait habituellement, par la somme $p_1 + p_2$, le point du plan qui forme le sommet opposé à O dans la parallélogramme où les deux autres sommets sont p_1 et p_2. Le concept de somme de deux points s'étend immédiatement en celui de somme d'un nombre fini quelconque de points. Soient $p_1, p_2, \ldots, p_n, \ldots$ une suite infinie de points situés dans le plan; la série infinie $\sum_{n=1}^{\infty} p_n$ sera alors dite convergente et avec la somme p si le point $q_N = \sum_{n=1}^{N} p_n$ se rapproche du point limite fini et déterminé p, N croissant indéfiniment; c. à d. si $\lim_{N=\infty} q_N = p$.

Soient M_1, aux éléments (points) m_1, et M_2 aux éléments m_2, deux ensembles de points donnés, situés dans le plan; je désigne par la somme $M_1 + M_2$ l'ensemble de points constitué par tous les points de forme $m_1 + m_2$; en d'autres termes: l'ensemble de points $M_1 + M_2$ contient tous les points susceptibles d'être construits comme somme d'un élément de M_1 et d'un élément de M_2, et ces points seulement.

Le concept de somme de deux ensembles de points s'étend immédiatement en celui de somme d'un nombre fini quelconque d'ensembles de points.

Supposons donnée une suite infinie d'ensembles de points: M_1 aux éléments m_1, M_2 aux éléments m_2, ..., M_n aux élé-

37

ments m_n, ... La série infinie $\sum_{n=1}^{\infty} M_n$ sera dite convergente dans les cas où toute série infinie $\sum_{n=1}^{\infty} m_n$ (où m_n désigne un point quelconque de M_n) sera convergente; d'une série $\sum_{n=1}^{\infty} M_n$ convergente nous dirons qu'elle représente (a la somme) M, M désignant l'ensemble de points qui aura pour éléments m tous les points de la forme $\sum_{n=1}^{\infty} m_n$ (chaque point n'étant compté qu'une fois) et ces points seulement.

Suivent des remarques et des théorèmes relatifs à la somme d'une infinité d'ensembles de points. Citons, à titre d'exemple, le théorème suivant: Étant donnée une série convergente $\sum_{n=1}^{\infty} M_n$ d'ensembles de points M_n situés tout entiers dans le fini (c.-à-d. à l'intérieur d'un cercle ayant l'origine pour centre et de rayon R_n) et fermés (c.-à-d. contenant ses points limite) l'ensemble de points $M = \sum_{n=1}^{\infty} M_n$ constituera également un ensemble de points fermé, situé tout entier dans le fini.

Citons en outre le théorème que voici: Étant donnée une suite infinie d'ensembles de points $M_1, M_2, \ldots, M_n, \ldots$ telle que $\sum_{n=1}^{\infty} M_n$ soit convergente; désignons par M^* l'ensemble de points composé de points m^* pour lesquels il soit possible de prendre, dans l'ensemble de points $L_r = \sum_{n=1}^{r} M_n$ un point l_r tel que $\lim_{r=\infty} l_r = m^*$. L'ensemble de points M^* comprendra, comme éléments m^*, tous les points contenus dans l'ensemble de points $M = \sum_{n=1}^{\infty} M_n$ ou constituant des points limite de l'ensemble M, et ces points seulement. De là résulte spécialement qu'au cas où l'ensemble de points $M = \sum_{n=1}^{\infty} M_n$ est un ensemble de points fermé — et, à fortiori, si chacun des ensembles de

points M_n est fermé et situé tout entier dans le fini, M sera égal à M^*.

Citons enfin le théorème suivant qui nous aidera à comprendre la structure des ensembles de points M_n d'une série convergente $\sum_{n=1}^{\infty} M_n$: Soit M_n ($n = 1, 2, \ldots$) un ensemble de points situé tout entier dans le fini et soit $\sum_{n=1}^{\infty} M_n$ convergente. Il existera une suite de points $c_1, c_2, \ldots, c_n, \ldots$ où $\sum_{n=1}^{\infty} c_n$ sera convergente et une suite correspondante de nombres positifs $r_1, r_2, \ldots, r_n, \ldots$ où $\sum_{n=1}^{\infty} r_n$ sera convergente tel que l'ensemble de points M_n ($n = 1, 2, \ldots.$) sera situé tout entier à l'intérieur du cercle de centre c_n et de rayon r_n.

De ce dernier théorème se déduit immédiatement la remarque importante: Dans l'étude des séries convergentes $\sum_{n=1}^{\infty} M_n$ dont chaque terme soit un ensemble de points situé tout entier dans le fini, on pourra se borner à considérer les séries absolument convergentes (une serie est dite absolument convergente quand le terme général M_n se trouve contenu tout entier dans un cercle ayant l'origine pour centre et de rayon r_n tel que la série à termes positifs, $\sum_{n=1}^{\infty} r_n$ soit convergente), car à toute série convergente $\sum_{n=1}^{\infty} M_n$ dont les termes soient tout entiers situés dans le fini correspondra évidemment, en vertu du théorème précédent une série absolument convergente $\sum_{n=1}^{\infty} M'_n$ où l'ensemble de points M'_n sera congruent à l'ensemble de points M_n et tel que l'ensemble de points $M = \sum_{n=1}^{\infty} M_n$ se déduit immédiatement de l'ensemble de points $M' = \sum_{n=1}^{\infty} M'_n$ à l'aide d'un simple déplacement parallèle.

Le § 1 se termine par la considération du cas spécial où chacun des ensembles de points M_n de la série convergente $\sum_{n=1}^{\infty} M_n$ est une courbe de Jordan. On montre aisément, en

se basant sur les théorèmes précédents qu'il suffit de considérer le cas où l'origine O se trouve intérieurement à chacune des courbes M_n et, aussi, que dans ce cas $\sum_{n=1}^{\infty} M_n$ sera absolument convergente.

§ 2. Sur l'addition de deux courbes convexes.

Ce paragraphe contient le démonstration du théorème suivant:

Soient M_1 et M_2 deux courbes convexes contenant toutes deux l'origine O comme point intérieur et soit l'aire de M_1 [1] $\geqq$ l'aire de M_2. L'ensemble de points $M_1 + M_2$ sera ou un domaine (y compris la frontière) limité par une seule courbe convexe Y_2 et tel que la courbe M_1 se trouve située tout entière à l'intérieur de la courbe Y_2, ou bien $M_1 + M_2$ représentera un domaine (y compris la frontière) limité par deux courbes convexes Y_2 et I_2 dont I_2 sera situé tout entière à l'intérieur de Y_2 et de telle sorte que la courbe M_1 soit contenue toute entière à l'intérieur de la courbe Y_2 tout en contenant I_2 tout entière à son intérieur. En outre, en désignant par ς_i $(i = 1, 2)$, respectivement par e_i $(i = 1, 2)$, le maximum, respectivement le minimum, de la distance $\overline{Om_i}$, où m parcourt la courbe M_i, la courbe Y_2 sera contenue tout entière à l'intérieur d'un cercle (y compris la circonférence) de centre O et de rayon $\varsigma_1 + \varsigma_2$ et contiendra à son intérieur (y compris la frontière) un cercle de centre O et de rayon $e_1 + e_2$, de même qu'au cas où $M_1 + M_2$ est limité par deux courbes convexes, la courbe I_2 se trouvera contenue tout entière dans un cercle (y compris la circonférence) de centre O et de rayon $\varsigma_1 - e_2$. Enfin, l'aire du domaine compris entre M_1 et Y_2 sera plus grande que l'aire de M_2 de même qu'au cas où $M_1 + M_2$ est limité par deux courbes convexes l'aire du domaine compris entre M_1 et I_2 sera plus grand

[1] Par l'aire d'une courbe convexe nous entendons l'aire du domaine, forcément existant, qui est situé intérieurement à la courbe en question.

que l'aire de M_2; il en résulte qu'au cas où M_1+M_2 est limité par une seule courbe convexe, l'aire du domaine M_1+M_2 sera plus grande que l'aire de M_1+ l'aire de M_2, tandis qu'au cas où M_1+M_2 est limité par deux courbes convexes, l'aire du domaine M_1+M_2 sera $> 2\cdot$(l'aire de M_2).

§ 3. Sur l'addition d'une infinité de courbes convexes.

Extension du théorème sur l'addition de deux courbes convexes, démontré au § 2, en un théorème plus générale portant sur l'addition d'un nombre fini arbitraire de courbes convexes. Démonstration de la proposition générale suivante sur l'addition d'une infinité de courbes convexes:

Étant donnée une courbe convexe fermée M_n $(n=1,2,\ldots)$ contenant l'origine O à son intérieur et telle que $\sum_{n=1}^{\infty} M_n$ soit convergente, soit l'aire de $M_1 \geqq$ l'aire de M_n $(n=2,3,\ldots)$; l'ensemble de points $M=\sum_{n=1}^{\infty} M_n$ sera un domaine (y compris la frontière) limité, ou par une seule courbe convexe fermée Y, ou par deux courbes convexes fermées Y et I dont I se trouvera tout entière intérieurement à Y. En outre nous aurons, en désignant par ς_n et par e_n, respectivement, le maximum et le minimum, respectivement, de la distance $\overline{Om_n}$, pour m_n parcourant M_n, la courbe Y située à l'intérieur d'un cercle (y compris la circonférence) de centre O et de rayon $\sum_{n=1}^{\infty}\varsigma_n$ et contenant à son intérieur (y compris la frontière) un cercle de centre O et de rayon $\sum_{n=1}^{\infty} e_n$, dé même que nous aurons, au cas où M se trouve limité par les deux courbes Y et I, la courbe I située à l'intérieur d'un cercle (y compris la circonférence) de centre O et de rayon $\varsigma_1-\sum_{n=2}^{\infty} e_n$. Enfin, la courbe Y entourera la courbe M_1 tout entière

et l'aire du domaine compris entre M_1 et Y sera $> \sum_{n=2}^{\infty}$ (aire M_n), de même qu'au cas où M se trouve limité par les deux courbes Y et I, la courbe I sera située tout entière à l'intérieur de la courbe M_1 et de telle sorte que l'aire du domaine compris entre M_1 et I sera $> \sum_{n=2}^{\infty}$ (aire M_n). Par conséquent, si M est limité par la seule courbe Y, l'aire du domaine M sera $> \sum_{n=1}^{\infty}$ (aire M_n), tandis que dans le cas où est limité par les deux courbes convexes Y et I, l'aire du domaine M sera $> 2 \cdot \sum_{n=2}^{\infty}$ (aire M_n). [De cette proposition il s'ensuit spécialement qu'en supposant $\varsigma_1 \leqq \sum_{n=2}^{\infty} e_n$, ou bien, en faisant l'aire de $M_1 \leqq \sum_{n=2}^{\infty}$ (aire M_n), on aura forcément M limité par une seule courbe convexe fermée.]

Om Addition af konvekse Kurver med givne Sandsynlighedsfordelinger.

(En mængdeteoretisk Undersøgelse).

Af Harald Bohr.

1. Den Opgave at addere konvekse Kurver »belagt« med Sandsynlighed møder man ved Studiet af Primtalteoriens Funktioner*). Vi skal i denne lille Afhandling nøjes med at betragte Addition af to saadanne Kurver (og desuden kun beskæftige os med Eksistensspørgsmaal). De Hjælpemidler, der benyttes, er hentet fra den moderne Mængdelære; naar der i det følgende tales om en »maalelig« Mængde eller en »integrabel« Funktion, skal disse Begreber derfor overalt tages i den *Lebesgue*'ske Betydning.

2. Vi tænker os givet to vilkaarlige lukkede konvekse Kurver K_1 og K_2, beliggende i en kompleks z-Plan og begge omsluttende Planens Nulpunkt O; for Simpelheds Skyld vil vi antage, at Kurverne ingen Knæk har, altsaa at der i ethvert Punkt er en bestemt Tangent, og at de ingen retlinede Stykker indeholder. Kurven K_i ($i = 1,2$) tænker vi os opgivet ved en kontinuert Parameterfremstilling $z_i = f_i(t_i)$, hvor t_i gennem-

*) En Oversigt over de Undersøgelser, der har ført til geometriske Problemer af den nævnte Art, er givet af Forfatteren i et Foredrag: »Über diophantische Approximationen und ihre Anwendungen auf Dirichlet'sche Reihen, besonders auf die Riemann'sche Zetafunktion«, holdt paa den skandinaviske Matematikerkongres i Helsingfors 1922.

Dedicated to J. Hjelmslev.

løber et Interval $0 \leqq t_i \leqq 1$ [eller, idet $f_i(0) = f_i(1)$, et Interval, der er foldet omkring en Cirkel af Længden 1]. Denne Parameterfremstilling skal opfattes saaledes, at den ikke blot bestemmer os Kurven K_i som Helhed, men tillige giver os en **Sandsynlighedsfordeling** paa K_i, idet vi ved Sandsynligheden for, at et Kurvepunkt z_i falder paa en vilkaarligt given Bue B paa K_i, vil forstaa det Tal β, der angiver Længden af det tilsvarende Parameterinterval B. Vi vil forudsætte, at den givne Afbildning af Kurven K_i paa Intervallet $0 \leqq t_i \leqq 1$ er saaledes beskaffen, at den nævnte »Buesandsynlighed« kan føres tilbage til en »**Punktsandsynlighed**«, d. v. s. at der eksisterer en i Punkterne z_i paa K_i defineret Funktion $s_i(z_i)$ saaledes, at Sandsynligheden β for at et Punkt falder paa en given Bue B paa K_i (altsaa Længden β af det tilsvarende Parameterinterval B) er lig med $\int s_i(z_i)\,dl_i$ taget over Buen B, idet dl_i angiver Buedifferentialet paa K_i. For at simplificere Fremstillingen vil vi endvidere antage, at denne Punktsandsynlighed $s_i(z_i)$ er en begrænset Funktion af z_i, hvoraf følger, at Forholdet mellem Længden af et vilkaarligt Parameterinterval og Længden af den tilsvarende Bue paa K_i er $<$ en fast Konstant.

3. **Vi adderer** nu Kurverne K_1 og K_2, d. v. s. vi danner den Punktmængde Ω i z-Planen, hvis Punkter z fremkommer ved Addition af et vilkaarligt Punkt z_1 paa K_1 og et vilkaarligt Punkt z_2 paa K_2. Om denne Punktmængde Ω har Forfatteren tidligere vist*), at den er et afsluttet Omraade, der begrænses af en enkelt lukket konveks Kurve eller af to lukkede konvekse Kurver, af hvilke den ene omslutter den anden. Vi skal nu undersøge »**Sandsynligheden**« for, at Sumpunktet z falder indenfor en vis given Del af Ω. Hertil karakteriserer vi en Sum $z_1 + z_2$ af et Punkt z_1 paa K_1 og et Punkt z_2 paa K_2 ved at opgive de to til z_1 og z_2 svarende Punkter t_1 og t_2 i Parameterintervallerne $0 \leqq t_1 \leqq 1$ og $0 \leqq t_2 \leqq 1$, eller anderledes udtrykt ved det Punkt i Enhedskvadratet E i en Plan med et fastlagt retvinklet Koordinatsystem, der har Koordinaterne (t_1, t_2). [Herved skal naturligvis »tilsvarende« Punkter paa modsatte Kvadratsider opfattes som et og samme Punkt; den geometriske Sammenhæng opfattes derfor bedst

*) H. Bohr: Om Addition af uendelig mange konvekse Kurver, Videnskabernes Selskabs Oversigter, 1913 (S. 325—358).

ved at tænke sig Punkterne afbildet paa en Torus i Stedet for paa et Kvadrat]. Herved er bestemt en Afbildning af Punkterne $T: (t_1, t_2)$ i Enhedskvadratet E paa Punkterne z i Omraadet Ω; denne Afbildning er imidlertid ikke enentydig — som den Afbildning Talen var om ved den enkelte konvekse Kurve — idet der vel til ethvert Punkt T i E svarer et og kun eet Punkt z i Ω (som varierer kontinuert med T), men ikke omvendt til et givet z svarer kun eet T, idet et Punkt z i Ω i Almindelighed paa flere Maader kan dannes som en Sum $z_1 + z_2$. Lad nu M være en vilkaarlig maalelig Mængde i z-Planen og M den »tilsvarende« Punktmængde i Enhedskvadratet E; ved Sandsynligheden for, at Sumpunktet z falder i M, vil vi da forstaa det Tal μ, der angiver Maalet af Punktmængden M. Vort Formaal er at bevise — eller rettere skitsere Beviset for — følgende Sætning:

Til enhver maalelig Mængde M i Ω eksisterer der en Sandsynlighed μ, d. v. s. den tilsvarende Mængde M i E er ligeledes maalelig; og denne »Mængdesandsynlighed« $\mu = \mu(M)$ kan føres tilbage til en »Punktsandsynlighed« $S(z)$, d. v. s. der findes en — i Almindelighed ikke begrænset — Punktfunktion $S(z)$, saaledes at Maalet af M er lig med Integralet af $S(z)$ udstrakt over Mængden M.

4. Vi betragter først en særlig simpel Type paa en Punktmængde M i z-Planen, nemlig et vilkaarligt akseparallelt Rektangel R, som vi antager aabent (d. v. s. Randen regnes ikke med). Hertil svarer der øjensynlig, fordi z er en kontinuert Funktion af (t_1, t_2), en aaben Punktmængde R i Enhedskvadratet E (d. v. s. ethvert Punkt i R er »indre« Punkt i R), og R er derfor — som enhver aaben Punktmængde — maalelig. Herved er defineret en »Rektangelfunktion« $\varphi(R)$, d. v. s. til ethvert (aabent, akseparallelt) Rektangel R svarer der et bestemt Tal $\varphi(R)$, nemlig Maalet ρ paa den tilsvarende Punktmængde R. For at vise, at denne Rektangelsandsynlighed $\varphi(R)$ kan føres tilbage til en Punktsandsynlighed $S(z)$, og samtidig dermed vise, at Sætningen ovenfor gælder i sin fulde Udstrækning (d. v. s. ikke blot for Rektangler R, men ogsaa for vilkaarlige maalelige Mængder M), behøver vi, som man let indser udfra *Lebesgue*'s fundamentale Undersøgelse over additive Mængdefunktioner*), blot at vise, at $\varphi(R)$ opfylder følgende to Betingelser:

*) Se f. Eks. Carathéodory's udmærkede Lærebog: Vorlesungen über reelle Funktionen, Leipzig und Berlin (1918). Se særlig § 453.

A. Deles et Rektangel R ved en Linie parallel med en af Siderne i to mindre Rektangler R_1 og R_2, da er $\varphi(R) = \varphi(R_1) + \varphi(R_2)$.

Denne Betingelse er øjensynlig opfyldt for vor Funktion $\varphi(R)$. Thi idet den til Rektanglet R svarende Punktmængde R jo simpelthen fremkommer ved til de til R_1 og R_2 svarende Punktmængder R_1 og R_2 at føje den Mængde Λ, der svarer til Delingsliniestykket L, gælder det blot om at vise, at denne Mængde Λ har Maalet o. Og at dette er Tilfældet, følger straks — idet man, som bekendt, kan bestemme Maalet af en maalelig »todimensional« Mængde Λ ved først at maale den paa en (variabel) lodret Linie liggende »endimensionale« Delmængde og derefter udføre en simpel Integration — deraf, at Λ for det første er m a a l e l i g (fordi den jo dannes udfra de maalelige Mængder R, R_1, R_2 ved Processen $\mathsf{R} - \mathsf{R}_1 - \mathsf{R}_2$) og for det andet skæres af enhver lodret Linie $t_1 =$ konst. i e n l i n e æ r P u n k t m æ n g d e a f M a a l e t o (nemlig i højst to Punkter, idet den konvekse Kurve i z-Planen, der fremkommer af K_2 ved at parallelforskyde den saaledes, at Nulpunktet falder i det til den betragtede faste t_1-Værdi svarende Punkt z_1 paa K_1, jo kun skærer L i højst to Punkter).

B. Rektangelfunktionen $\varphi(R)$ er »absolut kontinuert«, d. v. s. til ethvert givet $\varepsilon > 0$ skal der eksistere et $\delta > 0$, saaledes at hvis $R_1, \cdots R_n$ er et vilkaarligt endeligt Antal Rektangler uden fælles Punkter, hvis samlede Areal A er $< \delta$, da er $\varphi(R_1) + \cdots + \varphi(R_n) < \varepsilon$.

Beviset for, at vor Rektangelfunktion $\varphi(R)$ ogsaa opfylder Betingelsen B, kræver en lidt nøjere Undersøgelse af den til Grund liggende Afbildning af Omraadet Ω paa Enhedskvadratet E. For at finde de forskellige Maader, hvorpaa et givet Punkt z i Ω kan dannes som en Sum $z = z_1 + z_2$, altsaa for at finde de Punkter (t_1, t_2) i E, der svarer til z, drejer vi Kurven K_2 180^0 om Punktet O og parallelforskyder den dernæst saaledes, at O falder i z; de søgte »Opløsninger« af z i $z_1 + z_2$ vil da øjensynlig netop svare til de forskellige Punkter, hvori denne Kurve, som vi vil betegne med $\overline{K}_2(z)$, skærer Kurven K_1 (se Fig. 1). Dette Antal Skæringspunkter — og dermed Antallet af z's Billedpunkter i E — kan være meget stort, naar z ligger saaledes, at $\overline{K}_2(z)$ »næsten« rører K_1 (og det kan blive uendelig stort, naar $\overline{K}_2(z)$ virkelig rører K_1). Der er imidlertid, for enhver Beliggenhed af z, kun et begrænset Antal Skæringspunkter mellem K_1 og $\overline{K}_2(z)$, hvori Skæringsvinklen v (d. v. s. den mindste Vinkel mellem Tangenterne) er $>$ en given Vinkel v_0, nemlig, som en elementær Betragtning viser, færre end $2\pi : v_0$. Vi opfatter nu denne Skæringsvinkel v som en Funktion af de tilsvarende Vektorer z_1 og z_2, d. v. s. af det tilsvarende

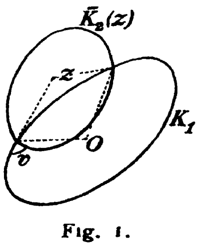


Fig. 1.

Punkt (t_1, t_2) i E; denne Funktion $v = v(t_1, t_2)$, der aabenbart ogsaa kan karakteriseres som Vinklen imellem Tangenterne til de givne Kurver K_1 og K_2 i de to til Parameterværdierne t_1 og t_2 svarende Punkter z_1 og z_2, er øjensynlig en kontinuert Funktion af (t_1, t_2). De »kritiske« Punkter i E er nu dem, hvori $v(t_1, t_2) = 0$, d. v. s. hvor t_1 og t_2 svarer til Røringspunkterne for parallele Tangenter til K_1 og K_2. Idet K_1 og K_2 er konvekse, og deres Tangenter derfor drejer sig monotont, bestaar den af disse kritiske Punkter dannede Punktmængde i E aabenbart af to (paa Torus lukkede) Kurver — svarende til de to forskellige Tangenter til K_2 parallele med en given Tangent til K_1 —, langs hvilke Kurver Ordinaten t_2 er en monoton Funktion af Abscissen t_1 (se Fig. 2, hvor en af Kurverne er angivet ved den stærkt optrukne Linie). Disse »skikkelige« monotone Kurver har naturligvis Maalet 0, og vi kan udskære smalle Parallelstrimler omkring dem (den skraverede Strimmel paa Fig. 2), der omslutter dem saa snævert, f. Eks. med et samlet Areal $< \varepsilon : 2$, at vi i det følgende kan se helt bort fra disse Strimler, d. v. s. kun behøver at betragte den Del E^* af E, der ligger udenfor dem, og i hvis Punkter Funktionen (Vinklen) $v(t_1, t_2)$ aabenbart (af Kontinuitetsgrunde) er $>$ et fast positivt Tal $v_0 = v_0(\varepsilon)$.

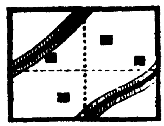


Fig. 2.

Vi tænker os nu givet en vilkaarlig Samling af Rektangler $R_1, \cdots R_n$ i z-Planen (som vi kan antage alle er »meget smaa« og ikke »altfor aflange«, idet vi ellers blot behøver at skære dem itu i flere mindre Rektangler), og vil da først undersøge den Punktmængde R* i E^*, hvori et enkelt af disse Rektangler, R, afbildes. Til et vilkaarligt Punkt z i R svarer der kun et begrænset Antal Punkter i E^* (idet der jo findes færre end $2\pi : v_0$ Skæringspunkter mellem K_1 og $\overline{K}_2(z)$, hvori Skæringsvinklen v er $> v_0$), og man finder endvidere ved en nærmere Analyse (se Fig. 3), at de Skæringspunkter mellem K_1 og $\overline{K}_2(z)$, hvori Vinklen v er $> v_0$, ved en »meget lille« Forskydning Δz af z kun forskydes smaa Buestykker Δl_1 og Δl_2 paa K_1 og $\overline{K}_2$, der er af højst »samme Størrelsesorden« som Forskydningen Δz, og at de til z svarende Punkter (t_1, t_2) i E^* derfor ogsaa (fordi $\Delta t_i : \Delta l_i$ er begrænset) kun undergaar Forskydninger af denne Størrelsesorden Δz. Heraf slutter man, at den til det betragtede lille Rektangel R svarende Punktmængde R* i E^* kan indesluttes i et begrænset Antal N smaa ligestore Kvadrater (de sorte Kvadrater paa Fig. 2), hvis Sider er af samme Størrelsesorden som Rektanglets Diameter d, d. v. s. som er $< k \cdot d$, hvor k, ligesom N, kun afhænger af v_0 (d. v. s. af ε), men ikke af Rektanglets Beliggenhed, og Maalet af denne Punktmængde R* er

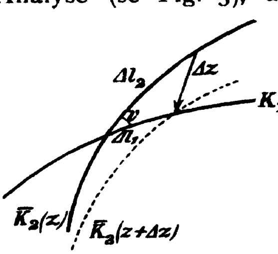


Fig. 3.

derfor $< N(kd)^2$, altsaa $< c \cdot a$, hvor a angiver Arealet af Rektanglet R og $c = Nk^2$ kun afhænger af ε. Har den givne Rektangelsamling i z-Planen det samlede Areal A, vil den tilsvarende Punktmængde i E^* derfor have et Maal $< c \cdot A$, og dette Maal kan altsaa gøres vilkaarlig lille ($< \varepsilon : 2$) ved blot at vælge Tallet A tilstrækkelig lille ($< \varepsilon : 2c$). Hermed er Beviset for Betingelsen B fuldført.

5. I en senere Afhandling skal jeg komme udførligt tilbage til det her behandlede Problem og bl. a. ogsaa betragte Addition af et vilkaarligt Antal, n, konvekse Kurver — hvorved de omhandlede Sandsynligheder defineres som Maalet af Punktmængder i et n-dimensionalt Rum — samt angive eksplicite Udtryk for de optrædende Punktsandsynligheder.

OM SANDSYNLIGHEDSFORDELINGER VED ADDITION AF KONVEKSE KURVER

AF

HARALD BOHR OG BØRGE JESSEN

MED 34 FIGURER

D. KGL. DANSKE VIDENSK. SELSK. SKRIFTER, NATURVIDENSK. OG MATHEM. AFD., 8. RÆKKE, XII. 3

KØBENHAVN
HOVEDKOMMISSIONÆR: ANDR. FRED. HØST & SØN, KGL. HOF-BOGHANDEL
BIANCO LUNOS BOGTRYKKERI
1929

FORORD

AF

HARALD BOHR

Udgangspunktet for Undersøgelserne i den foreliggende Afhandling er nogle Arbejder af mig for en Række Aar tilbage vedrørende den Riemannske Zetafunktion. I disse Arbejder blev det vist, ved Hjælp af Læren om Diofantiske Approximationer, at Spørgsmaalet om Fordelingen af Zetafunktionens Værdier paa det nøjeste hang sammen med visse rent geometriske Spørgsmaal om *Addition af konvekse Kurver*. Undersøgelserne, som delvis blev udført i Samarbejde med Professor Courant i Göttingen, førte vel til Resultater, der beskrev den paagældende Værdifordeling i store Træk, men de blev ikke den Gang ført saa fuldstændig igennem, som Metoderne tillod; dog fik jeg Lejlighed til i et Foredrag paa den skandinaviske Matematikerkongres i Helsingfors 1922 at angive Hovedlinierne for en saadan afsluttende Undersøgelse. Arbejdet blev imidlertid afbrudt ved, at jeg kom ind paa andre Problemer, og først for et Aarstid siden har jeg genoptaget det. Det drejede sig i første Linie om et geometrisk Forarbejde vedrørende Addition af konvekse Kurver *med givne Sandsynlighedsfordelinger*. Jeg var saa heldig til dette Arbejde at kunne knytte Hr. Jessen som Medarbejder. For det oprindelige Formaal, Anvendelsen paa Zetafunktionens Teori, vilde det have været tilstrækkeligt at gennemføre disse geometriske Undersøgelser for visse specielle Typer af konvekse Kurver. Naar det foreliggende Arbejde imidlertid ikke indskrænker sig til at behandle saadanne specielle Kurver, men har en langt videregaaende Karakter, skyldes det Hr. Jessen, hvem det er lykkedes paa naturlig Maade at afgrænse et Omraade af almindelige konvekse Kurver, for hvilke de paagældende Undersøgelser kunde gennemføres, uden at dette førte til Komplikationer i Beviserne, ja snarere medførte større Overskuelighed. Afhandlingen har herved faaet en væsentlig anden Karakter end oprindelig tænkt, og turde, som den foreligger, paaregne en vis almindelig geometrisk Interesse, ganske uafhængig af Anvendelserne indenfor Funktionsteorien. Ogsaa Gennemførelsen af en Række geometriske Enkeltundersøgelser skyldes udelukkende Hr. Jessen, ligesom det er ham, der har foretaget Udarbejdelsen af den hele Afhandling og paataget sig alt Arbejdet med Tegningen af de talrige Figurer og Korrekturlæsningen.

I nogle følgende fælles Arbejder vil Hr. Jessen og jeg udførligt behandle Anvendelserne af de her vundne Resultater paa Problemet om Zetafunktionens Værdifordeling, der som ovenfor nævnt var det oprindelige Udgangspunkt for vort Arbejde.

KAPITEL I.

Om konvekse Kurver.

Forinden vi paabegynder Behandlingen af vort egentlige Emne: Addition af konvekse Kurver med givne Sandsynlighedsfordelinger, skal vi i dette Kapitel (tildels uden Beviser) give en kort Oversigt over de Sætninger om konvekse Jordankurver, som vi især kommer til at anvende i det følgende.

Almindelige Bemærkninger om konvekse Kurver.

1. Ved en *Jordankurve* forstaas en lukket kontinuert Kurve uden Dobbeltpunkter, eller præcisere: en Punktmængde, der kan afbildes enentydig og kontinuert paa en Cirkelperiferi. En Jordankurve deler som bekendt Mængden af alle de af Planens Punkter, der ikke er beliggende paa selve Kurven, i to Omraader, et indenfor Kurven og et udenfor Kurven beliggende Omraade, for hvilke Kurven selv er den fælles Begrænsning. Betegnes Kurven ω, vil vi for disse Omraader anvende Betegnelserne $I(\omega)$ og $Y(\omega)$. Med $\overline{I}(\omega)$ og $\overline{Y}(\omega)$ vil vi betegne de afsluttede Mængder, som fremgaar af $I(\omega)$ og $Y(\omega)$ ved Tilføjelse af Begrænsningen ω.

En ret Linie kaldes *Støttelinie* for en afsluttet Punktmængde i Planen, naar den indeholder mindst et Punkt af Mængden, og denne er beliggende helt i den ene af de to afsluttede Halvplaner, som begrænses af Linien. En lukket Polygon kaldes konveks, naar enhver Linie, som forbinder to paa hinanden følgende Vinkelspidser, er Støttelinie for Polygonen; denne Definition udelukker ikke, at tre eller flere paa hinanden følgende Vinkelspidser i Polygonen ligger paa ret Linie. *En Jordankurve kaldes konveks, naar enhver Polygon, der opstaar ved at man forbinder en Række efter hinanden følgende Punkter* $P_1, P_2, \ldots, P_n, P_{n+1} = P_1$ *af Kurven, er konveks* (se Fig. 1).

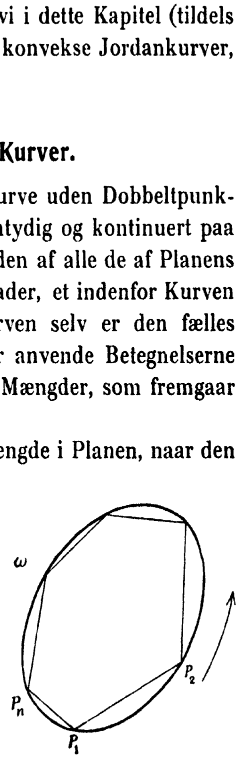


Fig. 1.

Lad der være givet en konveks Jordankurve ω. Det er en simpel Følge af Konveksiteten, at der i ethvert Punkt P af Kurven findes en Halvtangent til hver Side, bestemt som Grænsestilling for en Halvlinie, der udgaar fra P, og som indeholder et Punkt af Kurven, der fra en bestemt Side konvergerer mod P. De to Halvtangenter i Punktet P danner en Vinkel mindre end eller lig med π, hvori Kurven er beliggende. Er Vinklen lig med π, falder Halvtangen-

terne i hinandens Forlængelse og danner Kurvens Tangent i P, som er en Støttelinie (og den eneste Støttelinie) til Kurven gennem P. I modsat Fald har Kurven et Knæk i Punktet P, og der findes uendelig mange Støttelinier til Kurven gennem P. Har en konveks Jordankurve i ethvert af sine Punkter P en bestemt Tangent, vil denne variere monotont, naar Punktet gennemløber Kurven; omvendt vil en Jordankurve, som i ethvert Punkt har en bestemt Tangent, sikkert være konveks, naar Tangenten varierer monotont med Røringspunktet. En Støttelinie til en Jordankurve ω er altid tillige Støttelinie for Omraadet $\bar{I}(\omega)$.

Analoge Definitioner og Sætninger til de her angivne kan opstilles for vilkaarlige kontinuerte Kurver. Man viser let, at en lukket kontinuert Kurve (defineret som det entydige og kontinuerte Billede af en Cirkelperiferi), naar den er konveks, (d. v. s. naar enhver indskreven Polygon, hvis Vinkelspidser svarer til en Række efter hinanden følgende Punkter paa Cirkelperiferien, er konveks), og naar den ikke netop reducerer sig til et Punkt eller til et ret Liniestykke, aldrig vil kunne indeholde andre Dobbeltpunkter end saadanne, som fremkommer ved, at et Punkt paa Kurven svarer til en hel Bue paa Cirkelperiferien, og derfor som Punktmængde betragtet vil kunne opfattes som en konveks Jordankurve. Dette er Grunden til, at vi i det følgende hyppigt taler om konvekse Jordankurver blot som lukkede konvekse Kurver.

2. En Punktmængde kaldes konveks, naar den med hvilkesomhelst to Punkter tillige indeholder det Liniestykke, der forbinder dem. En konveks Jordankurve er, opfattet som Punktmængde, ikke konveks. Vi vil imidlertid vise, at *en nødvendig og tilstrækkelig Betingelse for, at en Jordankurve ω er konveks, er den, at det af den begrænsede indre Omraade $I(\omega)$ er en konveks Punktmængde.*

Betingelsen er *nødvendig*. Lad ω være en konveks Jordankurve. Vi betragter samtlige Støttelinier til Kurven og de af dem begrænsede afsluttede Halvplaner, hvori Kurven er beliggende. Hver af disse er en konveks Mængde. Det samme gælder derfor om den ligeledes afsluttede Mængde M, der udgøres af de fælles Punkter for samtlige Halvplaner. Ethvert Punkt af Kurven tilhører M, thi det tilhører alle Halvplanerne. Det er et Randpunkt for M, thi det er et Randpunkt for mindst en af Halvplanerne bestemt ved en Støttelinie gennem Punktet. Et Punkt af $I(\omega)$ vil, tillige med en vis Omegn, tilhøre samtlige Halvplaner og vil altsaa ogsaa være indre Punkt af M. Heraf følger imidlertid, at ethvert Punkt P af $Y(\omega)$ maa falde udenfor M. Tilhørte det nemlig M, kunde vi forbinde det med et Punkt Q af $I(\omega)$. Da dette tillige med en vis Omegn tilhører M, vilde alle Punkter af Liniestykket PQ, undtagen maaske P, være indre Punkter af M. Nu ligger der paa PQ mindst et Punkt af ω; dette maatte altsaa være indre Punkt af M i Modstrid med, hvad vi ovenfor viste. Heraf følger da, at Jordankurven ω maa være identisk med Randen af den konvekse Mængde M, Omraadet $I(\omega)$ med Mængden af indre Punkter i M, Omraadet $Y(\omega)$ med Mængden af Punkter udenfor M. Nu indeholder et Liniestykke, der forbinder to indre Punkter af M, lutter indre Punkter af M. Omraadet $I(\omega)$ er altsaa en konveks Punktmængde.

Betingelsen er *tilstrækkelig*. Lad ω være en Jordankurve, og lad $I(\omega)$ være en konveks Mængde; ω er Begrænsningen for denne Mængde. Man viser uden Vanskelighed, ved Hjælp af Konveksiteten, at der gennem hvert Punkt af ω gaar en ret Linie, som ikke indeholder noget Punkt af $I(\omega)$. Denne Linie vil være Støttelinie for ω. Lad nu $P_1, P_2, \ldots, P_n, P_{n+1} = P_1$ være en Polygon, som fremkommer ved, at man forbinder en Række efter hinanden følgende Punkter af Kurven. Vi vil vise, at denne er konveks. Lad $P_\nu P_{\nu+1}$ være en Side i Polygonen. Enten er den rette Linie $P_\nu P_{\nu+1}$ Støttelinie for ω og altsaa ogsaa for Polygonen, eller den er ikke Støttelinie for ω; i saa Fald indeholder Kurven kun de to Punkter P_ν og $P_{\nu+1}$ af Linien; var der nemlig et tredje Punkt, maatte et af de tre Punkter ligge mellem de to andre, og Støttelinien til Kurven gennem dette Punkt maatte være den betragtede Linie mod Forudsætning. Buen $P_{\nu+1} P_\nu$ paa Jordankurven, som rummer Polygonen, falder altsaa sikkert helt i den ene af de afsluttede Halvplaner, som begrænses af Linien $P_\nu P_{\nu+1}$, og denne er sikkert en Støttelinie for Polygonen. Hermed er den opstillede Sætning bevist.

Af Beviset kan vi aflede en simpel Følgesætning. Bemærker man, at der ved Beviset for, at den angivne Betingelse er nødvendig, kun er anvendt den ene Egenskab ved Jordankurven ω, at der *gennem hvert af dens Punkter gaar en Støttelinie til Kurven*, ser man, at *denne Egenskab ved en given Jordankurve er tilstrækkelig til at sikre, at Kurven er konveks.*

Da ethvert konvekst Omraade, som er beliggende helt i det endelige, begrænses af en Jordankurve, ser man, ved Anvendelse af den beviste Sætning, at den ovenfor givne Definition af en konveks Jordankurve er identisk med følgende: *En konveks Kurve er Begrænsningen for et konvekst Omraade, der er beliggende helt i det endelige.*

I Tilslutning til den første Definition definerer man Længden af en lukket konveks Kurve som øvre Grænse for Længden af alle indskrevne konvekse Polygoner. Denne Størrelse eksisterer altid. Paa tilsvarende Maade defineres en Buelængde paa Kurven og, almindeligere, et (lineært) Jordan'sk Maal for Punktmængder paa Kurven. Det plane Jordan'ske Maal for en lukket konveks Kurve er Nul. Ved Arealet af en lukket konveks Kurve vil vi i det følgende stedse forstaa Arealet (d. v. s. det plane Jordan'ske Maal) af det af Kurven begrænsede Omraade. Dette Maal eksisterer altid.

Addition af konvekse Kurver.[1]

3. Lad der i en Plan *med fastlagt Begyndelsespunkt* O være givet et endeligt Antal Punkter $P_0, P_1, \ldots, P_N$. Ved Summen $P = \sum_{n=0}^{N} P_n$ af disse Punkter vil vi paa sædvanlig Maade forstaa Endepunktet P for den Vektor OP, der bestemmes som

[1] Dette Afsnit gengiver i sammentrængt Form de vigtigste Resultater af en udførligere Afhandling af H. Bohr: Om Addition af uendelig mange konvekse Kurver. D. Kgl. Danske Vidensk. Selsk. Forhandlinger 1913.

Sum af Vektorerne OP_0, OP_1, ..., OP_N. Betegner $M_0, M_1, \ldots, M_N$ et endeligt Antal af Punktmængder i Planen, vil vi ved Summen M af disse Punktmængder forstaa Mængden af samtlige Punkter $P = \sum_{n=0}^{N} P_n$, hvor Punkterne $P_0, P_1, \ldots, P_N$ tilhører hver sin af de givne Mængder. Det til et givet Punkt P symmetriske Punkt m. H. t. Begyndelsespunktet betegner vi $-P$. Subtraktion af Punktet P skal være ensbetydende med Addition af Punktet $-P$. Den til en given Mængde M symmetriske Mængde m. H. t. Begyndelsespunktet betegner vi paa tilsvarende Maade $-M$; $-M$ er den Punktmængde, der fremkommer, naar M drejes 180° om Begyndelsespunktet O. Subtraktion af Mængden M skal være ensbetydende med Addition af Mængden $-M$. Er P et givet Punkt og M en given Mængde, vil Punktmængden $P+M$ fremgaa af M ved den ved Vektoren OP bestemte Parallelforskydning; Mængden $P-M$ vil fremgaa af M ved en Drejning paa 180° omkring Midtpunktet af Liniestykket OP. Af Reglerne for Vektorers Addition følger umiddelbart at den indførte Addition af Punktmængder tilfredsstiller saavel den kommutative som den associative Lov.

4. Lad

$$\omega_0, \omega_1, \ldots, \omega_N, \ldots \tag{1}$$

være en Følge af konvekse Kurver. Vi vil betragte den Følge af Punktmængder

$$\Sigma_0 = \omega_0, \ \Sigma_1 = \omega_0 + \omega_1, \ \ldots, \ \Sigma_N = \sum_{n=0}^{N} \omega_n, \ \ldots, \tag{2}$$

som fremkommer, idet Kurverne adderes i den opskrevne Rækkefølge. For ethvert n fremgaar Mængden Σ_{n+1} af den foregaaende Mængde Σ_n ved Addition af Kurven ω_{n+1}; en nødvendig og tilstrækkelig Betingelse for, at et vilkaarligt Punkt P i Planen tilhører Σ_{n+1}, er derfor den, at Kurven $P - \omega_{n+1}$ indeholder mindst et Punkt af Σ_n.

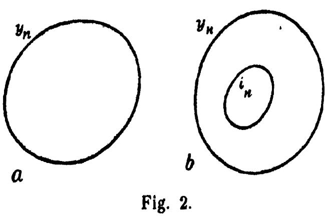


Fig. 2.

Punktmængderne Σ_n, $n = 0, 1, \ldots, N, \ldots$ er alle begrænsede. Vi vil vise, at *de er afsluttede Mængder, som hver for sig enten bestaar af et enkelt afsluttet konvekst Omraade $\bar{I}(y_n)$ eller af et afsluttet konvekst Omraade $\bar{I}(y_n)$ minus et aabent konvekst Omraade $I(i_n)$. I det første Tilfælde* (Fig. 2a) *begrænses Σ_n af en enkelt konveks Kurve y_n, i det andet Tilfælde* (Fig. 2b) *af to konvekse Kurver y_n og i_n.*

For $n = 0$ er denne Sætning sikkert rigtig; Kurverne y_0 og i_0 falder sammen med ω_0. For at vise Sætningens almindelige Gyldighed antager vi den rigtig for et eller andet n og viser herudfra dens Rigtighed ogsaa for $n+1$. Komplementærmængden Σ_n^* til Σ_n bestaar enten af et enkelt Omraade $Y(y_n)$, hvor y_n er en lukket konveks Kurve, eller af to Omraader $Y(y_n)$ og $I(i_n)$, hvor y_n og i_n er lukkede konvekse Kurver, af hvilke den første omslutter den sidste. Betingelsen for, at et Punkt

P tilhører Komplementærmængden Σ^*_{n+1} til Σ_{n+1}, er den, at Kurven $P-\omega_{n+1}$ tilhører Σ^*_n. Dette er muligt paa tre væsentlig forskellige Maader:

1. den kan tilhøre Omraadet $I(i_n)$;
2. den kan tilhøre Omraadet $Y(y_n)$, idet den omslutter y_n;
3. den kan tilhøre Omraadet $Y(y_n)$ uden at omslutte y_n.

Den første Mulighed bortfalder, saafremt Σ^*_n kun bestaar af det ene Omraade $Y(y_n)$. I det første Tilfælde tilhører $\bar{I}(P-\omega_{n+1})$ Omraadet $I(i_n)$; i det andet Tilfælde tilhører $\bar{Y}(P-\omega_{n+1})$ Omraadet $Y(y_n)$, medens endelig i det tredje Tilfælde $\bar{I}(P-\omega_{n+1})$ tilhører $Y(y_n)$.

Medens der altid findes Punkter P, for hvilke den tredje Mulighed indtræffer, vil det afhænge af de givne Kurver, om der findes Punkter, for hvilke den første eller den anden Mulighed indtræffer, og disse to Muligheder kan aldrig forekomme samtidig; den første fordrer nemlig, at Arealet af ω_{n+1} skal være mindre end Arealet af i_n, medens den anden Mulighed fordrer, at det skal være større end Arealet af y_n; falder Arealet af ω_{n+1} mellem Arealerne af i_n og y_n, har vi et Eksempel paa et Tilfælde, hvor alene den tredje Mulighed indtræffer.

Mængden af Punkter, for hvilke den tredje Mulighed indtræffer, danner et Omraade, som vi vil vise er det ydre Omraade for en konveks Jordankurve. Komplementærmængden M til dette Omraade er en afsluttet Punktmængde. Betingelsen for, at et Punkt P tilhører M, er den, at $\bar{I}(P-\omega_{n+1})$ ikke tilhører $Y(y_n)$, at altsaa $\bar{I}(P-\omega_{n+1})$ og $\bar{I}(y_n)$ har mindst et fælles Punkt. M er saaledes beliggende helt i det endelige. Lad (se Fig. 3) P_1 og P_2 være to vilkaarlige Punkter af M, og lad $\bar{I}(P_1-\omega_{n+1})$ og $\bar{I}(P_2-\omega_{n+1})$ have henholdsvis Punktet Q_1 og Punktet Q_2 fælles med $\bar{I}(y_n)$. Lad P være et Punkt af Strækningen P_1P_2. Punktmængden $\bar{I}(P-\omega_{n+1})$ indeholder saavel Punktet $Q_1+(P-P_1)$ som Punktet $Q_2+(P-P_2)$, følgelig ogsaa det Liniestykke, der forbinder dem. Dette Liniestykke skærer Liniestykket Q_1Q_2 i et Punkt Q, som er et fælles Punkt for $\bar{I}(P-\omega_{n+1})$ og $\bar{I}(y_n)$. P tilhører altsaa M. Punktmængden M er følgelig konveks og begrænses af en konveks Jordankurve; betegner vi denne y_{n+1}, er M identisk med Mængden $\bar{I}(y_{n+1})$, Omraadet $Y(y_{n+1})$ med Mængden af Punkter, for hvilke den tredje af de angivne Muligheder indtræffer.

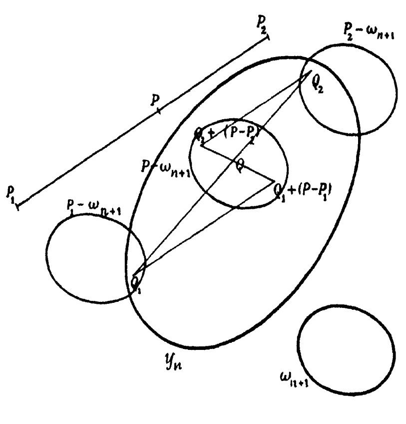


Fig. 3.

Indtræffer for samtlige Punkter P af Σ^*_{n+1} denne tredje Mulighed, er Σ^*_{n+1} identisk med $Y(y_{n+1})$, Σ_{n+1} følgelig med det afsluttede Omraade $\bar{I}(y_{n+1})$. Findes der derimod Punkter, for hvilke enten den første eller den anden Mulighed ind-

træffer, vil den af disse udgjorte Mængde aabenbart være et aabent konvekst Omraade, som tilhører $\bar{I}(y_{n+1})$; Begrænsningen for dette Omraade vil være en konveks Jordankurve i_{n+1}, som falder helt indenfor y_{n+1}, og Σ_{n+1}^{*} vil være sammensat af de to Omraader $Y(y_{n+1})$ og $I(i_{n+1})$. Hermed er den opstillede Sætning bevist.

Findes der i Planen ingen Punkter, for hvilke den første eller den anden Mulighed indtræffer, men vel Punkter, for hvilke Kurven $P-\omega_{n+1}$ enten tilhører Omraadet $\bar{I}(i_n)$ eller tilhører Omraadet $\bar{Y}(y_n)$, idet den omslutter y_n, vil det være praktisk at medregne den af disse Punkter udgjorte Mængde, der bestaar enten af et enkelt Punkt eller af Punkterne af et afsluttet Liniestykke, til Begrænsningen for Σ_{n+1} som en Rand i det Indre. Dette Tilfælde danner en Overgang mellem de to mulige Typer af Mængder Σ_{n+1}. Et Punkt P af Σ_{n+1} vil med denne udvidede Definition af Randen da og kun da være et indre Punkt for Σ_{n+1}, naar Kurven $P-\omega_{n+1}$ indeholder indre Punkter af Σ_n (eller for $n=0$ skærer ω_0).

Betegner b_n, $n = 0, 1, \ldots, N, \ldots$ en Bue paa Kurven ω_n, kan man paa tilsvarende Maade betragte de Mængder $\Sigma_0' = b_0$, $\Sigma_1' = b_0 + b_1, \ldots, \Sigma_N' = \sum_{n=0}^{N} b_n, \ldots$, som fremkommer ved Addition af Buerne $b_0, b_1, \ldots, b_N, \ldots$. Man viser let, at disse Punktmængder vil være sammenhængende Mængder maalelige i Jordan'sk Forstand. Denne Bemærkning kommer vi senere til at anvende.

5. Lad $P_0, P_1, \ldots, P_N, \ldots$ være en Følge af Punkter i Planen. Den uendelige Række $\sum_{n=0}^{\infty} P_n$ skal da siges at være konvergent, og dens Sum at være Punktet P, saafremt det ved Rækkens Afsnit bestemte Punkt $\sum_{n=0}^{N} P_n$ for $N \to \infty$ nærmer sig til det entydig bestemte Grænsepunkt P. Lad $M_0, M_1, \ldots, M_N, \ldots$ være en Følge af Punktmængder i Planen. Den uendelige Række $\sum_{n=0}^{\infty} M_n$ skal siges at være konvergent, saafremt *enhver* uendelig Række $\sum_{n=0}^{\infty} P_n$, hvor P_n, $n = 0, 1, \ldots, N, \ldots$ er et Punkt af Mængden M_n, er konvergent; Mængden M af de ved disse Rækker fremstillede Punkter skal betegnes som den givne Rækkes Sum.

Er $\sum_{n=0}^{\infty} M_n$ konvergent, er det muligt svarende til ethvert Tal $\varepsilon > 0$ at bestemme et positivt helt Tal $N_0 = N_0(\varepsilon)$, saaledes, at for ethvert Tal $N \geqq N_0$ og ethvert positivt helt Tal p Mængden $\sum_{n=N+1}^{N+p} M_n$ er beliggende helt indenfor en Cirkel med Begyndelsespunktet O som Centrum og Radius ε; vi udtrykker dette ved at sige, at den uendelige Række $\sum_{n=0}^{\infty} P_n$, hvor P_n gennemløber Punktmængden M_n, er *ligelig konvergent.* I modsat Fald maatte der nemlig eksistere et bestemt Tal $e > 0$ og en Følge af hele positive Tal $N_1 < N_1 + p_1 < N_2 < N_2 + p_2 < \ldots < N_R < N_R + p_R < \ldots$ saaledes, at Mængderne $\sum_{n=N_r+1}^{N_r+p_r} M_n$, $r = 1, 2, \ldots, R, \ldots$ alle indeholdt Punkter udenfor

eller paa Randen af Cirklen med O som Centrum og Radius e; men dette vilde umiddelbart føre til Konstruktion af en divergent Række $\sum_{n=0}^{\infty} P_n$, hvor P_n, $n = 0, 1, \ldots, N, \ldots$ tilhørte Mængden M_n.

Lad $\delta_{N,p}$ betegne øvre Grænse for Afstanden fra Begyndelsespunktet til et Punkt af Mængden $\sum_{n=N+1}^{N+p} M_n$ og lad δ_N betegne øvre Grænse for $\delta_{N,p}$, naar p gennemløber de hele positive Tal; da udsiger den fundne Sætning, at Størrelsen δ_N konvergerer mod Nul, naar N vokser ud over alle Grænser. Heraf følger specielt, at den største Afstand fra Begyndelsespunktet til et Punkt af Mængden M_n vil konvergere mod Nul for $n \to \infty$.

Endvidere følger let, at hvis $\sum_{n=0}^{\infty} M_n$ er konvergent, og enhver af Punktmængderne M_n er beliggende helt i det endelige, da vil Punktmængden $M = \sum_{n=0}^{\infty} M_n$ ligeledes være beliggende helt i det endelige. Forudsætter man yderligere om Punktmængderne M_n, at de er afsluttede Mængder, viser man let, at ogsaa Mængden M er en afsluttet Punktmængde.

6. Lad $\omega_0, \omega_1, \ldots, \omega_N, \ldots$ være en Følge af lukkede konvekse Kurver, og lad Rækken

$$(3) \qquad \sum_{n=0}^{\infty} \omega_n$$

være konvergent. Kurverne ω_n er helt i det endelige beliggende afsluttede Punktmængder. Summen Σ af Rækken (3) er derfor ligeledes afsluttet og beliggende helt i det endelige. Vi vil vise, *at Punktmængden* Σ ligesom Rækkens Afsnit $\Sigma_N = \sum_{n=0}^{N} \omega_n$ *enten bestaar af et enkelt afsluttet konvekst Omraade* $\bar{I}(y)$ *eller af et afsluttet konvekst Omraade* $\bar{I}(y)$ *minus et aabent konvekst Omraade* $I(i)$.

Lad Q_n, $n = 0, 1, \ldots, N, \ldots$ være et fast Punkt paa Kurven ω_n. I Stedet for at addere Kurverne ω_n vil vi addere de parallelforskudte Kurver $\omega'_n = \omega_n - Q_n$, som alle gaar gennem Begyndelsespunktet. Idet Rækken $\sum_{n=0}^{\infty} Q_n$ er konvergent, har vi

$$\Sigma - \sum_{n=0}^{\infty} Q_n = \sum_{n=0}^{\infty} (\omega_n - Q_n) = \sum_{n=0}^{\infty} \omega'_n = \Sigma';$$

Rækken $\sum_{n=0}^{\infty} \omega'_n$ er altsaa konvergent, og dens Sum Σ' fremgaar af Σ ved en Parallelforskydning. Afsnittene i Rækken $\sum_{n=0}^{\infty} \omega'_n$ betegner vi $\Sigma'_0, \Sigma'_1, \ldots, \Sigma'_N, \ldots$; de danner en voksende Følge af Mængder. Ethvert Punkt i Planen, som for et eller andet n tilhører Σ'_n, vil ogsaa tilhøre Σ'; Punktmængden Σ' vil derfor kunne bestemmes som det afsluttede Hylster for Grænsemængden for Σ'_n for $n \to \infty$.

Komplementærmængden Σ'^{*} til Σ' udgøres af de Punkter i Planen, som tillige med en vis Omegn tilhører Komplementærmængden $\Sigma_n'^{*}$ til Σ_n' for alle $n = 0, 1, \ldots, N, \ldots$. Punktmængden $\Sigma_n'^{*}$ bestaar enten af et enkelt Omraade $Y(y_n')$, hvor y_n' er en konveks Jordankurve, eller af to Omraader $Y(y_n')$ og $I(i_n')$, hvor y_n' og i_n' er konvekse Jordankurver, af hvilke den første omslutter den sidste. Omraaderne $Y(y_n')$ danner en aftagende Følge. Et Punkt P, som tilhører samtlige Mængder $\Sigma_n'^{*}$, kan enten tilhøre samtlige Omraader $Y(y_n')$ eller samtlige disse Omraader til et vist Trin og dernæst samtlige Omraader $I(i_n')$. Det kan ikke først tilhøre et Omraade $I(i_n')$ og dernæst for et større n et Omraade $Y(y_n')$. Mængden $\Sigma_y'^{*}$ af Punkter, som tillige med en vis Omegn tilhører samtlige Omraader $Y(y_n')$, vil være komplementær til det afsluttede Hylster for den Mængde, som bestaar af samtlige Punkter, som for et eller andet n tilhører $\bar{I}(y_n')$. Denne Mængde er konveks; betegner vi dens Begrænsning y' vil $\Sigma_y'^{*}$ være identisk med det ydre Omraade for den konvekse Kurve y'. Tilhører ethvert Punkt af Σ'^{*} samtlige Omraader $Y(y_n')$, er Σ'^{*} identisk med $\Sigma_y'^{*}$, Punktmængden Σ' følgelig med det afsluttede konvekse Omraade $\bar{I}(y')$. I modsat Fald bestaar Σ'^{*} foruden af Omraadet $Y(y')$ af en Punktmængde $\Sigma_i'^{*}$ bestaaende af de Punkter, som tillige med en vis Omegn tilhører samtlige Omraader $Y(y_n')$ til et vist Trin og dernæst samtlige Omraader $I(i_n')$. Denne Mængde vil være den aabne Kærne for Mængden af Punkter, som tilhører samtlige Omraader $Y(y_n')$ til et vist Trin og dernæst samtlige Omraader $I(i_n')$. Denne Mængde er sikkert konveks; betegner vi dens Begrænsning i', er Omraadet $I(i')$ identisk med Mængden $\Sigma_i'^{*}$. Mængden Σ' bestaar i dette Tilfælde af det afsluttede Omraade $\bar{I}(y')$ minus det aabne Omraade $I(i')$.

Hermed er den opstillede Sætning bevist.

Et Punkt i Planen, som tillige med en vis Omegn tilhører samtlige Mængder Σ_n', vil være et indre Punkt af Σ'. Benytter man dette som Definition, opnaar man i særlige Tilfælde, hvor Σ begrænses af en enkelt konveks Kurve, i Lighed med ovenfor en Udvidelse af Randen af Σ til ogsaa at omfatte et afsluttet Liniestykke i det Indre af Σ.

7. Vi vil endnu, med særligt Henblik paa en senere Anvendelse (§ 47), foretage en Almindeliggørelse af Begrebet Sum af en uendelig Række $\sum_{n=0}^{\infty} \omega_n$ til ogsaa at omfatte visse *divergente Rækker.*

Lad os for ethvert N og ethvert positivt Tal p med $\Sigma_{N,N+p}$ betegne Summen $\sum_{n=N+1}^{N+p} \omega_n$ af Kurverne $\omega_{N+1}, \ \ , \omega_{N+p}$. De Rækker $\sum_{n=0}^{\infty} \omega_n$, som vi vil betragte, er dem, for hvilke det er muligt svarende til ethvert tilstrækkelig stort N (d. v. s. ethvert $N \geqq N_0$) at bestemme et helt Tal $p_0 = p_0(N)$ og et positivt Tal ϱ_N, som konvergerer mod Nul for $N \to \infty$, saaledes at for ethvert $p \geqq p_0$ Mængden $\Sigma_{N,N+p}$ indeholder Punkter af den afsluttede Cirkelskive Γ_N med Centrum i Begyndelsespunktet og Radius ϱ_N. For konvergente Rækker er denne Betingelse sikkert opfyldt; vælger vi $\varrho_N = \delta_N$ (se § 5) vil endda for ethvert p samtlige Punkter af $\Sigma_{N,N+p}$ tilhøre Γ_N.

For ethvert $N \geqq N_0$ betragter vi Mængden Σ_N' af Punkter af Σ_N, hvis Afstand

fra Randen for Σ_N opfattet i den i § 4 angivne udvidede Betydning er større end $2\varrho_N$; for tilstrækkelig store N eksisterer denne Mængde og er en aaben Mængde begrænset enten af en enkelt konveks Kurve y'_N eller af to konvekse Kurver y'_N og i'_N. Betegner P et Punkt af Σ'_N, indeholder for ethvert Punkt Q af $P+\Gamma_N$ og ethvert $p \geqq p_0$ Mængden $Q-\Sigma_{N,N+p}$ Punkter af Σ_N. Cirkelskiven $P+\Gamma_N$ tilhører altsaa det Indre af samtlige Mængder Σ_{N+p} med $p \geqq p_0$. Da $\varrho_{N+p} \to 0$ for $p \to \infty$ følger heraf, at Mængden Σ'_N tilhører samtlige Mængder Σ'_{N+p} fra et vist Trin. Heraf følger nu uden Vanskelighed ved Hjælp af den i forrige Paragraf anvendte Betragtning, at der vil eksistere en bestemt Grænsemængde Σ' for Σ'_N for $N \to \infty$ bestemt ved Mængden af Punkter, som for et eller andet N tilhører den tilsvarende Mængde Σ'_N og bestaaende enten af et enkelt aabent konvekst Omraade (som nu ikke behøver at være begrænset) eller af et aabent konvekst Omraade minus et afsluttet konvekst Omraade, som specielt kan udarte til et Punkt eller til et ret Liniestykke. Denne Mængde Σ' vil være identisk med Mængden af Punkter, som tillige med en vis Omegn tilhører det Indre for samtlige Mængder Σ'_N fra et vist Trin; thi ethvert Punkt, for hvilket dette er Tilfældet, vil aabenbart for et tilstrækkelig stort N tilhøre Σ'_N. Det afsluttede Hylster for den fundne Grænsemængde Σ' betegner vi som *Summen Σ af den uendelige Række* $\sum_{n=0}^{\infty} \omega_n$; Randen for Σ' skal være Randen for Σ. Af Karakteriseringen af de indre Punkter for Σ fremgaar, at den indførte Sum for konvergente Rækker stemmer overens med den sædvanlige.

KAPITEL II.

Indledende Definitioner og Sætninger om Sandsynlighed.

Inden vi i næste Kapitel afgrænser det Omraade af plane konvekse Kurver, som vi vil lægge til Grund for Undersøgelsen af Sandsynligheder ved Addition af konvekse Kurver, skal vi i dette Kapitel indføre nogle af de Begreber, som kommer til at spille en Rolle i det følgende. Vi vil definere, hvad vi vil forstaa ved en Sandsynlighedsfordeling paa en konveks Kurve, og vi vil se, hvorledes man ved Addition af konvekse Kurver, paa hvilke der er givet bestemte Sandsynlighedsfordelinger, kan komme til at tale om Sandsynlighedsfordelinger paa de ved Additionen fremkomne Punktmængder, samt indenfor visse Grænser undersøge disses Egenskaber. Disse Betragtninger gennemføres naturligst for almindelige konvekse Kurver. Da vi imid-

lertid kun kommer til at anvende dem paa Kurver af det afgrænsede Omraade, har vi ikke lagt Vægt paa at give Undersøgelsen væsentlig større Almindelighed, end det for disse Anvendelser er nødvendigt.

Buesandsynlighed og Mængdesandsynlighed paa en konveks Kurve.

8. Vi betragter en konveks Jordankurve ω (se Fig. 4). Lad der være givet en Funktion $f(b)$ defineret for enhver Bue b paa ω, ω selv medregnet, for hvilken det gælder, at dens Værdi er den samme, enten vi betragter en aaben Bue eller de Buer, der fremkommer af den ved Tilføjelse enten af et af Endepunkterne eller af begge. Funktionen siges at være *kontinuert*, saafremt stedse $f(b')$ konvergerer mod $f(b)$, naar den variable Bue b' konvergerer mod den faste Bue b paa ω. Den siges at være *additiv*, saafremt stedse $f(b_1) + f(b_2) = f(b_0)$, naar b_1 og b_2 er to Buer, der ligger i hinandens Forlængelse og tilsammen udgør Buen b_0. Der findes Funktioner af den betragtede Art, der er saavel kontinuerte som additive, f. Eks. Buelængden paa Kurven.

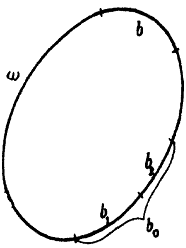


Fig. 4.

Lad os nu om en saadan Funktion yderligere antage, at den stedse er positiv, og at dens Værdi svarende til selve Kurven er 1. Lad os betegne den

$$w(b).$$

Denne Funktion bestemmer da, hvad vi vil kalde en *kontinuert Buesandsynlighed paa den givne konvekse Kurve*, som fremkommer, idet vi for enhver Bue b af ω betegner den tilsvarende Funktionsværdi $w(b)$ som Sandsynligheden for, at et vilkaarligt Punkt P af ω falder paa Buen b.

Da Funktionen er additiv og stedse positiv, maa vi for enhver ægte Bue b paa ω have $0 < w(b) < 1$. Visheden for, at Punktet P tilhører ω, er udtrykt i Forudsætningen $w(\omega) = 1$. Additiviteten giver Udtryk for Reglen om Sandsynligheders Addition, idet den viser, at Sandsynligheden for, at et Punkt P af ω enten tilhører den ene eller den anden af to Buer b_1 og b_2, der ligger i hinandens Forlængelse, er Summen af Sandsynlighederne for, at det tilhører hver af de to Buer.

Lad os betragte et bestemt Punkt P_0 af ω (se Fig. 5), og lad Punktet P_1 gennemløbe Kurven i en bestemt Retning fra P_0 til P_0. For enhver Stilling af P_1 betragter vi Sandsynligheden for, at et vilkaarligt Punkt af ω tilhører den allerede gennemløbne Bue P_0P_1. Denne Sandsynlighed er, opfattet som Funktion af P_1, kontinuert og stedse

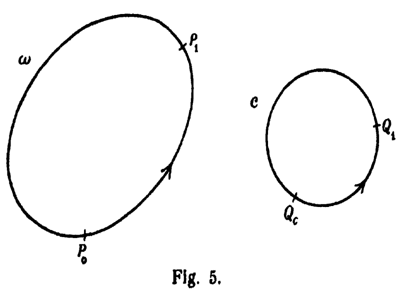


Fig. 5.

voksende. Den vokser fra 0 til 1, naar P_1 gennemløber ω. Lad os nu betragte en Cirkel c med Omkreds 1, paa hvilken der er givet et fast Punkt Q_0, og lad Q_1 være et Punkt, som gennemløber c i en bestemt Retning fra Q_0 til Q_0. Lader vi dette Punkt følges med Punktet P_1, saaledes at Længden af den af Q_1 gennemløbne Bue stadig er lig med den til Buen $P_0 P_1$ paa ω svarende Sandsynlighed, opnaar vi en enentydig og kontinuert Afbildning af ω paa c. Til enhver Bue paa ω svarer ved denne Afbildning en Bue paa Cirklen, hvis Længde er den til den givne Bue svarende Sandsynlighed. Omvendt ser man, at enhver enentydig og kontinuert Afbildning af Kurven ω paa en Cirkel med Omkreds .1 vil definere os en Buesandsynlighed paa Kurven af den betragtede Art. Indfører vi som Parameter θ paa Cirkelperiferien Længden af Buen $Q_0 Q_1$ fra det faste Punkt Q_0 til det variable Punkt Q_1, fremkommer en Afbildning af Kurven paa Parameterintervallet $0 \leqq \theta < 1$, som lige saa vel som Afbildningen paa Cirklen er egnet til Beskrivelse af Buesandsynligheden paa Kurven.

9. Denne Fremstilling af Sandsynlighedsfordelingen paa Kurven giver Anledning til Betragtning af Sandsynligheder svarende ikke blot til Buer, men almindeligere til alle Punktmængder paa Kurven, for hvilke den tilsvarende Mængde af Parameterværdier er maalelig, idet Sandsynligheden svarende til en saadan Mængde m ligefrem sættes lig med Maalet $w(m)$ af denne Punktmængde. Ved maalelig forstaar vi her som ogsaa stedse i det følgende maalelig i Jordan'sk Forstand, ved en Punktmængdes Maal dens Jordan'ske Maal. Naar vi opererer med dette Maal og ikke med det langt mere fintmærkende Lebesgue'ske, skyldes det dels, at det om alle de Mængder, vi kommer til at betragte, gælder, at de er maalelige allerede i Jordan'sk Forstand, dels, at den foreliggende Undersøgelse som en Undersøgelse over kontinuerte Funktioner bevæger sig paa klassisk Grund og ikke paa noget Punkt forudsætter de Lebesgue'ske Teorier. Idet Maalet for et Liniestykke simpelthen er Liniestykkets Længde, ser vi, at denne nye Definition fremtræder som en naturlig Udvidelse af den oprindelige. Den indførte Mængdefunktion betegnes som en *Mængdesandsynlighed paa den givne konvekse Kurve.*

Mængdesandsynligheder i Planen.

10. Lad der i en Plan med fastlagt Begyndelsespunkt O være givet en Følge

(1) $$\omega_0, \omega_1, \ldots, \omega_N, \ldots$$

af konvekse Jordankurver. Vi vil betragte de ovenfor definerede Punktmængder

(2) $$\Sigma_0 = \omega_0, \ \Sigma_1 = \omega_0 + \omega_1, \quad , \ \Sigma_N = \sum_{n=0}^{N} \omega_n, \ \ldots,$$

der fremkommer, naar Kurverne adderes i den opskrevne Rækkefølge.

Lad os antage, at der paa hver af Kurverne ω_n er givet en Mængdesandsyn-

lighed $w_n(m)$ af den ovenfor betragtede Art bestemt ved en Afbildning af Kurven paa et Parameterinterval $0 \leqq \theta_n < 1$. Dette fører umiddelbart til en Afbildning af hver af Mængderne Σ_N paa Enhedsterningen Q_N $(0 \leqq \theta_n < 1)$ i det tilsvarende $\theta_0, \theta_1, \ldots, \theta_N$-Rum, idet vi lader et Punkt P af Σ_N svare til et Punkt $(\theta_0, \theta_1, \ldots, \theta_N)$ af Q_N, naar de til Koordinaterne θ_n for dette Punkt svarende Punkter P_n af de enkelte Kurver ω_n har Summen P. Hvert Punkt af Q_N faar ved denne Afbildning netop et tilsvarende Punkt i Σ_N, medens (for $N \geqq 1$) den overvejende Del af Σ_N's Punkter svarer til flere Punkter af Q_N.

Ved Hjælp af denne Afbildning kan vi nu definere en *Mængdesandsynlighed i Σ_N-Planen.* Lad os betragte en Punktmængde M i denne Plan. Svarende til denne betragter vi Mængden Ω af Punkter i Q_N, hvis tilsvarende Punkt i Σ_N tilhører Mængden M. Saafremt denne Mængde Ω er maalelig (i JORDAN'sk Forstand), betegner vi dens Maal

$$W_N(M)$$

som *Sandsynligheden for, at det vilkaarlige Punkt P af Σ_N tilhører M.* Vi har stedse $0 \leqq W_N(M) \leqq 1$. Til Σ_N selv svarer selve Q_N, som er maalelig og har Maalet 1. Dette udtrykker Visheden for, at Punktet P tilhører Σ_N. Endvidere gælder Reglen om Sandsynligheders Addition. Har vi to Mængder M_1 og M_2 i Σ_N-Planen uden noget fælles Punkt, til hvilke der svarer bestemte Sandsynligheder $W_N(M_1)$ og $W_N(M_2)$, vil der til Foreningsmængden $M_0 = M_1 + M_2$ af de to Mængder svare den bestemte Sandsynlighed $W_N(M_0) = W_N(M_1) + W_N(M_2)$. Thi til M_0 svarer en Delmængde Ω_0 af Q_N, der kan dannes som Foreningsmængde af de til M_1 og M_2 svarende Mængder Ω_1 og Ω_2. Disse er begge maalelige og har intet Punkt fælles. Heraf følger imidlertid at ogsaa Ω_0 er maalelig, og at dens Maal $W_N(M_0)$ er Summen $W_N(M_1) + W_N(M_2)$ af Maalene for Ω_1 og Ω_2.

Af Hensyn til en senere Anvendelse bemærker vi, at Mængdesandsynligheden $W_N(M)$, ligesom Punktmængden Σ_N selv, er uafhængig af den Orden, hvori vi har adderet Kurverne $\omega_0, \omega_1, \ldots, \omega_N$. Punktmængden Ω i den $N+1$-dimensionale Enhedsterning, hvis Maal vi benyttede som Definition paa Sandsynligheden for, at et Punkt P af Σ_N tilhørte Mængden M i Σ_N-Planen, vil nemlig ved Omordning af Kurverne blive erstattet med en Mængde Ω^* kongruent med Ω. Thi Omordningen af Kurverne $\omega_0, \omega_1, \ldots, \omega_N$ svarer simpelthen til en Omordning af Koordinaterne $\theta_0, \theta_1, \ldots, \theta_N$. De to Mængder Ω og Ω^* vil følgelig samtidig være maalelige og altid med samme Maal.

11. Af Reglen om Sandsynlighedernes Addition fremgaar først rigtig det berettigede i at kalde de betragtede Mængdefunktioner $W_N(M)$ Sandsynligheder. Den viser tillige, at vi kan faa et Overblik over Sandsynlighedsfordelingerne, selv om vi indskrænker os til kun at betragte særlig simple Punktmængder i Planen, idet vi altid ved Sammensætning af disse kan naa til mere komplicerede. Vi vil gennemføre Betragtningerne for det Tilfælde, at disse Punktmængder er *Rektangler*, idet vi herved forstaar Punktmængder, som i et passende retvinklet Koordinatsystem (XY) kan fremstilles ved Uligheder af Formen $x_0 \leqq x < x_1$, $y_0 \leqq y < y_1$.

Vi vil alene betragte det Tilfælde, hvor *ingen af Kurverne ω_n indeholder rette Liniestykker.* Lad der i Planen være givet et fast retvinklet Koordinatsystem (XY), og lad os undersøge Mængdesandsynlighederne $W_N(M)$ for akseparallelle Rektangler $R(x_0 < x < x_1, y_0 \leq y < y_1)$ i dette System.

Vi vil vise, at der *for ethvert N eksisterer en bestemt Rektangelsandsynlighed $W_N(R)$, defineret for alle de betragtede Rektangler,* om hvilken det gælder, at der *til ethvert Tal $\varepsilon > 0$ svarer et Tal $\delta > 0$, saaledes at for ethvert Rektangel, hvis Areal er mindre end δ, den tilsvarende Sandsynlighed er mindre end ε.* Heraf vil specielt følge, at Funktionen $W_N(R)$ er *kontinuert,* d. v. s. at $W_N(R')$ vil konvergere mod $W_N(R)$, naar R' er et akseparallelt Rektangel, der konvergerer mod R. Thi R' vil gaa over i R ved Addition eller Subtraktion af (højst) fire akseparallelle Rektangler, hvis Areal konvergerer mod Nul. Endvidere vil vi vise, at *$W_N(R)$ da og kun da er positiv, naar R i sit Indre indeholder et Punkt af Σ_N.*[1]

Vi fører Beviset for den opstillede Sætning ved Induktion. Lad os først betragte Tilfældet $N = 0$, hvor Mængden Σ_0 er identisk med den konvekse Kurve ω_0, Enhedsterningen Q_0 med Intervallet $0 \leq \theta_0 < 1$. Punktmængden Ω af Q_0, som svarer til et Rektangel R i Σ_0-Planen, bestaar øjensynlig af et endeligt Antal Intervaller, thi de Punkter af ω_0, som tilhører R, danner et endeligt Antal (højst fire) Buer paa Kurven. Sandsynligheden $W_0(R)$ eksisterer altsaa sikkert. Lige saa klart er det, at denne Sandsynlighed maa være mindre end en vilkaarlig positiv Størrelse ε, naar blot Rektanglets Areal er mindre end en tilsvarende positiv Størrelse δ. Thi er Rektanglets Areal mindre end δ, er mindst en af dets Sider mindre end $\sqrt{\delta}$, og δ kan vælges saa lille, at Sandsynligheden svarende til de højst to Buer, som en Parallelstrimmel af Bredden $\sqrt{\delta}$ har fælles med ω_0, er mindre end ε uafhængig af Parallelstrimlens Beliggenhed. Dette følger af vor Forudsætning, at ω_0 ingen rette Liniestykker indeholder. Heraf følger ogsaa, at $W_0(R)$ da og kun da er positiv, naar R i sit Indre indeholder Punkter af ω_0.

Vi antager nu Sætningen rigtig for et eller andet N og vil herudfra vise dens Rigtighed ogsaa for $N+1$. Lad (se Fig. 6) R betegne et bestemt akseparallelt Rektangel. Vi betragter den tilsvarende Punktmængde Ω i den $N+2$-dimensionale Enhedsterning Q_{N+1}. Vi skal vise, at denne Punktmængde er maalelig. Samtidig vil vi angive et Udtryk for dens Maal $W_{N+1}(R)$ ved Hjælp af Funktionen W_N. Punktmængden Σ_{N+1} bestemmes som $\Sigma_{N+1} = \Sigma_N + \omega_{N+1}$. Lad P betegne Midtpunktet af R og lad P_{N+1} være et Punkt af ω_{N+1}. De Punkter af Σ_N, som ved Addition af Punktet P_{N+1} giver Punkter af Σ_{N+1}, som tilhører R, maa alle tilhøre det Rektangel $R - P_{N+1}$, som fremkommer af R, naar man parallelforskyder det Vektoren $-OP_{N+1}$. Gennemløber P_{N+1} Kurven ω_{N+1}, vil Midtpunktet $P - P_{N+1}$ for dette Rektangel gennemløbe Kurven $P - \omega_{N+1}$. For ethvert Punkt $P - P_{N+1}$ af denne Kurve betragter

[1] Det følgende Bevis er hentet fra en Afhandling af H. Bohr og R. Courant: Neue Anwendungen der Theorie der Diophantischen Approximationen auf die Riemann'sche Zetafunktion. Journ. f. Math. Bd. 144. (1914), hvor det dog kun er gennemført for et specielt Tilfælde.

vi nu den tilsvarende Rektangelsandsynlighed $W_N(R-P_{N+1})$. Denne Sandsynlighed er ifølge vor Antagelse en kontinuert Funktion af Rektanglet $R-P_{N+1}$, følgelig ogsaa af dets Midtpunkt $P-P_{N+1}$. Det vil atter sige, at den er en kontinuert Funktion af Parameteren θ_{N+1} for Punktet P_{N+1} paa ω_{N+1}. Vi vil vise, at dens Integral

$$I = \int_0^1 W_N(R-P_{N+1})\, d\theta_{N+1}$$

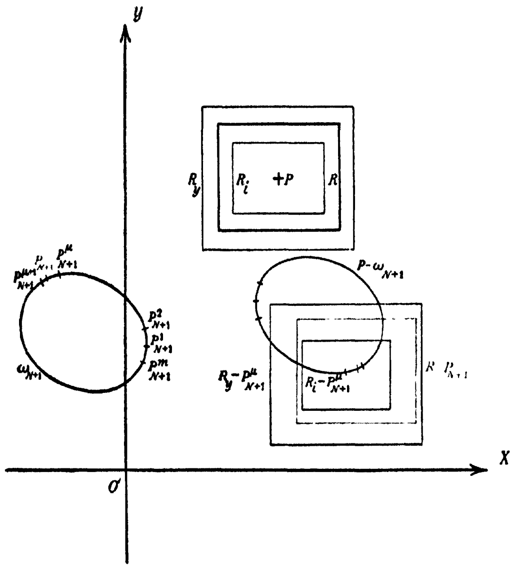


Fig. 6.

netop bestemmer Rektangelsandsynligheden $W_{N+1}(R)$.

Lad os med A_i og A_y betegne henholdsvis det indre og det ydre Maal for Punktmængden Ω. For at vise, at Ω er maalelig med Maalet I, er det tilstrækkelig at vise, at for ethvert $\varepsilon > 0$

$$I-\varepsilon < A_i; \quad A_y < I+\varepsilon.$$

Thi da $A_i \leq A_y$ følger heraf $A_i = A_y = I$.

For at vise, at dette er Tilfældet, betragter vi to med R fast forbundne akseparallelle Rektangler R_i og R_y, ligeledes med Midtpunkt P, af hvilke det førstes Rand falder indenfor, det andets udenfor R. $R_i - P_{N+1}$ og $R_y - P_{N+1}$ betegner disse Rektangler parallelforskudt Vektoren $-OP_{N+1}$. Ifølge Antagelse kan vi vælge R_i og R_y saa nær ved R, at for *ethvert* P_{N+1}

$$(3) \quad W_N(R-P_{N+1}) - \frac{\varepsilon}{2} < W_N(R_i - P_{N+1}); \quad W_N(R_y - P_{N+1}) < W_N(R-P_{N+1}) + \frac{\varepsilon}{2}.$$

Lad os nu betragte en Række efter hinanden følgende Punkter $P^1_{N+1}, P^2_{N+1}, \ldots, P^m_{N+1}, P^{m+1}_{N+1} = P^1_{N+1}$ paa Kurven ω_{N+1}. Lad os med $\Delta^\mu \theta_{N+1}$ $(\mu = 1, 2, \ldots, m)$ betegne Sandsynligheden svarende til Buen $P^\mu_{N+1}\, P^{\mu+1}_{N+1}$ paa ω_{N+1}. Størrelsen

$$\sum_{\mu=1}^{m} W_N(R-P^\mu_{N+1})\, \Delta^\mu \theta_{N+1}$$

giver os en Tilnærmelsesværdi for Integralet I. Vi bestemmer nu Punkterne $P^1_{N+1}, P^2_{N+1}, \ldots, P^m_{N+1}, P^{m+1}_{N+1} = P^1_{N+1}$ saaledes, at for det første

$$(4) \quad \left| I - \sum_{\mu=1}^{m} W_N(R-P^\mu_{N+1})\, \Delta^\mu \theta_{N+1} \right| < \frac{\varepsilon}{2},$$

for det andet Rektanglet $R-P_{N+1}$, for ethvert Punkt P_{N+1} af Buen $P^{\mu}_{N+1}\,P^{\mu+1}_{N+1}$ $(\mu = 1, 2, \ldots, m)$ paa ω_{N+1}, indeholder Rektanglet $R_i - P^{\mu}_{N+1}$ svarende til Buens Begyndelsespunkt P^{μ}_{N+1}, medens det indeholdes i Rektanglet $R_y - P^{\mu}_{N+1}$. Dette sidste vil altid være Tilfældet, naar blot Længden af hver Bue $P^{\mu}_{N+1}\,P^{\mu+1}_{N+1}$ paa ω_{N+1} er mindre end den mindste Afstand fra et Punkt af Randen for R til et Punkt af Randen for R_i eller R_y.

Da er det indre og ydre Maal for den Del af Ω, for hvilken Koordinaten θ_{N+1} svarer til Punkter af Buen $P^{\mu}_{N+1}\,P^{\mu+1}_{N+1}$, mindst lig med $W_N(R_i - P^{\mu}_{N+1})\,\Delta^{\mu}\theta_{N+1}$, højst lig med $W_N(R_y - P^{\mu}_{N+1})\,\Delta^{\mu}\theta_{N+1}$. Følgelig er

$$\sum_{\mu=1}^{m} W_N(R_i - P^{\mu}_{N+1})\,\Delta^{\mu}\theta_{N+1} \leq A_i; \quad A_y \leq \sum_{\mu=1}^{m} W_N(R_y - P^{\mu}_{N+1})\,\Delta^{\mu}\theta_{N+1}.$$

Ved Anvendelse af (3) følger heraf

$$\sum_{\mu=1}^{m} W_N(R - P^{\mu}_{N+1})\,\Delta^{\mu}\theta_{N+1} - \frac{\varepsilon}{2} < A_i; \quad A_y < \sum_{\mu=1}^{m} W_N(R - P^{\mu}_{N+1})\,\Delta^{\mu}\theta_{N+1} + \frac{\varepsilon}{2}$$

og, ved Hjælp af (4),

$$I - \varepsilon < A_i; \quad A_y < I + \varepsilon,$$

hvormed Eksistensen af $W_{N+1}(R)$ og samtidig Identiteten

$$\text{(5)} \qquad \underline{W_{N+1}(R) = \int_0^1 W_N(R - P_{N+1})\, d\theta_{N+1}}$$

er bevist.

Heraf følger umiddelbart den anden Del af den opstillede Sætning; thi er $W_N(R) < \varepsilon$ for ethvert Rektangel, hvis Areal er mindre end δ, er ogsaa $W_{N+1}(R) < \varepsilon$ for ethvert saadant Rektangel. Anderledes formuleret udsiger denne Del af Sætningen, at Rektangelsandsynligheden gaar *ligelig* mod Nul med Rektanglets Areal. Den sidste Bemærkning viser da, at Ligelighedsgraden ikke forringes ved Overgang fra N til $N+1$; *en en Gang opnaaet Regelmæssighed i Sandsynlighedsfordelingen bevares under den fortsatte Addition af konvekse Kurver.*

Endelig viser Identiteten (5) Rigtigheden af den i Sætningens sidste Del udtalte Paastand, at $W_{N+1}(R)$ da og kun da er positiv, naar R i sit Indre indeholder Punkter af Σ_{N+1}. Thi dette er ensbetydende med, at mindst et af Rektanglerne $R - P_{N+1}$ i sit Indre indeholder Punkter af Σ_N, altsaa ifølge Forudsætning ensbetydende med, at den kontinuerte Funktion $W_N(R - P_{N+1})$ under Integraltegnet for

mindst en Værdi af Parameteren θ_{N+1} er positiv. Men det er netop Betingelsen for, at Integralet er positivt. Hermed er den opstillede Sætning fuldstændig bevist.

Forudsætningen om, at de betragtede Kurver ikke indeholder rette Liniestykker, er, som man kan vise, uden Betydning for Eksistensen af Rektangelsandsynlighederne $W_N(R)$. Derimod er den nødvendig for i Almindelighed at sikre os, at disse bliver kontinuerte.

Vi skal i Kapitel IV vende tilbage til Spørgsmaalet om Sandsynlighedsfordelinger ved Addition af konvekse Kurver, men under speciellere Forudsætninger end de hidtil anvendte. Udfra den Antagelse, at Sandsynlighederne paa de enkelte Kurver er differentiable Mængdefunktioner med kontinuerte Differentialkvotienter, og følgelig lader sig fremstille som Integraler af kontinuerte »Punktsandsynligheder«, vil vi vise, at ogsaa de paa Grundlag af disse indførte Mængdesandsynligheder i Planen bliver differentiable med kontinuerte Differentialkvotienter og saaledes ogsaa kan fremstilles som Integraler af kontinuerte Punktsandsynligheder. Dette bliver dog, som det allerede fremgaar af den sidste Bemærkning, først muligt, naar vi giver Afkald paa at betragte vilkaarlige konvekse Kurver. Afgrænsningen af Kurveomraadet giver Anledning til almindeligere Betragtninger over lukkede konvekse Kurver. Da disse er af en ret afrundet Karakter, har vi af systematiske Grunde samlet dem i et selvstændigt Kapitel III.

KAPITEL III.

Afgrænsning af Kurveomraadet.

Indre og ydre Radius for en konveks Jordankurve.

12. Lad (se Fig. 7) ω være *en konveks Jordankurve uden Knæk, som ikke indeholder noget ret Liniestykke.* I ethvert Punkt P af Kurven findes en bestemt Tangent t, som kun har det ene Punkt P fælles med Kurven, og som varierer kontinuert med P. Vi betragter samtlige Cirkler, som rører t i P, og som falder paa samme Side af t som ω; deres Centrer udfylder Halvnormalen n til t i P. Som Grænsetilfælde medregnes blandt Cirklerne Punktet P selv og Linien t. Blandt de af disse Cirkler, som har den Egenskab, at deres Indre tilhører det Indre af ω, findes en største, som vi betegner $\gamma_i(P)$; dens Radius betegner vi $\varrho_i(P)$; $\varrho_i(P)$ vil normalt være et positivt Tal, men kan i udartede Tilfælde være Nul. Paa samme Maade findes der blandt de af Cirklerne, hvis Ydre tilhører det Ydre af ω, en mindste, som vi betegner $\gamma_y(P)$; dens Radius betegner vi $\varrho_y(P)$; $\varrho_y(P)$ vil normalt være endelig, men kan i udartede Tilfælde være uendelig. Vi lader nu P gennemløbe ω og bestemmer nedre Grænse r_i for alle $\varrho_i(P)$, øvre Grænse r_y for alle $\varrho_y(P)$. *Størrelserne r_i og r_y betegnes henholdsvis som indre og ydre Radius for den givne Kurve ω.*

Ved den fortsatte Undersøgelse af Sandsynlighedsfordelinger ved Addition af konvekse Kurver vil vi indskrænke os til at betragte saadanne Kurver, hvis Radier tilfredsstiller Betingelserne

(1) $\underline{0 < r_i;\ r_y < \infty.}$

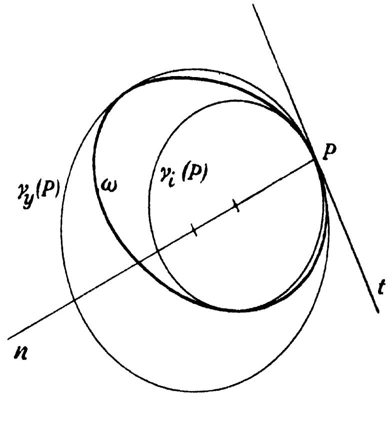


Fig. 7.

Disse Kurver skal siges at udgøre *Klassen K*. Gennem ethvert Punkt P af en Kurve af Klassen K kan der (se Fig. 8) lægges to egentlige Cirkler $c_i(P)$ og $c_y(P)$ med Radier r_i og r_y, saaledes at ethvert indre Punkt for $c_i(P)$ er indre Punkt for ω, ethvert ydre Punkt for $c_y(P)$ er ydre Punkt for ω, og dette er ikke muligt for nogen større Radius end r_i og nogen mindre Radius end r_y[1]. Omvendt vil enhver Jordankurve ω, for hvilken dette er Tilfældet, tilhøre Klassen K og vil have de tilsvarende Radier r_i og r_y; thi i ethvert Punkt P af ω vil Cirklerne $c_i(P)$ og $c_y(P)$ have samme Tangent, og denne Tangent vil være en Støttelinie for ω; men en Jordankurve, som har den Egenskab, at der gennem hvert af dens Punkter gaar en Støttelinie til Kurven, er, som vi i § 2 har set, konveks. Betingelserne (1) udelukker, at Kurven kan have Knæk eller indeholde rette Liniestykker.

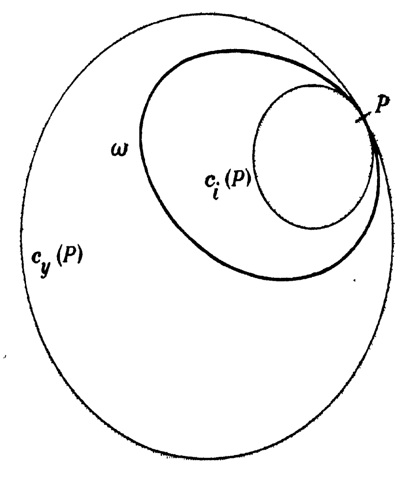


Fig. 8.

Ved de følgende Betragtninger over Kurver af Klassen K vil vi stedse komme til at anvende den her givne Definition af indre og ydre Radius. Drejer det sig imidlertid om at afgøre, hvorvidt en paa Forhaand forelagt Kurve er af Klassen K, kan Definitionen virke noget upraktisk. Vi vil derfor nøjere se, hvad det er den dækker, og saaledes naa til andre Bestemmelser af Radierne r_i og r_y.

Oskulationscirkler og Krumningscirkler.

13. Lad ω være en konveks Jordankurve uden Knæk, som ikke indeholder rette Liniestykker. Bestemmelsen af dens indre og ydre Radius efter den ovenfor givne Definition fordrer en Betragtning af Kurvens

[1] Kinematisk er Radierne r_i og r_y bestemt henholdsvis som Radius i den største Cirkel, som kan rulle i ω, og som Radius i den mindste Cirkel, hvori ω kan rulle.

Forløb »i det Store«; vi vil vise, hvorledes denne Betragtning gennem Indførelsen af Begreberne Oskulationscirkel og Krumningscirkel kan erstattes med en Betragtning af Kurvens Forløb »i det Smaa«.

Lad P være et Punkt af ω, t Tangenten i P. Som *Oskulationscirkel* til ω i Punktet P betegnes enhver Cirkel, der kan fremkomme som Grænsecirkel for en Følge af Cirkler, der rører t i P, og som indeholder et Punkt af ω forskellig fra P, der konvergerer mod P. Findes der saadanne Følger af Cirkler, hvis Radier enten gaar mod Nul eller vokser ud over alle Grænser, regner vi enten Punktet P selv eller Tangenten t for Oskulationscirkel i Punktet.

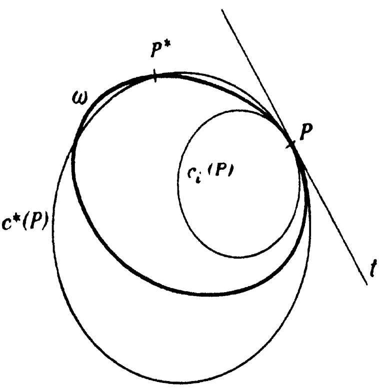


Fig. 9.

Vi vil vise, at *indre og ydre Radius for den givne Kurve bestemmes henholdsvis som nedre og øvre Grænse for Radierne i samtlige Kurvens Oskulationscirkler*, at der m. a. O., naar disse Grænser betegnes henholdsvis r_i^0 og r_y^0, gælder de to Relationer

$$(2) \qquad r_i = r_i^0; \quad r_y = r_y^0.$$

Vi vil nøjes med at bevise den første Relation.

Lad (se Fig. 9) P være et vilkaarligt Punkt af ω, t Tangenten i P; vi betragter Cirklen $c_i(P)$ med Radius r_i, som rører t i P og falder paa samme Side af t som ω. Den aabne Cirkelskive $I(c_i(P))$ tilhører Omraadet $I(\omega)$. Lad $c(P)$ med Radius $r(P)$ være en Oskulationscirkel til ω i Punktet P bestemt som Grænsestilling for en Følge af Cirkler $c^*(P)$, der rører t i P, og som indeholder et Punkt P^* af ω, der konvergerer mod P. For Radius $r^*(P)$ i enhver af disse Cirkler har vi

$$r^*(P) \geq r_i;$$

heraf følger imidlertid straks ved Grænseovergangen, at ogsaa

$$r(P) \geq r_i.$$

Da dette gælder for enhver Oskulationscirkel til ω, er $r_i^0 \geq r_i$.

Vi vil nu vise, at ogsaa omvendt $r_i^0 \leq r_i$. Lad os (se Fig. 10) for et vilkaarligt Punkt P af ω betragte den Cirkel $c_i^0(P)$ med Radius r_i^0, som rører Tangenten t til ω i P og falder paa samme Side af t som ω; vi vil vise, at det Indre af denne Cirkel tilhører Omraadet $I(\omega)$; hermed vil Beviset for den opstillede Sætning være fuldført.

Lad q være en Cirkel med Radius $\varrho < r_i^0$, som rører $c_i^0(P)$ indvendig i P. Fjerner vi Punktet P af q, fremkommer den »opskaarne« Cirkel q. Ethvert Punkt af $I(c_i^0(P))$ tilhører for en passende Værdi af ϱ denne opskaarne Cirkel. Den af ω begrænsede afsluttede konvekse Mængde $\bar{I}(\omega)$ har t til Støttelinie. Lad Q være et Punkt

af den opskaarne Cirkel q. Halvlinien h fra P gennem Q har et afsluttet Liniestykke PR fælles med $\bar{I}(\omega)$. Naar Q gennemløber den opskaarne Cirkel q fra P til P, vil PR efterhaanden optage ethvert Punkt af $\bar{I}(\omega)$ og bortset fra P hvert Punkt netop en Gang. R vil altsaa gennemløbe Kurven ω opskaaret i Punktet P. Vi betragter nu *Liniestykket* QR regnet med Fortegn positivt bort fra P. *Det varierer kontinuert med* Q. Vi vil vise 1. *at det i mindst et Punkt er positivt;* 2. *at det aldrig bliver Nul.* Heraf vil følge, at QR har konstant Fortegn og altsaa stedse er positivt; men det vil atter sige, at Q stedse er indre Punkt af $\bar{I}(\omega)$ og følgelig ligger indenfor ω.

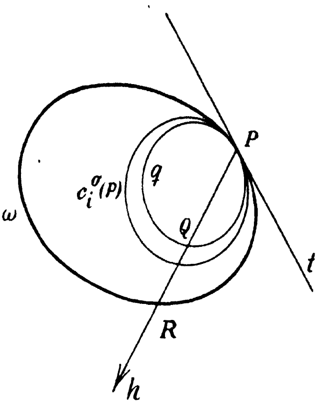


Fig. 10.

1. (Fig. 11). Lad P^* og Q^* være to Punkter af ω og q forskellige fra P med den fælles Projektion T^* paa t. Lad P^*, Q^*, T^* konvergere mod P. Da er

$$\frac{1}{2}\lim\frac{(PT^*)^2}{T^*Q^*} = \varrho < r_i^0 \leq \frac{1}{2}\liminf\frac{(PT^*)^2}{T^*P^*}.$$

Vi kan altsaa bestemme et positivt Tal ε saaledes, at $T^*Q^* > T^*P^*$ naar blot $PT^* < \varepsilon$. De saaledes bestemte Punkter P^* falder udenfor q; lad R være et af dem, og lad Halvlinien h fra P gennem R skære q i Q; da er QR positiv.

2. (Fig. 12). For at vise, at q og ω kun har det ene Punkt P fælles, betragter vi i Almindelighed Mængden D af fælles Punkter for q og ω. Denne Mængde er afsluttet; den maa være endelig, thi i modsat Fald havde den et Fortætningspunkt, i hvilket q og ω havde samme Tangent, og hvori q var en oskulerende Cirkel for ω; men det kan ikke indtræffe. Lad os nu antage, at D indeholdt andre Punkter end P. Lad S være et saadant, hvis Afstand fra P er den mindst mulige. En af Buerne PS paa q er $\leq \pi$ og indeholder kun de to Punkter P og S af D. Vi betegner den q^*. Linien PS har kun de to Punkter P og S fælles med ω. Disse Punkter begrænser to aabne Buer ω^* og ω^{**} paa ω, hvoraf den ene, ω^*, falder paa samme Side af Linien PS som q^*, den anden, ω^{**}, paa den modsatte Side. I Omegnen af P falder ω^* udenfor q; den falder altsaa helt udenfor q. Gennem S trækkes en Linie s parallel med t. Halvtangenten til ω^* i Punktet S kan ikke falde udenfor Parallelstrimlen t, s; thi da Tangenten til ω i S er Støttelinie for ω, vilde dette medføre, at Buen ω^{**} tilhørte den aabne Parallelstrimmel t, s; i Omegnen af P falder denne Bue udenfor q, i Omegnen af S vilde den falde indenfor q; den vilde følgelig have et Punkt fælles med q, som laa P nærmere end S, men et saadant eksisterer

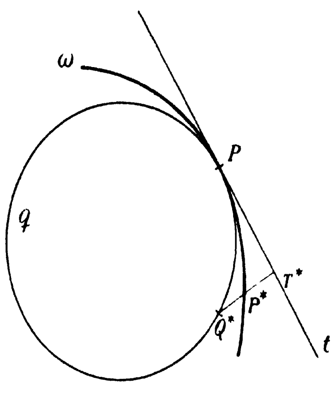


Fig. 11.

ikke. Heraf følger da, at *Buen ω^* tilhører den aabne Parallelstrimmel t, s.* Vi betragter nu et Punkt U^*, som gennemløber den afsluttede Bue ω^*. Gennem U^* trækker vi en Halvlinie u^* parallel med den fælles negative Halvtangent for q^* og ω^* i P. Den skærer q^* i et Punkt V^*. Længden af Liniestykket U^*V^* er en kontinuert Funktion af U^*; den er positiv for alle indre Punkter af Buen, Nul i P og S; den antager da sit Maksimum i et indre Punkt U af Buen. Det tilsvarende Punkt V er et indre Punkt af q^*. Parallelforskyder vi nu ω^* Stykket UV til Stillingen ω_1^*, kommer U til at falde i V, og alle Punkter U^* i en vis Omegn af U kommer til at falde indenfor eller paa q. Tangenten til q i V bliver Tangent til ω_1^*. Nu viser imidlertid et Ræsonnement nøjagtig som det under 1. anstillede, at alle Punkter af ω_1^* i en vis Omegn af V tvært imod falder udenfor q. Hermed er vi naaet til en Modstrid, og Beviset for den opstillede Sætning er fuldført.

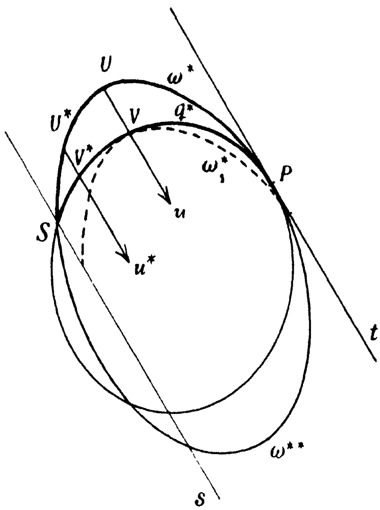


Fig. 12.

14. Lad P være et Punkt af den givne Kurve ω, t Tangenten i P. Lad P^* være et Punkt af ω forskellig fra P, som konvergerer mod P, og lad t^* være ω's Tangent i P^*. Vi betragter Forholdet $\frac{PP^*}{\angle(tt^*)}$ mellem Korden PP^* og Totalkrumningen $\angle(tt^*)$ for den forsvindende Bue PP^* af ω. Som *Krumningscirkel* til ω i Punktet P betegner vi da enhver Cirkel, der rører t i P og falder paa samme Side af t som ω, og hvis Radius, der betegnes som *Krumningsradius* i Punktet, kan fremkomme som Grænseværdi for Størrelsen $\frac{PP^*}{\angle(tt^*)}$.

Begreberne Oskulationscirkel og Krumningscirkel er ikke sammenfaldende. Vi vil imidlertid vise, at det i fuldkommen Analogi med den ovenfor beviste Sætning gælder, at *indre og ydre Radius i den givne Kurve ω bestemmes som nedre og øvre Grænse for samtlige Kurvens Krumningsradier*[1]. Betegner vi disse Grænser r_i^k og r_y^k, gælder m. a. O. de to Relationer

$$r_i = r_i^k; \quad r_y = r_y^k. \tag{3}$$

Vi vil nøjes med at bevise den første Relation.

Er $r_i = 0$ har vi sikkert $r_i^k \geq r_i$. Vi vil vise, at det samme er Tilfældet naar $r_i > 0$. Lad (se Fig. 13) P være et vilkaarligt Punkt af ω, t Tangenten i P. Vi betragter Cirklen $c_i(P)$ med Radius r_i, hvis Indre $I(c_i(P))$ tilhører $I(\omega)$. Dens Centrum A_i falder paa Halvnormalen n til t i P. Gennem A_i trækkes en Linie t_i parallel

[1] Smlgn. W. Blaschke, Kreis und Kugel, (1916), S. 114—17.

med t. Lad R og S være de Punkter af ω, hvori Tangenten er vinkelret paa t; lad P^* være et Punkt af Buen RPS forskellig fra P, og lad t^* og n^* betegne Tangenten og Normalen til ω i P^*. n og n^* skærer hinanden i et Punkt Q^*. Lad nu yderligere P^* falde paa samme Side af t_i som P. Vi betragter Cirklen $c_i(P^*)$ med Radius r_i, hvis Indre $I(c_i(P^*))$ tilhører $I(\omega)$; den skærer ikke t. Dens Centrum A_i^* falder derfor paa den modsatte Side af t_i som t. Nu falder A_i^* tillige paa en Cirkel a_i^* med Centrum P^* og Radius r_i. Da $P^*A_i > r_i$ vil samtlige Punkter af denne Cirkel, som falder paa modsat Side af t_i som t, falde paa samme Side af n som P^*. Halvlinien n^*, som gaar gennem A_i^*, skærer følgelig n i et Punkt paa Forlængelsen af PA_i ud over A_i, og vi har $PQ^* > r_i$. Lader vi nu P^* konvergere mod P, faar vi

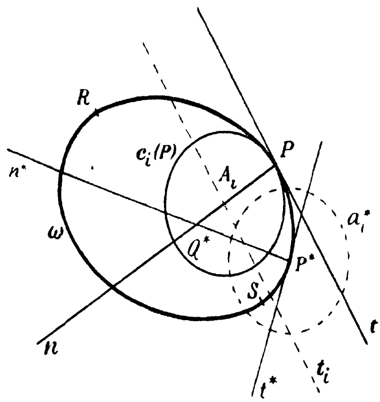


Fig. 13

$$\liminf \frac{PP^*}{\angle(tt^*)} = \liminf \frac{PP^*}{\angle(nn^*)}$$

$$= \liminf \frac{PP^*}{\sin(PQ^*P^*)} = \liminf \frac{PQ^*}{\sin(PP^*Q^*)} = \liminf PQ^* \geqq r_i.$$

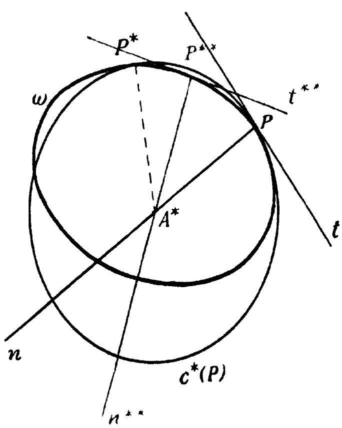


Fig. 14.

Men da P var et vilkaarligt Punkt af ω, følger heraf straks, at $r_i^k \geqq r_i$.

Vi vil nu vise, at ogsaa omvendt $r_i^k \leqq r_i$; hermed vil Beviset for den opstillede Sætning være fuldført. Ifølge den foregaaende Sætning er dette ensbetydende med, at $r_i^k \leqq r_i^o$. Vi viser Rigtigheden af denne Relation, idet vi viser, at *enhver Oskulationscirkel til en konveks Kurve tillige er Krumningscirkel til Kurven i samme Punkt*[1].

Lad (se Fig. 14) P være et Punkt af Kurven med Tangenten t, $c(P)$ en Oskulationscirkel i Punktet bestemt som Grænsestilling for en Følge af Cirkler $c^*(P)$, som rører t i P, og som gaar gennem et Punkt P^* af Kurven, som konvergerer mod P. Centrerne A^* for disse Cirkler falder paa den indadrettede Halvnormal n til Kurven i Punktet P. Radius $r(P)$ i $c(P)$ fremkommer som Grænseværdi for Radius $r^*(P) = PA^*$ i $c^*(P)$, naar P^* konvergerer mod P. Da $A^*P = A^*P^*$ vil der svarende til ethvert af de betragtede Punkter P^* findes et indre Punkt P^{**} af Buen PP^*, hvis Afstand fra A^* er enten mindst mulig eller

[1] J. Hjelmslev, Über die Grundlagen der kinematischen Geometrie, Acta math. 47, (1925), S. 155.

størst mulig. Lad Tangenten i dette Punkt være t^{**}; den indadrettede Halvnormal n^{**} i P^{**} gaar gennem A^*. Naar P^* konvergerer mod P, vil ogsaa P^{**} konvergere mod P, og vi faar

$$r(P) = \lim r^*(P) = \lim \frac{PA^*}{\sin(PP^{**}A^*)} = \lim \frac{PP^{**}}{\sin(PA^*P^{**})} = \lim \frac{PP^{**}}{\angle(nn^{**})} = \lim \frac{PP^{**}}{\angle(tt^{**})}.$$

Altsaa er $r(P)$ Krumningsradius, $c(P)$ Krumningscirkel til Kurven i Punktet P, og den opstillede Sætning er bevist.

15. Af de beviste Sætninger følger specielt, at

$$r_i^o = r_i^k; \quad r_y^o = r_y^k. \tag{4}$$

For enhver lukket konveks Kurve uden Knæk, som ikke indeholder rette Liniestykker, er nedre og øvre Grænse for Oskulationscirklernes Radier lig med nedre og øvre Grænse for Krumningscirklernes Radier. Vi vil vise, at den samme Sætning gælder for *vilkaarlige, aabne eller afsluttede, konvekse Buer.*

En konveks Bue uden Knæk, som ikke indeholder rette Liniestykker, kan altid deles i simple Buer, d. v. s. Buer, hvis Totalkrumning er mindre end eller lig med π. Vi kan derfor nøjes med at bevise Sætningen for saadanne Buer. Endvidere kan vi nøjes med at betragte afsluttede Buer; thi en aaben Bue vil altid være sammensat af en Følge af afsluttede Buer. Beviset føres nu simpelthen saaledes, at vi viser, at en afsluttet simpel Bue altid vil være Del af en konveks Jordankurve, som kan vælges saadan, at øvre og nedre Grænse for Radierne i Krumningscirklerne og Oskulationscirklerne er de samme for Jordankurven som for den givne Bue. Lad Buen være b, dens Endepunkter A og B. Er dens Totalkrumning netop π, vil vi ved til Buen at føje den Bue b', der fremgaar af b ved en Drejning paa 180^0 omkring Midtpunktet af Liniestykket AB, faa en Jordankurve af den ønskede Art. Er Buens Totalkrumning mindre end π, kan den samme Metode anvendes, naar vi blot først gennem en passende Forlængelse af Buen ved Hjælp af kongruente Buer har forøget dens Totalkrumning til π.

Lad os paa en konveks Jordankurve ω betragte de to eventuelt flertydige Funktioner $r^o(P)$ og $r^k(P)$ af det variable Punkt P paa Kurven, som bestemmer Radierne henholdsvis i Oskulationscirklerne og i Krumningscirklerne i Punktet P. For ethvert Punkt P af ω vil Værdierne $r^o(P)$ udgøre en Delmængde af Værdierne $r^k(P)$. Lad os betragte Funktionen $r^o(P)$; for enhver aaben Bue b paa ω bestemmer vi nedre og øvre Grænse $r_i^o(b)$ og $r_y^o(b)$ for $r^o(P)$, naar P gennemløber b. Lad P_0 være et bestemt Punkt af Kurven ω; øvre Grænse for $r_i^o(b)$ for samtlige Buer b, som indeholder P_0, betegner vi $\varrho^o(P_0)$, nedre Grænse for $r_y^o(b)$ over disse Buer betegner vi $P^o(P_0)$. Funktionerne $\varrho^o(P)$ og $P^o(P)$ er entydige Funktioner paa Kurven ω. De betegnes henholdsvis som den nedre og den øvre *Limesfunktion* for Funktionen $r^o(P)$. Paa tilsvarende Maade defineres Limesfunktionerne $\varrho^k(P)$ og $P^k(P)$ for Funktionen $r^k(P)$. For enhver Bue b paa ω er

$$r_i^o(b) = r_i^k(b); \quad r_y^o(b) = r_y^k(b).$$

Følgelig er for ethvert Punkt P af ω

$$\varrho^o(P) = \varrho^k(P); \quad P^o(P) = P^k(P). \tag{5}$$

Funktionerne $r^o(P)$ og $r^k(P)$ har m. a. O. de samme Limesfunktioner. Omvendt er denne Egenskab ved disse Funktioner tilstrækkelig til at sikre, at de paa enhver Bue har samme øvre og nedre Grænse.[1]

Sætninger om Skæringspunkter og Skæringsvinkler.

16. Lad der i en Plan med fastlagt Begyndelsespunkt O være givet to konvekse Jordankurver ω og ω^* *af Klassen* K (§ 12) med de tilsvarende Radier r_i og r_y og r_i^* og r_y^*, og lad os antage, at

$$\underline{r_y \geqq r_i > r_y^* \geqq r_i^*}. \tag{6}$$

Lad P være et variabelt Punkt i Planen; for enhver Stilling af Punktet P betragter vi den Kurve $P+\omega^*$, som (§ 4) fremgaar af ω^* ved Parallelforskydningen OP.

Vi vil vise følgende Sætning: *Kurverne ω og $P+\omega^*$ har stedse højst to Punkter fælles. Rører de hinanden, er Røringspunktet det eneste fælles Punkt, og, omvendt, har Kurverne kun et fælles Punkt, rører de hinanden i dette Punkt.*

Vi indfører for Kurven $P+\omega^*$ den afkortede Betegnelse ω'.

Lad os først betragte det Tilfælde, hvor Kurverne ω og ω' kun har et Punkt Q fælles. ω' kan ikke indeholde saavel et Punkt R af $I(\omega)$ som et Punkt S af $Y(\omega)$, thi i saa Fald vilde der paa hver af de aabne Buer RS paa ω' findes et Punkt af ω. ω' maa altsaa, bortset fra Punktet Q, enten helt tilhøre Omraadet $I(\omega)$ eller Omraadet $Y(\omega)$. Men heraf følger umiddelbart, at ω og ω' i Punktet Q har samme Tangent.

Har omvendt ω og ω' Punktet Q fælles, og har de i dette Punkt samme Tangent t, kan de kun have det ene Punkt Q fælles. Falder Kurverne paa hver sin Side af t, er dette indlysende. Falder de paa samme Side af t, betragter vi (Fig. 15) de to Cirkler $c_i(Q)$ og $c'_y(Q)$ med Radier r_i og r_y^*, som rører t i Q, og som falder paa samme Side af t som ω og ω'. ω' tilhører den afsluttede Cirkelskive $\bar{I}(c'_y(Q))$; da $r_y^* < r_i$ tilhører denne bortset fra Punktet Q den aabne Cirkelskive $I(c_i(Q))$, som atter tilhører $I(\omega)$. ω' tilhører altsaa bortset fra Punktet Q Omraadet $I(\omega)$ og har saaledes kun det ene Punkt Q fælles med ω.

Vi gaar nu over til at betragte det Tilfælde, hvor ω og ω' har mere end et fælles Punkt. Kurverne kan ikke have samme Tangent i noget af de fælles Punkter. Heraf følger straks, at Kurverne maa have et endeligt Antal Punkter fælles. I modsat

[1] Med Hensyn til Rækkevidden af de i dette Afsnit beviste Sætninger bemærkes det, at Forudsætningen om den betragtede Kurve ω, at den hverken maatte have Knæk eller indeholde rette Liniestykker, forsaavidt er uvæsentlig, som man ved passende Definitioner let kan opnaa, at Resultaterne bliver almengyldige. Man maa blot i Stedet for Tangenter betragte Støttelinier til Kurven, i Stedet for Røringspunkter Støttepunkter, og man maa tage Hensyn til, at der gennem et Punkt af Kurven kan gaa flere Støttelinier, paa en Støttelinie til Kurven kan ligge flere Støttepunkter.

Fald havde nemlig Mængden af fælles Punkter et Fortætningspunkt, som ligeledes var et fælles Punkt for ω og ω', men i dette Punkt havde Kurverne samme Tangent.

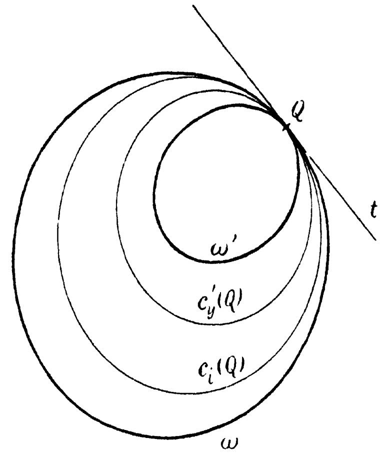


Fig. 15.

For at vise at Antallet af Skæringspunkter netop er to, antager vi, i den Hensigt at naa til en Modstrid, at der var flere, altsaa mindst tre. Disse Punkter deler ω i lige saa mange Buer. Højst en af disse Buer har en Totalkrumning $\geqq \pi$. Der vil altsaa sikkert findes to paa hinanden følgende Buer QR og RS (se Fig. 16), som begge har en Totalkrumning $< \pi$. Ingen af disse Buer har noget indre Punkt fælles med ω'. Bortset fra Endepunkterne maa Buerne følgelig hver for sig enten tilhøre Omraadet $I(\omega')$ eller Omraadet $Y(\omega')$; og de maa tilhøre hver sit af de to Omraader, thi ellers havde ω og ω' i Punktet R samme Tangent. Lad os antage, at det er Buen QR paa ω, som bortset fra Endepunkterne tilhører $Y(\omega')$. Vi betegner den b. Samtidig med denne Bue betragter vi den Bue QR paa ω', som ikke indeholder S. Vi betegner den b'. Linien QR har kun de to Punkter Q og R fælles med ω og ω'; den deler hver af Kurverne i to Buer, en paa hver Side af Linien. Buerne b og b' falder paa samme Side af Linien QR, nemlig begge paa den modsatte Side af QR som S.

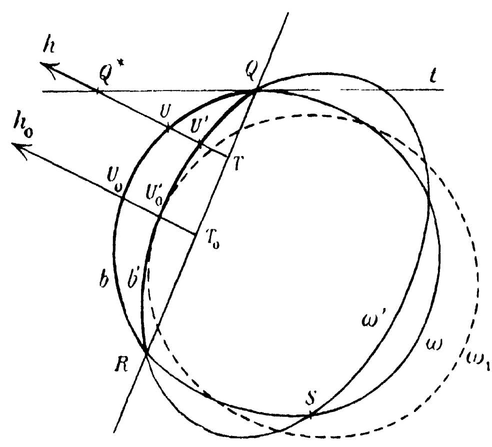


Fig. 16.

Lad T være et indre Punkt af Liniestykket QR, h en Halvlinie, som udgaar fra Punktet, og som er rettet ind i den Halvplan, begrænset af Linien QR, hvori b og b' ligger. Da Totalkrumningen for Buen b er $< \pi$, kan h vælges saaledes, at den *ikke er parallel med nogen Tangent til* b. T tilhører saavel $I(\omega)$ som $I(\omega')$. Halvlinien h vil derfor skære b og b' hver i netop et Punkt. Disse Punkter betegner vi U og U'. De er indre Punkter af b og b'. Punkterne T, U, U' er indbyrdes forskellige. De følger efter hinanden i Ordenen T, U', U paa h, thi det afsluttede Liniestykke TU' tilhører $\bar{I}(\omega')$, medens U tilhører $Y(\omega')$. Vi lader nu T gennemløbe det aabne Liniestykke

QR, medens vi holder Retningen af h fast. Længden af Liniestykket UU' er da en kontinuert Funktion af T. Konvergerer T mod et af Liniestykkets Endepunkter, f. Eks. Q, vil denne Funktion konvergere mod Nul. Thi Buen b falder paa samme Side af sin Tangent t i Q som T. Det betragtede Liniestykke UU' er derfor en Del af Liniestykket TQ^*, hvor Q^* er Skæringspunktet mellem h og t, og dette Liniestykke konvergerer mod Nul, naar T konvergerer mod Q. Funktionen har et Maksimum, som den antager for et Punkt T_0 af QR. Lad de tilsvarende Punkter paa b og b' være U_0 og U'_0. Vi parallelforskyder nu Kurven ω Stykket $U_0U'_0$ til Stillingen ω_1. Herved føres U_0 over i Punktet U''_0, medens Punkter af ω i en vis Omegn af U_0 føres over i Punkter indenfor eller paa ω'. Kurverne ω' og ω_1 faar altsaa samme Tangent i Punktet U'_0. Men da følger det af Ræsonnementet ovenfor, at ω_1 bortset fra Punktet U'_0 falder helt udenfor ω'. Hermed er vi naaet til en Modstrid, og Beviset er fuldført.[1]

17. Vi vil nærmere undersøge, for hvilke Punkter P i Planen Kurverne ω og $P+\omega^*$ har enten 0, 1 eller 2 Punkter fælles. Vi vil vise følgende Sætning:

Mængden af Punkter P, for hvilke ω og $P+\omega^*$ har netop et fælles Punkt, bestaar af to Jordankurver ω_i og ω_y, hvor ω_y omslutter ω_i. For ethvert Punkt P, som tilhører enten $I(\omega_i)$ eller $Y(\omega_y)$, har ω og $P+\omega^*$ intet Punkt fælles; for ethvert Punkt P, som tilhører saavel $Y(\omega_i)$ som $I(\omega_y)$, har ω og $P+\omega^*$ to Punkter fælles.

Mængden af Punkter P, for hvilke ω og $P+\omega^*$ har netop et fælles Punkt, betegner vi M_1. Lad P_1 være et Punkt af M_1, og lad ω og $P_1+\omega^*$ have det fælles Punkt P. Ifølge Sætningen ovenfor vil ω og $P_1+\omega^*$ i Punktet P have samme Tangent t. Kurven $P-\omega^*$, som fremgaar af $P_1+\omega^*$ ved en halv Omdrejning omkring Midtpunktet af Liniestykket PP_1, gaar gennem P_1; dens Tangent t_1 i P_1 er parallel med t.

Lad omvendt (se Fig. 17) P være et Punkt af ω med Tangenten t, og lad os paa Kurven $P-\omega^*$ bestemme de to Punkter, hvori Tangenten er parallel med t. I det ene Punkt P_i falder $P-\omega^*$ paa modsat Side af sin Tangent t_i som ω af t, i det andet Punkt P_y paa samme Side af Tangenten t_y som ω af t. Da tilhører saavel P_i som P_y Mængden M_1. Thi Kurverne $P_i+\omega^*$ og $P_y+\omega^*$ rører begge ω i Punktet P og har følgelig kun dette ene Punkt fælles med ω.

Lader vi P variere kontinuert paa ω, vil P_i og P_y variere kontinuert. Gennemløber P Kurven ω, vil P_i og P_y gennemløbe to lukkede Kurver uden Dobbeltpunkter, m. a. O. to Jordankurver ω_i og ω_y. ω_i og ω_y har ingen fælles Punkter; tilsammen udgør de Mængden M_1.

Mængden af Punkter P, for hvilke ω og $P+\omega^*$ ikke har noget fælles Punkt, betegner vi M_0, Mængden af Punkter, for hvilke disse Kurver har to fælles Punkter, betegner vi M_2. Saavel M_0 som M_2 er, som man uden Vanskelighed ser, aabne

---

[1] Vor Forudsætning (6) om Kurverne ω og ω^*, at $r_i > r_y^*$, kan (smlgn. § 14) siges at udtrykke, at Kurven ω er »helt igennem« svagere krummet end Kurven ω^*. Hensigten med Beviset ovenfor er at gøre den (i og for sig ikke overraskende) Kendsgerning øjensynlig, at af to simple konvekse Buer med fælles Endepunkter den stærkest krummede falder udenfor den svagere krummede

Punktmængder. Ethvert Randpunkt for M_0 eller M_2 vil følgelig tilhøre M_1. Vi vil vise, at M_1 er den fælles Begrænsning for M_0 og M_2.

Vi betragter paany (Fig. 17) de to Punkter P_i og P_y af ω_i og ω_y, som svarer til et vilkaarligt Punkt P af ω med Tangenten t. Kurverne $P_i + \omega^*$ og $P_y + \omega^*$ rører begge t i P. Den første falder paa samme Side af t som ω, den anden paa den modsatte Side. At P_i er Randpunkt for M_0 følger af, at $P_i + \omega^*$ tilhører en afsluttet

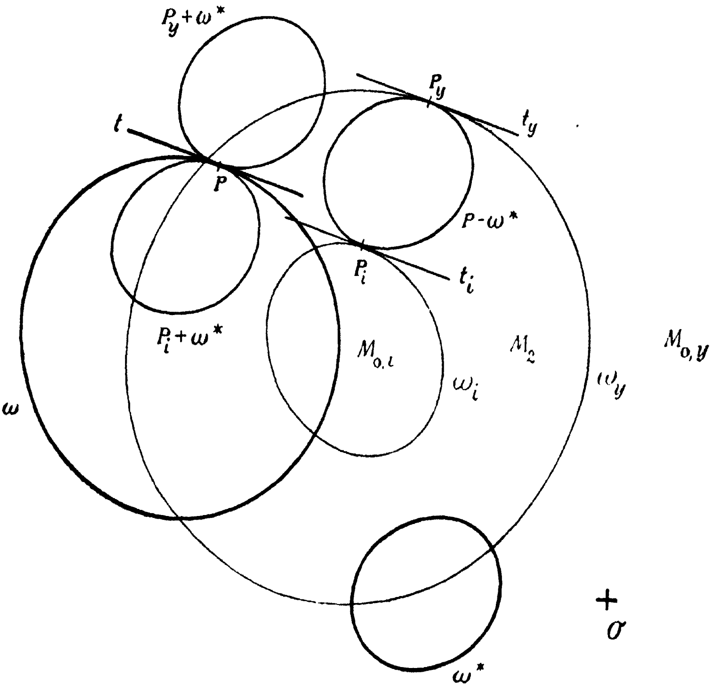


Fig. 17.

Cirkelskive med Radius r_y^*, som rører t i P, denne bortset fra Punktet P atter en aaben Cirkelskive med Radius r_i, som ligeledes rører t i P, og som tilhører Omraadet $I(\omega)$. En lille Forskydning af $P_i + \omega^*$ vinkelret paa t og bort fra t vil derfor bringe Kurven indenfor ω. At P_i er Randpunkt for M_2 følger af, at en lille Forskydning af $P_i + \omega^*$ den modsatte Vej vil bringe Kurven til Skæring med ω. At ogsaa P_y er Randpunkt saavel for M_0 som for M_2 følger af, at en lille Forskydning af $P_y + \omega^*$ vinkelret paa t og bort fra t vil bringe den udenfor ω, medens en lille Forskydning den modsatte Vej vil bringe den til Skæring med ω.

Det fremgaar af disse Betragtninger, at Kurven ω_i begrænser en Delmængde af M_0, nemlig Mængden $M_{0,i}$ af Punkter P, for hvilke Kurven $P + \omega^*$ tilhører $I(\omega)$; thi fòr et Randpunkt P_i for denne Mængde gælder det øjensynlig, at $P_i + \omega^*$ bortset fra et Punkt P

af ω tilhører $I(\omega)$, og, omvendt, tilhører $P_i + \omega^*$ bortset fra et Punkt P af ω Omraadet $I(\omega)$, er P_i et Randpunkt for $M_{0,i}$. Nu er $M_{0,i}$ en begrænset Mængde; følgelig er $M_{0,i} = I(\omega_i)$. Paa tilsvarende Maade ses, at ω_y begrænser en anden Delmængde af M_0, nemlig Mængden $M_{0,y}$ af Punkter P, for hvilke $P + \omega^*$ tilhører $Y(\omega)$. Nu er $M_{0,y}$ ikke begrænset; følgelig er $M_{0,y} = Y(\omega_y)$. De to Mængder $M_{0,i}$ og $M_{0,y}$, d. v. s. de to Omraader $I(\omega_i)$ og $Y(\omega_y)$, udgør tilsammen M_0; de har ingen Punkter fælles; følgelig er $I(\omega_i)$ en Delmængde af $\bar{I}(\omega_y)$; men heraf følger, at ω_y omslutter ω_i. Endelig ser vi, at M_2 vil bestaa af samtlige Punkter, som hverken tilhører $\bar{I}(\omega_i)$ eller $\bar{Y}(\omega_y)$, d. v. s. af samtlige Punkter, som tilhører saavel $Y(\omega_i)$ · som $I(\omega_y)$. Hermed er den opstillede Sætning fuldstændig bevist.

18. Vi vil nu vise, at *Kurverne ω_i og ω_y er konvekse Kurver af Klassen K, hvis Radier $r_{i,i}$, $r_{i,y}$ og $r_{y,i}$, $r_{y,y}$ tilfredsstiller Betingelserne*

$$\text{(7)}[1] \qquad \frac{r_i - r_y^* \leqq r_{i,i}; \quad r_{i,y} \leqq r_y - r_i^*}{r_i + r_i^* \leqq r_{y,i}; \quad r_{y,y} \leqq r_y + r_y^*}.$$

Vi vil nøjes med at gennemføre Beviset for Kurven ω_i.

Lad P_i være et vilkaarligt Punkt af ω_i; det svarer til et Punkt P af ω. Tangenterne t til ω i P og t_i til $P - \omega^*$ i P_i er parallelle. Paa den Halvnormal n_i til t_i i P_i, som falder paa samme Side af t_i som ω af t, bestemmer vi de to Punkter $A_{i,i}$ og $A_{i,y}$, hvis Afstand fra P_i er henholdsvis $r_i - r_y^*$ og $r_y - r_i^*$. Omkring $A_{i,i}$ og $A_{i,y}$ beskrives to Cirkler $c_{i,i}(P_i)$ og $c_{i,y}(P_i)$ med Radier $r_i - r_y^*$ og $r_y - r_i^*$. De rører t_i i P_i. For at vise den opstillede Sætning er det (smlgn. § 12) tilstrækkeligt at vise 1. at $I(c_{i,i}(P_i))$ tilhører $I(\omega_i)$, 2. at $Y(c_{i,y}(P_i))$ tilhører $Y(\omega_i)$.

1. (Fig. 18). Cirklen $c_i(P)$ med Radius r_i, som gaar gennem P, og for hvilken $I(c_i(P))$ tilhører $I(\omega)$, rører t i P. Dens Centrum A_i falder paa Halvnormalen n til t i P paa samme Side af t som ω. Paa Kurven ω^* betragter vi Punktet $P^* = P - P_i$ med Tangenten $t^* = P - t_i = t - P_i$. Cirklen $c_y^*(P^*)$ med Radius r_y^*, som gaar gennem P^*, og for hvilken $Y(c_y^*(P^*))$ tilhører $Y(\omega^*)$, rører t^* i P^*. Dens Centrum A_y^* falder paa Halvnormalen n^* til t^* i P^* paa samme Side af t^* som ω^*. Liniestykket OA_y^* er, som en simpel Betragtning viser, lig og parallel med Liniestykket $A_{i,i}\, A_i$.

Lad nu Q være et Punkt af $I(c_{i,i}(P_i))$; vi skal vise, at det tilhører $I(\omega_i)$. Det betyder ifølge det foregaaende, at Kurven $Q + \omega^*$ skal tilhøre $I(\omega)$. Kurven $Q + \omega^*$ tilhører den afsluttede Cirkelskive $\bar{I}(Q + c_y^*(P^*))$, hvis Centrum er $Q + A_y^*$. Dette Punkts Afstand fra A_i er lig med Afstanden fra Q til $A_{i,i}$, som er mindre end $r_i - r_y^*$. Cirkelskiven $\bar{I}(Q + c_y^*(P^*))$, hvis Radius er r_y^*, tilhører altsaa den aabne Cirkelskive $I(c_i(P))$, hvis Radius er r_i; denne tilhører atter $I(\omega)$. Følgelig tilhører $Q + \omega^*$ Omraadet $I(\omega)$, og Punktet Q er, som vi vilde vise, et indre Punkt for ω_i.

2. (Fig. 19). Cirklen $c_y(P)$ med Radius r_y, som gaar gennem P, og for hvilken

---

[1] Man viser yderligere, hvad vi dog ikke kommer til at anvende i det følgende, at

$$r_{i,i} \leqq \frac{r_i - r_i^*}{r_y - r_y^*} \leqq r_{i,y}; \quad r_{y,i} \leqq \frac{r_i + r_y^*}{r_y + r_i^*} \leqq r_{y,y}.$$

$Y(c_y(P))$ tilhører $Y(\omega)$, rører t i P; dens Centrum A_y falder paa Halvnormalen n til t i P paa samme Side af t som ω. Cirklen $c_i^*(P^*)$ med Radius r_i^*, som gaar gennem P^*, og for hvilken $I(c_i^*(P^*))$ tilhører $I(\omega^*)$, rører t^* i P^*. Dens Centrum A_i^* falder paa Halvnormalen n^* til t^* i P^* paa samme Side af t^* som ω^*. Liniestykket OA_i^* er lig og parallel med Liniestykket $A_{i,y}\,A_y$.

Vi skal vise, at Omraadet $Y(c_{i,y}(P_i))$ tilhører Omraadet $Y(\omega_i)$, eller at ω_i tilhører den afsluttede Cirkelskive $\bar{I}(c_{i,y}(P_i))$. Lad P_i' være et Punkt af ω_i. Vi skal vise, at dets Afstand fra $A_{i,y}$ højst er lig med $r_y - r_i^*$ eller, hvad der kommer ud paa det samme, at Afstanden fra Punktet $P_i' + A_i^*$ til Punktet A_y i det højeste er lig med $r_y - r_i^*$. Dette er ensbetydende med, at den aabne Cirkelskive $I(P_i' + c_i^*(P^*))$ tilhører $I(c_y(P))$. Nu tilhører den aabne Cirkelskive $I(P_i' + c_i^*(P^*))$ Omraadet $I(P_i' + \omega^*)$. Da P_i' er et Punkt af ω_i, tilhører $I(P_i' + \omega^*)$ Omraadet $I(\omega)$, som atter tilhører $I(c_y(P))$. Følgelig tilhører $I(P_i' + c_i^*(P^*))$ den aabne Cirkelskive $I(c_y(P))$, og Beviset for den opstillede Sætning er fuldført.

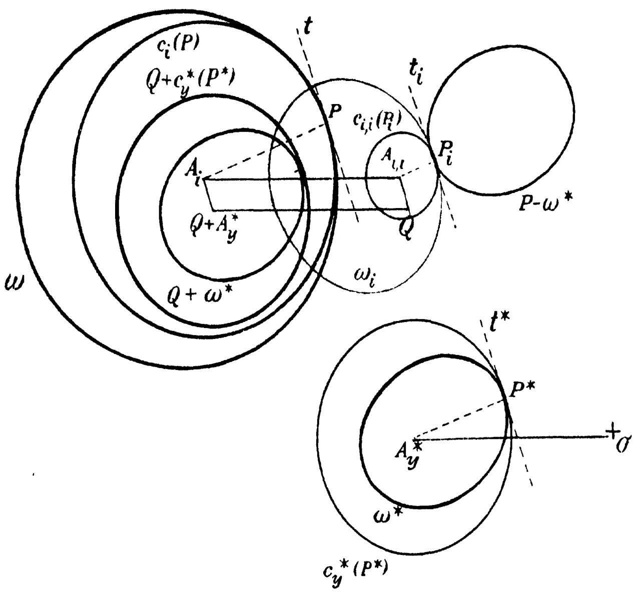


Fig. 18.

19. Vi betragter paany Mængden M_2 af Punkter P i Planen, for hvilke Kurverne ω og $P+\omega^*$ har to Punkter fælles. Lad (Fig. 20) P være et vilkaarligt Punkt af denne Mængde; de fælles Punkter for ω og $P+\omega^*$ betegner vi $P' = P + P^{*\prime}$ og $P'' = P + P^{*\prime\prime}$; $P^{*\prime}$ og $P^{*\prime\prime}$ er Punkter af ω^*. Punkterne P' og P'' er sikkert begge Skæringspunkter for ω og $P+\omega^*$; som Skæringsvinkler betegner vi de konvekse Vinkler p' og p'' mellem de indadrettede Halvnormaler til ω og $P+\omega^*$ i Punkterne P' og P''. Den mindste Afstand fra P til Begrænsningen for M_2 d. v. s. til et Punkt af ω_i eller ω_y betegner vi d; den er lig med den mindste Forskydning af Kurven $P+\omega^*$, som bringer den til at røre ω. Vi vil vise, at der for ethvert Punkt P af M_2 gælder de to Relationer

$$(8) \qquad \frac{\sqrt{d}}{\sin p'} < \sqrt{\frac{2 r_i r_y^*}{r_i - r_y^*}}; \quad \frac{\sqrt{d}}{\sin p''} < \sqrt{\frac{2 r_i r_y^*}{r_i - r_y^*}}.$$

Vi kan nøjes med at bevise den første Relation. Vi betragter Cirklerne $c_i(P')$ og $c_y(P')$ med Radier r_i og r_y, som rører ω i P' og falder paa samme Side af Tangenten i P' som ω selv. Deres Centrer betegner vi henholdsvis A'_i og A'_y. Endvidere betragter vi Cirklen $c^*_y(P^{*\prime})$ med Radius r^*_y, som rører ω^* i $P^{*\prime}$ og falder paa samme Side af Tangenten til ω^* i $P^{*\prime}$ som ω^*. Dens Centrum betegner vi $A^{*\prime}_y$. Cirklen

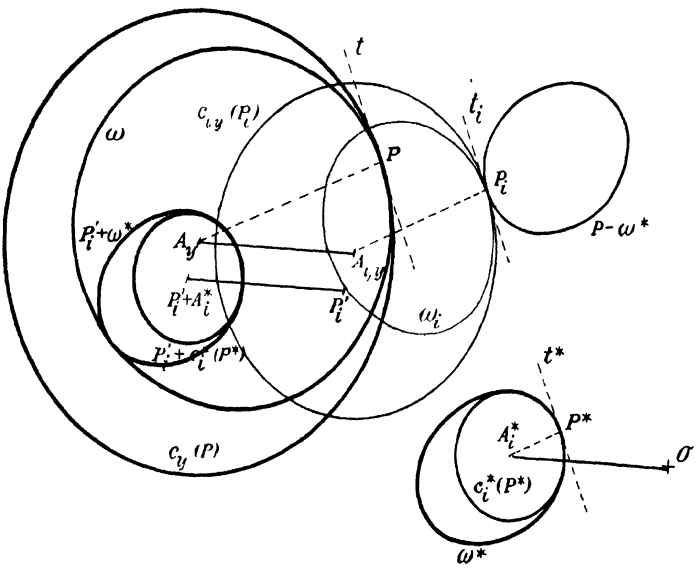


Fig. 19.

$P+c^*_y(P^{*\prime})$ har sit Centrum i Punktet $P+A^{*\prime}_y$; den skærer $c_i(P')$ og $c_y(P')$ i P' under Vinklen p'.

Vi parallelforskyder Cirklen $P+c^*_y(P^{*\prime})$ langs Liniestykket $P+A^{*\prime}_y\,A'_i$, til den kommer til at røre $c_i(P')$ indvendig. Herved kommer $P+\omega^*$ til at falde indenfor ω, P følgelig til at falde indenfor ω_i. Længden d'_i af Forskydningen er derfor mindst lig med Afstanden d_i fra P til ω_i. Ved Betragtning af Trekanten $P+A^{*\prime}_y\,P'\,A'_i$, hvori Vinklen ved P' er p' og Siden $P+A^{*\prime}_y\,A'_i$ har Længden $d'_i+r_i-r^*_y$, faas nu

$$2r_i r^*_y \sin p' = \sqrt{(2r_i+d'_i)\,(2r_i-2r^*_y+d'_i)\,(2r^*_y-d'_i)\,d'_i}$$

hvoraf, da $d'_i \geqq d_i \geqq d$

$$\frac{\sqrt{d}}{\sin p'} < \frac{k_i}{\sqrt{2r^*_y-d'_i}},$$

hvor $k_i = \dfrac{r_i r^*_y}{\sqrt{r_i(r_i-r^*_y)}}$.

Paa samme Maade ses, at den Parallelforskydning d_y' af Cirklen $P + c_y^*(P^{*'})$ langs Forlængelsen af Liniestykket $A_y' \; P + A_y^{*'}$ ud over $P + A_y^{*'}$, som bringer den til at røre $c_y(P')$ udvendig, vil bringe $P + \omega^*$ til at falde udenfor ω og følgelig P til at falde udenfor ω_y, saaledes at d_y' bliver større end eller lig med Afstanden d_y fra

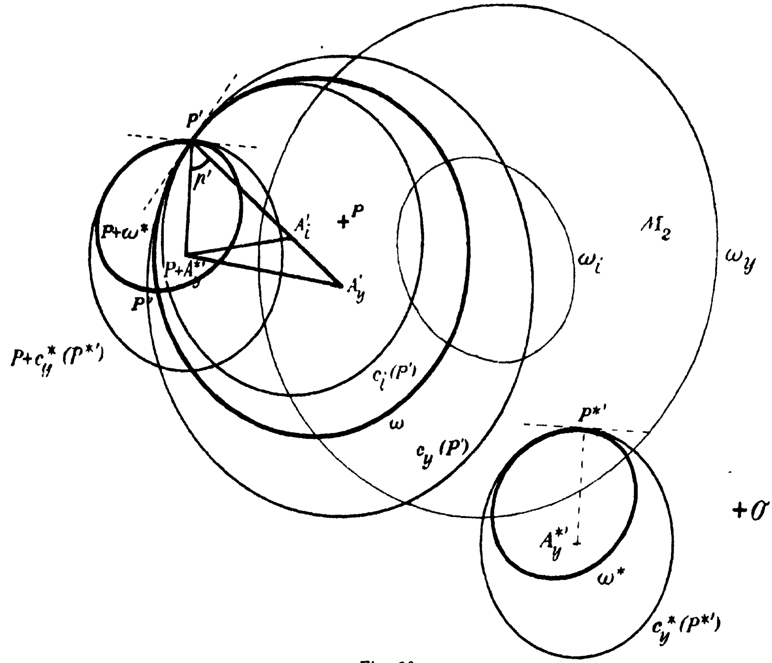


Fig. 20.

P til ω_y. Ved Betragtning af Trekanten $P + A_y^{*'} \; P' \; A_y'$, hvori Vinklen ved P' er p' og Siden $P + A_y^{*'} \; A_y'$ har Længden $r_y + r_y^* - d_y'$, faas nu

$$2 r_y r_y^* \sin p' = \sqrt{(2 r_y + 2 r_y^* - d_y')\,(2 r_y - d_y')\,(2 r_y^* - d_y')\, d_y'}$$

hvoraf, da $d_y' \geqq d_y \geqq d$ og $d_y' < 2 r_y^*$

$$\frac{\sqrt{d}}{\sin p'} < \frac{k_y}{\sqrt{2 r_y^* - d_y'}},$$

hvor $k_y = \dfrac{r_y r_y^*}{\sqrt{r_y (r_y - r_y^*)}}$.

Nu er for ethvert Punkt P af M_2 i hvert Fald den ene af Størrelserne $2 r_y^* - d_i'$

og $2r_y^* - d_y'$ større end f. Eks. $\frac{r_y^*}{2}$. Thi

1. er $p' \leq \frac{\pi}{2}$ har man

$$(d_i' + r_i - r_y^*)^2 \leq r_y^{*2} + r_i^2 < r_y^* r_i + r_i^2 < \left(\frac{r_y^*}{2} + r_i\right)^2,$$

altsaa $d_i' + r_i - r_y^* < \frac{r_y^*}{2} + r_i$ og $2r_y^* - d_i' > \frac{r_y^*}{2}$; og

2. er $p' > \frac{\pi}{2}$ har man

$$r_y + r_y^* - d_y' > r_y \text{ og } 2r_y^* - d_y' > r_y^*.$$

Da nu $r_y \geq r_i > r_y^*$ har man $k_i \geq k_y$. Det fremgaar da af de fundne Uligheder, at for ethvert Punkt P af M_2

$$\frac{\sqrt{d}}{\sin p'} < \frac{k_i}{\sqrt{\frac{r_y^*}{2}}} = \sqrt{\frac{2r_i r_y^*}{r_i - r_y^*}},$$

hvormed den opstillede Sætning er bevist.

Parallelkurver til en konveks Kurve.

20. Lad (se Fig. 21) ω være en konveks Jordankurve af Klassen K med Radierne r_i og r_y. Lad P være et vilkaarligt Punkt af Kurven med Tangenten t. Normalen n til t i P tænkes orienteret saaledes, at den regnes positiv bort fra ω. Lad δ være et vilkaarligt reelt Tal. Med $P(\delta)$ betegner vi det Punkt af n, hvis Afstand fra P regnet med Fortegn i Overensstemmelse med den angivne Orientering er δ. Gennemløber P Kurven ω, vil $P(\delta)$ gennemløbe en Kurve $\omega(\delta)$, som vi betegner *Parallelkurven til ω i Afstanden δ*. Vi har $\omega(0) = \omega$; er δ negativ, betegnes $\omega(\delta)$ som en indre Parallelkurve til ω; er δ positiv, betegnes $\omega(\delta)$ som en ydre Parallelkurve til ω.

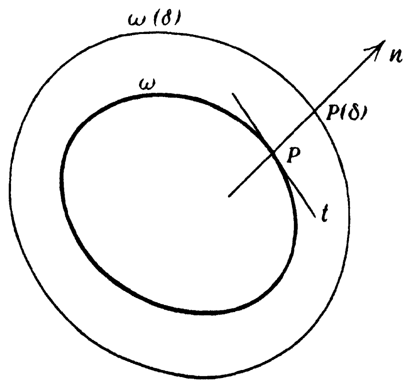


Fig. 21.

Lad ϱ være et positivt Tal mindre end r_i, og lad os betragte Parallelkurverne $\omega(-\varrho)$ og $\omega(\varrho)$. Cirklen ω^* med Centrum O og Radius ϱ er en konveks Jordankurve af Klassen K med Radierne ϱ, ϱ. Betingelsen for Anvendelsen af Betragtningerne i det foregaaende Afsnit er derfor tilstede. Nu er Punkterne $P(-\varrho)$ og $P(\varrho)$ ensbetydende med hvad vi ovenfor betegnede P_i og P_y, Kurverne $\omega(-\varrho)$ og $\omega(\varrho)$ følgelig med de konvekse Kurver ω_i og ω_y, som tilhører Klassen K. Radierne $r_i(-\varrho)$, $r_y(-\varrho)$

og $r_i(\varrho)$, $r_y(\varrho)$ i Kurverne $\omega(-\varrho)$ og $\omega(\varrho)$ tilfredsstiller Relationerne

$$r_i - \varrho \leqq r_i(-\varrho); \quad r_y(-\varrho) \leqq r_y - \varrho \tag{9a}$$

$$r_i + \varrho \leqq r_i(\varrho) \; ; \quad r_y(\varrho) \leqq r_y + \varrho. \tag{9b}$$

Som vi skal se, gælder i disse Relationer stadig Lighedstegnet.

Tangenterne $t(-\varrho)$ og $t(\varrho)$ til $\omega(-\varrho)$ og $\omega(\varrho)$ i Punkterne $P(-\varrho)$ og $P(\varrho)$ er parallelle med t; n er følgelig Normal til samtlige Kurver $\omega(\delta)$, hvor $-r_i < \delta < r_i$. Anderledes udtrykt betyder dette, at *Parallelkurverne* $\omega(\delta)$ *til* ω *for* $-r_i < \delta < r_i$ *er indbyrdes parallelle.* Den indre Parallelkurve til $\omega(\varrho)$ i Afstanden $\varrho < r_i(\varrho)$ er selve Kurven ω; vi har derfor de til (9a) analoge Relationer

$$r_i(\varrho) - \varrho \leqq r_i; \quad r_y \leqq r_y(\varrho) - \varrho,$$

som i Forbindelse med (9b) viser, at

$$r_i(\varrho) = r_i + \varrho; \quad r_y(\varrho) = r_y + \varrho.$$

Ved gentagen Anvendelse af dette Resultat ser man let, at ogsaa de ydre Parallelkurver $\omega(\delta)$ hvor $\delta \geqq r_i$ er konvekse Jordankurver af Klassen K med Radierne $r_i + \delta$, $r_y + \delta$. Nu er den ydre Parallelkurve til $\omega(-\varrho)$ i Afstanden ϱ netop Kurven ω. Følgelig har ω Radierne $r_i(-\varrho) + \varrho$ og $r_y(-\varrho) + \varrho$, og vi faar

$$r_i(-\varrho) = r_i - \varrho; \quad r_y(-\varrho) = r_y - \varrho.$$

Hermed er vist, *at Kurven* $\omega(\delta)$ *for ethvert* $\delta > -r_i$ *er en konveks Jordankurve af Klassen K med Radierne* $r_i + \delta$, $r_y + \delta$. Man viser uden Vanskelighed, at ogsaa de indre Parallelkurver $\omega(\delta)$ for $\delta < -r_y$ er konvekse Jordankurver af Klassen K, og at de har Radierne $-\delta - r_y$, $-\delta - r_i$.

Betegner d et positivt Tal mindre end r_i, vil samtlige Parallelkurver $\omega(\delta)$ for $-d \leqq \delta \leqq d$ udfylde en afsluttet Kurvering begrænset af Kurverne $\omega(-d)$ og $\omega(d)$. Gennem hvert Punkt af denne Kurvering gaar kun en af de betragtede Parallelkurver; thi af to af disse Kurver vil stedse den ene være ydre Parallelkurve til den anden; Kurverne vil derfor ikke have noget fælles Punkt.

Et specielt Tilfælde af konvekse Kurvers Addition.

21. Vi vil anvende de foregaaende Betragtninger paa et specielt Tilfælde af konvekse Kurvers Addition, som kommer til at spille en væsentlig Rolle ved Bestemmelsen af kontinuerte Punktsandsynligheder i Planen.

Vi tænker os givet tre konvekse Kurver ω_0, ω_1, ω_2 af Klassen K, hvis tilsvarende Radier $r_{0,i}$, $r_{0,y}$; $r_{1,i}$, $r_{1,y}$; $r_{2,i}$, $r_{2,y}$ tilfredsstiller Betingelserne

$$r_{0,i} \geqq 2 r_{1,y}; \quad r_{1,i} \geqq 2 r_{2,y}, \tag{10}$$

og vil undersøge de Mængder $\Sigma_0 = \omega_0$, $\Sigma_1 = \omega_0 + \omega_1$, $\Sigma_2 = \omega_0 + \omega_1 + \omega_2$, som fremkommer ved Kurvernes Addition.

Betingelsen for, at et Punkt P i Planen tilhører Σ_1, er den, at Kurverne ω_0 og $P-\omega_1$ har mindst et Punkt fælles. Nu har $-\omega_1$ og ω_1 som kongruente Kurver de samme Radier; vi kan derfor anvende den ovenfor beviste Sætning og ser, at Σ_1 er en afsluttet Kurvering begrænset af to konvekse Kurver ω_i og ω_y af Klassen K, hvis Radier $r_{i,i}$, $r_{i,y}$; $r_{y,i}$, $r_{y,y}$ tilfredsstiller Betingelserne

$$(11)\quad \begin{array}{l} r_{0,i}-r_{1,y} \leq r_{i,i};\; r_{i,y} \leq r_{0,y}-r_{1,i} \\ r_{0,i}+r_{1,i} < r_{y,i};\; r_{y,y} \leq r_{0,y}+r_{1,y}. \end{array}$$

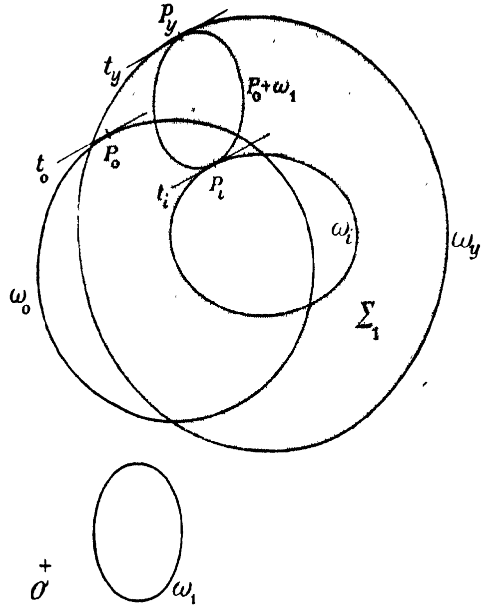


Fig. 22.

Lad (se Fig. 22) P_0 være et Punkt af ω_0; Kurven $P_0+\omega_1$ indeholder netop et Punkt P_i af ω_i og et Punkt P_y af ω_y; Tangenterne t_i og t_y til $P_0+\omega_1$ i disse Punkter er parallelle med Tangenten t_0 til ω_0 i P_0; de er tillige Tangenter henholdsvis for ω_i og ω_y. ω_i falder paa den modsatte Side af t_i som P_y. Den korteste Afstand fra P_y til ω_i er derfor mindst lig med Afstanden mellem t_i og t_y, som er større end eller lig med $2r_{1,i}$. Afstanden mellem Kurverne ω_i og ω_y er derfor mindst lig med denne Størrelse. Nu er Summen af Afstandene fra et Punkt P i Planen til Kurverne ω_i og ω_y mindst lig med Afstanden mellem ω_i og ω_y. Ethvert Punkt i Planen har derfor fra mindst en af disse Kurver en Afstand større end eller lig med $r_{1,i}$.

Ved Addition af Kurven ω_2 til Mængden Σ_1 fremkommer Mængden Σ_2. Betingelsen for, at et Punkt P i Planen tilhører Σ_2, er den, at Kurven $P-\omega_2$ indeholder mindst et Punkt af Σ_1.

Vi betragter først Mængden af Punkter P af Σ_2, for hvilke Kurverne ω_i og $P-\omega_2$ har mindst et Punkt fælles. Da ifølge (11) $r_{i,i} \geq 2r_{2,y}$ vil denne Mængde være en afsluttet Kurvering begrænset af to konvekse Kurver ω_{ii} og ω_{iy} af Klassen K, hvis korteste Afstand er mindst $2r_{2,i}$, og hvis Radier $r_{ii,i}$, $r_{ii,y}$; $r_{iy,i}$, $r_{iy,y}$ tilfredsstiller Betingelserne

$$(12a)\quad \begin{array}{l} r_{i,i}-r_{2,y} \leq r_{ii,i};\; r_{ii,y} \leq r_{i,y}-r_{2,i} \\ r_{i,i}+r_{2,i} \leq r_{iy,i};\; r_{iy,y} \leq r_{i,y}+r_{2,y}. \end{array}$$

Paa samme Maade vil Mængden af Punkter P af Σ_2, for hvilke Kurverne ω_y og $P-\omega_2$ har mindst et Punkt fælles, være en afsluttet Kurvering begrænset af to

konvekse Kurver ω_{yi} og ω_{yy} af Klassen K, hvis korteste Afstand er mindst $2r_{2,i}$, og hvis Radier $r_{yi,i}$, $r_{yi,y}$; $r_{yy,i}$, $r_{yy,y}$ tilfredsstiller Betingelserne

$$\begin{aligned} r_{y,i} - r_{2,y} &\leq r_{yi,i}; \quad r_{yi,y} \leq r_{y,y} - r_{2,i} \\ r_{y,i} + r_{2,i} &\leq r_{yy,i}; \quad r_{yy,y} \leq r_{y,y} + r_{2,y}. \end{aligned} \tag{12b}$$

Randen af Mængden Σ_2 udgøres af de to Kurver ω_{ii} og ω_{yy}; thi er P et Punkt af en af disse Kurver, vil $P - \omega_2$ indeholde netop et Punkt af Σ_1 og vil ved en vilkaarlig lille Forskydning kunne bringes helt udenfor Σ_1, ligesom omvendt ethvert Punkt P, for hvilket $P - \omega_2$ indeholder et Punkt af Σ_1 uden at indeholde noget indre Punkt af Σ_1, tilhører enten ω_{ii} eller ω_{yy}. Betegner P_i og P_y to til hinanden svarende Punkter af ω_i og ω_y, vil Kurven $P_i + \omega_2$ indeholde to Punkter P_{ii} og P_{iy} henholdsvis af ω_{ii} og ω_{iy}, medens Kurven $P_y + \omega_2$ vil indeholde to Punkter P_{yi} og P_{yy} henholdsvis af ω_{yi} og ω_{yy}. Tangenterne t_{ii}, t_{iy}, t_{yi}, t_{yy} til ω_{ii}, ω_{iy}, ω_{yi}, ω_{yy} i disse Punkter vil være indbyrdes parallelle. (Fig. 23).

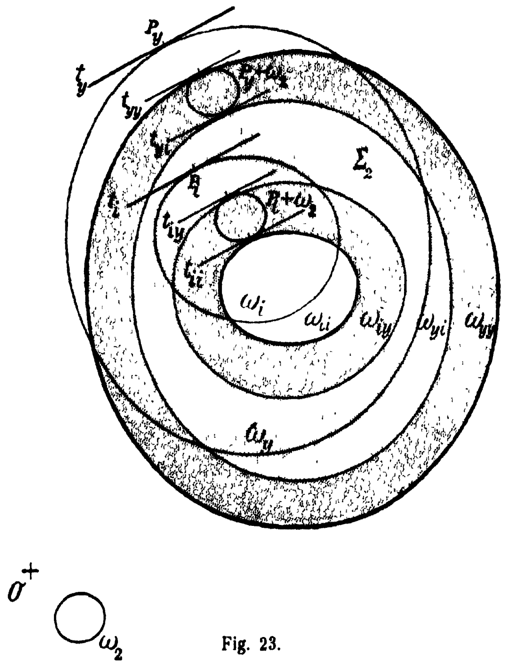


Fig. 23.

Som det fremgaar af Relationerne (10), vil ω_{iy} falde paa den modsatte Side af sin Tangent t_{iy} som P_{yi}, og Afstanden fra P_{yi} til ω_{iy} vil mindst være lig med $2r_{2,i}$. Kurven ω_{yi} omslutter derfor ω_{iy}, og Afstanden mellem ω_{iy} og ω_{yi} er mindst lig med $2r_{2,i}$. Samtlige Punkter P af Σ_2, for hvilke Kurven $P - \omega_2$ hverken skærer ω_i eller ω_y, d. v. s. for hvilke $P - \omega_2$ indeholder lutter indre Punkter af Σ_1, udgør den af Kurverne ω_{iy} og ω_{yi} begrænsede aabne Kurvering.

De indre Radier i Kurverne ω_{ii}, ω_{iy}, ω_{yi}, ω_{yy} er alle større end eller lig med $r_{2,i}$; betegner d et positivt Tal mindre end $r_{2,i}$, vil derfor Mængden af Punkter, hvis Afstand fra en af disse Kurver er mindre end eller lig med d, tilhøre en af de afsluttede Kurveringe, som begrænses af Kurverne $\omega_{ii}(-d)$ og $\omega_{ii}(d)$, $\omega_{iy}(-d)$ og $\omega_{iy}(d)$, $\omega_{yi}(-d)$ og $\omega_{yi}(d)$, $\omega_{yy}(-d)$ og $\omega_{yy}(d)$. Man indser let, at disse Kurveringe gaar helt fri af hinanden, og at de to og to i det mindste har Afstanden $2r_{2,i} - 2d$.

KAPITEL IV.

Punktsandsynlighed.

Vi vender nu tilbage til de i Kapitel II indførte Sandsynlighedsfordelinger i Planen. Vi betragtede dengang en Følge $\omega_0, \omega_1, \ldots, \omega_N, \ldots$ af lukkede konvekse Kurver, paa hvilke der var givet kontinuerte Buesandsynligheder, og viste, hvorledes dette førte til Betragtning af i det store og hele kontinuerte Rektangelsandsynligheder, svarende til de ved Kurvernes Addition fremkomne Punktmængder. Behandlingsmaaden gav tillige Anledning til Indførelse af Sandsynligheder svarende til almindeligere Punktmængder, saavel paa de enkelte Kurver som i Planen. Vi skal nu i dette Kapitel paa speciellere Grundlag gennem Indførelsen af Begrebet Punktsandsynlighed give en udførligere Behandling af disse Sandsynlighedsfordelinger.

Punktsandsynlighed paa en konveks Kurve.

22. Lad der paa en konveks Jordankurve ω være givet en kontinuert Funktion $f(P)$ af det variable Punkt P paa Kurven; vi antager om Funktionen, at den stedse er større end eller lig med Nul uden dog nogensinde at være Nul for samtlige Punkter af en Bue paa ω, og at dens Integral $\int_\omega f(P)\,d\omega$ over Kurven er 1. Funktionen $f(P)$ definerer da en kontinuert Buesandsynlighed paa Kurven af den tidligere (§ 8) betragtede Art, som fremkommer, idet vi for enhver Bue b paa ω betegner Funktionens Integral $\int_b f(P)\,db$ over Buen som Sandsynligheden $w(b)$ for, at et vilkaarligt Punkt af ω tilhører b. Denne Funktion af Buen b er nemlig saavel kontinuert som additiv, den er stedse positiv, og dens Værdi $w(\omega)$ svarende til Kurven selv er 1. Endvidere er den differentiabel, d. v. s. der eksisterer for ethvert Punkt P af ω en entydig bestemt Grænseværdi for Forholdet mellem Sandsynligheden svarende til en Bue, der konvergerer mod Punktet, og Buens Længde. Denne Grænseværdi, som er Funktionens Differentialkvotient i Punktet, er netop $f(P)$. Omvendt vil enhver kontinuert Buesandsynlighed, som er differentiabel med en kontinuert Differentialkvotient, være bestemt paa den angivne Maade ved Hjælp af denne. Funktionen $f(P)$ betegnes som en *kontinuert Punktsandsynlighed* paa Kurven ω.

23. Som ovenfor føres vi gennem en Afbildning af Kurven ω paa et Parameterinterval $0 \leqq \theta < 1$ til Betragtning af Sandsynligheder svarende ogsaa til saadanne Punktmængder paa Kurven, for hvilke den tilsvarende Mængde af Parameterværdier er maalelig i JORDAN'sk Forstand, idet Sandsynligheden svarende til en saadan Mængde sættes lig med Maalet for denne lineære Punktmængde. Disse Sandsynligheder kan ogsaa bestemmes ved Hjælp af Punktsandsynligheden $f(P)$. Til en Punktmængde paa ω sammensat af et endeligt Antal Buer svarer en Mængde af Parameterværdier sammensat af et endeligt Antal Intervaller, altsaa sikkert en maalelig Mængde; følgelig vil der til den givne Mængde paa Kurven svare en Sandsyn-

lighed bestemt som Summen af Integralerne af $f(P)$ over de enkelte Buer, hvoraf Mængden er sammensat. Denne Sum betegner vi naturligt som Funktionens Integral over den givne Mængde. Lad os nu betragte en vilkaarlig Punktmængde m paa Kurven ω. Vi definerer det indre og ydre (JORDAN'ske) Integral af Funktionen $f(P)$ over Mængden m som henholdsvis øvre og nedre Grænse for Funktionens Integral over saadanne Punktmængder, sammensat af et endeligt Antal Buer af ω, som henholdsvis tilhører og indeholder m. Er de to Integraler ligestore, betegner vi deres fælles Værdi simpelthen som Funktionens Integral $\int_m f(P)\,dm$ over Mængden m. Det ses straks, at de Punktmængder paa ω, for hvilke vi ad denne Vej faar defineret Integraler, er de samme som dem, for hvilke vi ovenfor fik defineret Sandsynligheder, og at for disse Mængder *Integral og Sandsynlighed stedse stemmer overens.*

Punktsandsynligheder i Planen. Definition.

24. Vi betragter nu en Følge

(1) $$\omega_0, \omega_1, \ldots, \omega_N, \ldots$$

af konvekse Jordankurver af Klassen K (se § 12), om hvis Radier

(2) $$r_{0,i},\ r_{0,y};\ r_{1,i},\ r_{1,y};\ \ldots;\ r_{N,i},\ r_{N,y};\ \ldots$$

vi antager, at de konvergerer mod Nul, naar N vokser ud over alle Grænser.

Lad der paa de enkelte Kurver ω_n være givet kontinuerte Punktsandsynligheder $f_n(P_n)$. Gennem de tilsvarende Afbildninger af Kurverne paa Parameterintervaller $0 \leq \theta_n < 1$ føres vi (se § 10) til Afbildning af de ved Kurvernes Addition fremkomne Punktmængder $\Sigma_N = \sum_{n=0}^{N} \omega_n$ paa Enhedsterninger $Q_N\ (0 \leq \theta_n < 1)$ i de tilsvarende $\theta_0, \theta_1, \ldots, \theta_N$-Rum og dermed til Indførelse af plane Sandsynlighedsfordelinger $W_N(M)$ svarende til disse Mængder.

Vi vil vise, hvorledes disse Mængdesandsynligheder for alle N fra et vist Trin, paa tilsvarende Maade som Sandsynlighedsfordelingerne paa de enkelte Kurver, lader sig bestemme ved Hjælp af kontinuerte Punktsandsynligheder i Planen.

Lad os betragte en kontinuert Funktion $F(P)$ af det variable Punkt P i Planen, om hvilken det er givet, at den ikke antager negative Værdier. Vi definerer det indre Integral $J_i(M)$ af denne Funktion over en given Punktmængde M i Planen som øvre Grænse for Funktionens Integral over alle Punktmængder, sammensat af et endeligt Antal Rektangler, som tilhører Mængden. Paa tilsvarende Maade definerer vi, naar M er begrænset, det ydre Integral $J_y(M)$ af Funktionen over Mængden M som nedre Grænse for Funktionens Integral over alle Punktmængder, sammensat af et endeligt Antal Rektangler, som indeholder Mængden; er M ikke begrænset, kan denne Definition ikke anvendes; vi definerer da Integralet $J_y(M)$ som øvre Grænse for det ydre Integral af $F(P)$ over alle begrænsede Delmængder af M. For begrænsede Punktmængder er de to Integraler, det indre og det ydre, altid endelige; derimod kan de for ubegrænsede Mængder antage Værdien uendelig. Er for en given Punktmængde

M det indre og det ydre Integral ligestore og endelige, betegner vi Funktionen som integrabel (i JORDAN'sk Forstand) over den givne Mængde; den fælles Værdi $J(M)$ for de to Integraler betegner vi da som det JORDAN'ske Integral $\iint_M F(P)\,dM$ af Funktionen $F(P)$ over Mængden M. Er i dette Tilfælde N en vilkaarlig Delmængde af M, har man

$$J(M) = J_i(N) + J_y(M-N).$$

Denne Relation viser, at man i de Tilfælde, hvor $F(P)$ er integrabel over hele Planen, simplere end ud fra den ovenfor givne Definition kan bestemme det ydre Integral af $F(P)$ over en vilkaarlig Punktmængde i Planen som den komplementære Værdi til det indre Integral af $F(P)$ over Komplementærmængden.

Hvad angaar Mængdesandsynlighederne $W_N(M)$, *er det nu vort Maal at vise, at de for alle* $N \geq N_0$, *hvor* N_0 *er et Tal, som alene afhænger af den givne Følge af konvekse Kurver, er fuldstændig bestemt som Integraler af kontinuerte Punktsandsynligheder, d. v. s. at der for ethvert* $N \geq N_0$ *findes en kontinuert Funktion*

$$F_N(P)$$

af det variable Punkt P i Planen, som ikke antager negative Værdier, og som er integrabel netop over de Mængder, for hvilke den tilsvarende Sandsynlighed $W_N(M)$ *er defineret og med denne Sandsynlighed til Integral. Skriver vi*

$$W_N(M) = \iint_M F_N(P)\,dM, \tag{3}$$

vil vi herved udtrykke den fuldstændige Identitet mellem de to Mængdefunktioner $W_N(M)$ *og* $\iint_M F_N(P)\,dM$ *ogsaa med Hensyn til Definitionsomraade.*

En saadan Funktion $F_N(P)$ vil altid være integrabel over hele Planen med Integralet 1; for alle Punkter udenfor eller paa Randen af Σ_N vil den have Værdien Nul.

25. Findes der for et bestemt N en kontinuert Punktsandsynlighed $F_N(P)$, vil denne øjensynlig være entydig bestemt ved Betingelsen (3); den vil endda være entydig bestemt allerede ved den svagere Betingelse

$$W_N(R) = \iint_R F_N(P)\,dR, \tag{4}$$

som fremgaar af (3), naar vi lader M betegne Rektangler alene, i hvilket Tilfælde Eksistensen af begge de betragtede Mængdefunktioner er sikker.

Vi vil derfor foreløbig indskrænke os til at søge denne Betingelse opfyldt. Senere skal vi vise, at en kontinuert Funktion $F_N(P)$, som tilfredsstiller Betingelsen (4), af sig selv vil tilfredsstille den stærkere Betingelse (3) og altsaa vil være en kontinuert Punktsandsynlighed af den ønskede Art.

Er det for et eller andet n lykkedes at bestemme en kontinuert Funktion $F_n(P)$ saaledes, at

$$W_n(R) = \iint_R F_n(P)\,dR$$

for ethvert Rektangel R i Planen, og sætter vi

$$(5) \qquad \underline{F_{n+1}(P) = \int_0^1 F_n(P-P_{n+1})\,d\theta_{n+1},}$$

vil $F_{n+1}(P)$ ligeledes være en kontinuert Funktion af P, og vi vil have

$$W_{n+1}(R) = \int_0^1 W_n(R-P_{n+1})\,d\theta_{n+1} = \int_0^1 d\theta_{n+1} \iint_{R-P_{n+1}} F_n(P)\,d(R-P_{n+1}) =$$

$$\int_0^1 d\theta_{n+1} \iint_R F_n(P-P_{n+1})\,dR = \iint_R dR \int_0^1 F_n(P-P_{n+1})\,d\theta_{n+1} = \iint_R F_{n+1}(P)\,dR.$$

Heraf følger, at dersom Betingelsen (4) kan opfyldes for et bestemt $N = N_0$, da vil den ogsaa kunne opfyldes for ethvert $N > N_0$, og Funktionerne $F_N(P)$ vil være bestemt udfra $F_{N_0}(P)$ ved Hjælp af Relationen (5). Vor Opgave er den at vise, at der findes et $N = N_0$, for hvilket Betingelsen kan opfyldes.

Antager vi dette bevist, følger af Relationen (5) i Forbindelse med den tidligere (§ 11) beviste Sætning, at for ethvert N Sandsynligheden $W_N(R) > 0$, naar blot R i sit Indre indeholder Punkter af Σ_N, den ikke helt uinteressante Egenskab ved Sandsynlighederne $F_N(P)$, at de, ihvert Fald for $N > N_0+2$, er positive i det Indre af de tilsvarende Mængder Σ_N (og ikke blot positive eller Nul), forudsat at vi opfatter Randen af Σ_N i den i § 4 angivne udvidede Betydning. Thi af $F_N(P) = 0$ følger for et saadant N af (5) $F_{N-1}(P-P_N) = 0$ for alle Punkter P_N af ω_N og altsaa $F_{N-2}(P-(P_N+P_{N-1})) = 0$ for alle Punkter P_N+P_{N-1} af den ved Addition af Kurverne ω_N og ω_{N-1} fremkomne Punktmængde $\omega_N+\omega_{N-1}$. For indre Punkter P af Σ_N indeholder imidlertid Punktmængden $P-(\omega_N+\omega_{N-1})$ sikkert Rektangler, som helt tilhører Σ_{N-2}.

Punktsandsynligheder i Planen. Konstruktion.

26. Vi vil først betragte det Tilfælde, hvor de til Kurverne ω_0, ω_1, ω_2, ω_3 svarende Radier tilfredsstiller Betingelserne

$$(6) \qquad \underline{r_{0,i} \geqq 2r_{1,y}; \quad r_{1,i} \geqq 2r_{2,y}; \quad r_{2,i} \geqq 2r_{3,y},}$$

et Tilfælde for hvilket vi tidligere (§ 21) har undersøgt Punktmængderne Σ_0, Σ_1, Σ_2. Vi vil vise, at Betingelsen (4) i dette Tilfælde kan tilfredsstilles allerede for $N = 3$.

Beviset er elementært, men ret kompliceret; det beror paa en direkte Konstruktion af en Funktion $F_3(P)$, som har den ønskede Egenskab. Som vi skal se, kan vi allerede for $N = 1$ angive en Funktion $F_1(P)$, som viser sig at være diskontinuert paa visse Kurver, men som dog, naar vi opererer med en passende Integraldefinition (den CAUCHY'ske), tilfredsstiller Betingelsen (4). Paa denne Funktion

anvender vi Betragtningen ovenfor og faar bestemt en Funktion

$$F_2(P) = \int_0^1 F_1(P - P_2)\, d\theta_2,$$

som ligeledes bliver diskontinuert paa visse Kurver, og som, idet vi stadig opererer med den CAUCHY'ske Integraldefinition, tilfredsstiller Betingelsen (4) for $N = 2$. Som Følge af Integrationen bliver $F_2(P)$ svagere diskontinuert end $F_1(P)$. Ved fornyet Anvendelse af den samme Betragtning denne Gang paa Funktionen $F_2(P)$ forsvinder al Diskontinuitet; Funktionen

$$F_3(P) = \int_0^1 F_2(P - P_3)\, d\theta_3$$

viser sig at være en kontinuert Funktion af P, som tilfredsstiller Betingelsen (4).

Konstruktion af $F_1(P)$.

27. Lad os betragte Punktmængden $\Sigma_1 = \omega_0 + \omega_1$, begrænset af de to konvekse Jordankurver ω_i og ω_y. Vi vil definere en Funktion $F_1(P)$ af det variable Punkt P i Planen. Tilhører P enten $I(\omega_i)$ eller $Y(\omega_y)$ (d. v. s. tilhører P ikke Σ_1), har Kurverne ω_0 og $P - \omega_1$ intet Punkt fælles; vi sætter da $F_1(P) = 0$. Tilhører P saavel $Y(\omega_i)$ som $I(\omega_y)$ (d. v. s. er P et indre Punkt af Σ_1), skærer Kurverne ω_0 og $P - \omega_1$ hinanden i to Punkter (se Fig. 24); betegner vi disse $P'_0 = P - P'_1$ og $P''_0 = P - P''_1$, er P'_1 og P''_1 de to Punkter af ω_1, som ved Addition henholdsvis til Punkterne P'_0 og P''_0 af ω_0 giver P; idet Vinklerne mellem ω_0 og $P - \omega_1$ i P'_0 og P''_0 betegnes henholdsvis p' og p'', vil vi definere Funktionsværdien i Punktet P som

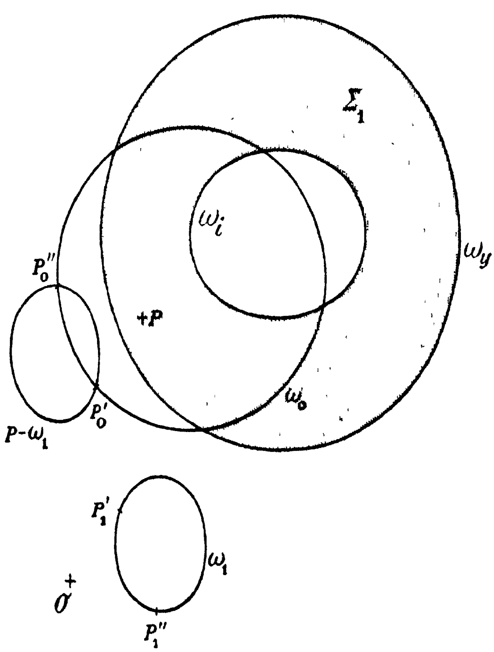


Fig. 24

(7) $$F_1(P) = \frac{f_0(P'_0) \cdot f_1(P'_1)}{\sin p'} + \frac{f_0(P''_0) \cdot f_1(P''_1)}{\sin p''};$$

her betegner $f_0(P'_0)$ og $f_0(P''_0)$ Punktsandsynlighederne paa ω_0 i Punkterne P'_0 og P''_0, og $f_1(P'_1)$ og $f_1(P''_1)$ betegner Punktsandsynlighederne paa ω_1 i Punkterne P'_1 og P''_1. Tilhører Punktet P endelig enten ω_i eller ω_y (d. v. s. er P et

Randpunkt for Σ_1), rører ω_0 og $P-\omega_1$ hinanden i et Punkt $P_0 = P - P_1$; vi sætter da $F_1(P) = \infty$.

Den hermed definerede Funktion $F_1(P)$ er *kontinuert undtagen paa Kurverne* ω_i *og* ω_y. Vi vil vise, at den, naar vi opererer med en passende Integraldefinition, tilfredsstiller Betingelsen (4), d. v. s. *at dens Integral over ethvert Rektangel* R *i Planen er lig med den til Rektanglet svarende Sandsynlighed* $W_1(R)$.

28. Lad (XY) være et vilkaarligt, men fast retvinklet Koordinatsystem i Planen, og lad os betragte de akseparallelle Rektangler i dette System. Vi vil definere, hvad vi vil forstaa ved Integralet $\iint_R F_1(P)\,dR$ af $F_1(P)$ over et Rektangel R i Systemet.

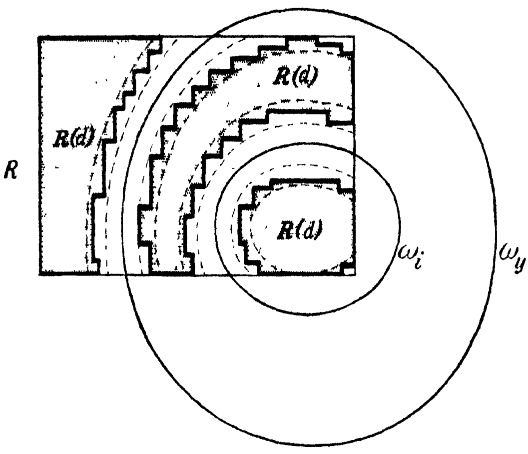


Fig. 25.

Lad d betegne et positivt Tal, som konvergerer mod Nul. For enhver Værdi af d betragter vi (Fig. 25) Mængden $A(d)$ af Punkter i R, hvis Afstand fra ω_i og ω_y er større end d. Paa Mængden $A(d)$ er $F_1(P)$ ligelig kontinuert og begrænset. For ethvert Punkt P af $A(d)$ betragter vi det akseparallelle Kvadrat med Diagonalen d, som har sit Midtpunkt i P. $A(d)$ vil tilhøre en Punktmængde $B(d)$ sammensat af et endeligt Antal af disse Kvadrater; den fælles Del for R og $B(d)$ betegner vi $R(d)$. Punktmængden $R(d)$ er sammensat af et endeligt Antal akseparallelle Rektangler; den tilhører $A\left(\frac{d}{2}\right)$; følgelig er $F_1(P)$ ligelig kontinuert og begrænset paa Mængden $R(d)$, og vi kan tale om Integralet

$$I(d) = \iint_{R(d)} F_1(P)\,dR(d)$$

af $F_1(P)$ over denne Mængde. Da $F_1(P)$ aldrig er negativ, vil Integralet $I(d)$, naar d konvergerer mod Nul, nærme sig til en bestemt Grænseværdi, som maaske kan være uendelig, og som vi betegner som Integralet

$$I = \iint_R F_1(P)\,dR$$

af $F_1(P)$ over Rektanglet R; man ser let, at den Vilkaarlighed, som er tilstede i Definitionen af Mængderne $R(d)$, er uden Betydning for Integralets Værdi.

29. For at vise at Integralet I altid er lig med den til Rektanglet R svarende Sandsynlighed $W_1(R)$, betragter vi for enhver af Mængderne $R(d)$ den tilsvarende Sandsynlighed $W_1(R(d))$ bestemt som Maalet for den Punktmængde $\Omega(d)$ i Enhedskvadratet $Q_1(0 \leqq \theta_0 < 1,\ 0 \leqq \theta_1 < 1)$ i θ_0, θ_1-Planen, der svarer til $R(d)$. Vi vil vise, at $W_1(R(d)) = I(d)$. Hermed vil Beviset være fuldført; thi idet d konvergerer mod Nul, vil $\Omega(d)$ konvergere mod en Mængde, som, bortset fra en Mængde af Maalet Nul hidrørende fra de Buer af ω_i og ω_y, som tilhører R, er lig med den til R svarende Delmængde Ω af Q_1. $W_1(R(d))$ vil følgelig[1] konvergere mod $W_1(R)$, og Rigtigheden af Identiteten

$$W_1(R) = \iint_R F_1(P)\,dR$$

vil være bevist.

Sandsynligheden $W_1(R(d))$ vil være bestemt som Summen af Sandsynlighederne svarende til de enkelte Rektangler, hvoraf $R(d)$ er sammensat; paa samme Maade vil $I(d)$ være bestemt som Summen af Integralerne af $F_1(P)$ over disse Rektangler. For at vise at $I(d) = W_1(R(d))$ er det derfor tilstrækkeligt at vise, at for ethvert af disse Rektangler Integral og Sandsynlighed stemmer overens. For de Rektangler, som tilhører enten $I(\omega_i)$ eller $Y(\omega_y)$, er saavel Integral som Sandsynlighed lig med Nul. Tilbage staar da blot at vise, at

$$W_1(R_0) = \iint_{R_0} F_1(P)\,dR_0,$$

naar (se Fig. 26) R_0 er et akseparallelt Rektangel, som tilhører Σ_1 og har en positiv Afstand fra ω_i og ω_y.

Funktionen $F_1(P)$ er for ethvert indre Punkt P af Σ_1 defineret som Summen af de to *Led*

$$F_1'(P) = \frac{f_0(P_0')\cdot f_1(P_1')}{\sin p'} \quad \text{og} \quad F_1''(P) = \frac{f_0(P_0'')\cdot f_1(P_1'')}{\sin p''}.$$

Det ligger derfor nær ved Integrationen af $F_1(P)$ at integrere ledvis. For at kunne gøre dette maa vi imidlertid først nøjagtig definere de to *Funktioner* $F_1'(P)$ og $F_1''(P)$, d. v. s. vi maa angive en Metode til samtidig for alle Punkter P at sondre mellem Punkterne P_0' og P_0''. Vi tænker os hertil fastlagt en Omløbsretning paa ω_0 (f. Eks. bestemt ved voksende θ_0) og indfører den Vedtægt stadig at lade P_0' være Begyndelsespunktet, P_0'' Endepunktet af den Bue af ω_0, der falder indenfor Kurven $P - \omega_1$. De derved bestemte Funktioner $F_1'(P)$ og $F_1''(P)$ er begge ligelig kontinuerte og begrænsede paa Rektanglet R_0, og vi har

---

[1] Medens der i den LEBESGUE'ske Maalteori gælder den almindelige Sætning, at Grænsemængden for en voksende Følge af maalelige Mængder altid er maalelig paany, og at dens Maal er Grænseværdien for de givne Mængders Maal, udsiger den tilsvarende Sætning i den JORDAN'ske Teori kun, at *saafremt* Grænsemængden for en voksende Følge af maalelige Mængder paany er maalelig, da vil dens Maal være Grænseværdien for de givne Mængders Maal.

$$\iint_{R_0} F_1(P)\,dR_0 = \iint_{R_0} F_1'(P)\,dR_0 + \iint_{R_0} F_1''(P)\,dR_0 .$$

Vi betragter nu (Fig. 26) den til R_0 svarende Delmængde Ω_0 af Q_1. Ethvert Punkt P af R_0 kan paa to Maader bestemmes som en Sum af to Punkter P_0 og P_1 henholdsvis af ω_0 og ω_1, nemlig som Sum af Punkterne P_0' og P_1' og som Sum af Punkterne P_0'' og P_1''; det svarer derfor til to Punkter (θ_0', θ_1') og (θ_0'', θ_1'') af Ω_0.

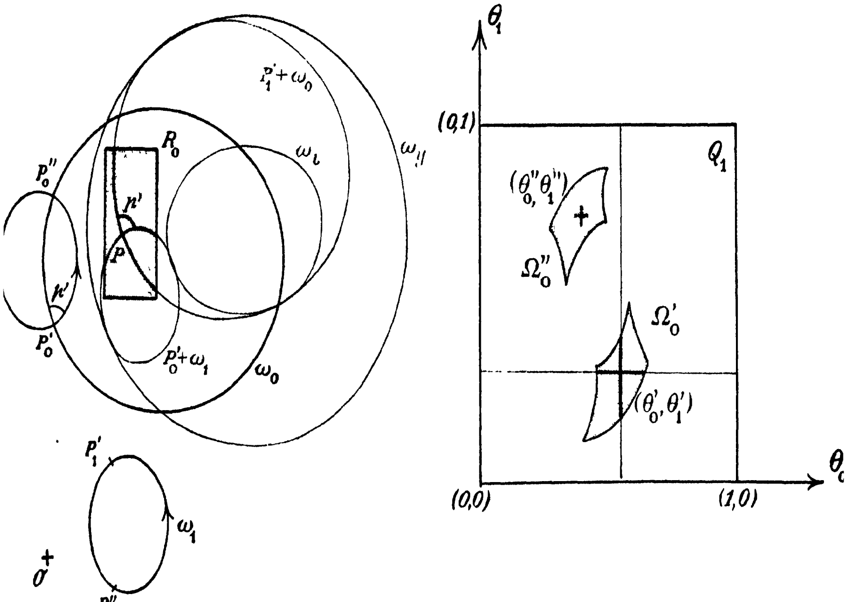


Fig. 26.

Vor Vedtægt giver os en tilsvarende Deling af Ω_0 i to Mængder Ω_0' og Ω_0'', den første bestaaende af Punkterne (θ_0', θ_1'), den anden af Punkterne (θ_0'', θ_1''). Ω_0' og Ω_0'' har en positiv mindste Afstand; de er derfor begge maalelige. Betegner vi deres Maal $W_1'(R_0)$ og $W_1''(R_0)$, er

$$W_1(R_0) = W_1'(R_0) + W_1''(R_0).$$

Vi vil vise, at

$$W_1'(R_0) = \iint_{R_0} F_1'(P)\,dR_0 \quad \text{og} \quad W_1''(R_0) = \iint_{R_0} F_1''(P)\,dR_0 ,$$

hvormed Beviset vil være fuldført. Vi kan nøjes med at bevise den første Relation. Punktmængden Ω_0' er afbildet enentydig og kontinuert paa Rektanglet R_0. Lad (θ_0', θ_1') være det Punkt af Ω_0', som svarer til det vilkaarlige Punkt P af R_0. Til de rette Liniestykker $\theta_1 = \theta_1'$ og $\theta_0 = \theta_0'$ gennem (θ_0', θ_1'), som er parallelle henholdsvis med θ_0-Aksen og med θ_1-Aksen, svarer de to Kurver $\omega_0 + P_1'$ og $P_0' + \omega_1$

gennem P; deres Tangenter i Punktet P skærer hinanden under samme Vinkel p' som Tangenterne til ω_0 og $P-\omega_1$ i Punktet P'_0; til Bueelementerne $d(\omega_0+P'_1)$ og $d(P'_0+\omega_1)$ i Punktet P svarer Linieelementer $d\theta_0 = f_0(P'_0)\,d(\omega_0+P'_1)$ og $d\theta_1 = f_1(P'_1)\,d(P'_0+\omega_1)$ i Punktet (θ'_0, θ'_1). Anvender vi derfor ved Bestemmelsen af $\iint_{R_0} F'_1(P)\,dR_0$ Planelementet $dR_0 = d(\omega_0+P'_1)\,d(P'_0+\omega_1)\sin p'$, faar vi

$$\iint_{R_0} F'_1(P)\,dR_0 = \iint_{\Omega'_0} d\theta_0\,d\theta_1 = W'_1(R_0).$$

Hermed er den opstillede Sætning bevist.

30. Lad d være et positivt Tal mindre end $r_{1,i}$. Funktionen $F_1^d(P)$, som er lig med $F_1(P)$ for alle Punkter P, hvis Afstand fra ω_i og ω_y er større end d, men som iøvrigt er lig med Nul, er en begrænset Funktion af Punktet P. Den er kontinuert undtagen paa Parallelkurverne $\omega_i(d)$ og $\omega_y(-d)$ til ω_i og ω_y. Begge disse Kurver er konvekse Jordankurver; Funktionen $F_1^d(P)$ er derfor integrabel over ethvert Rektangel R i Planen, og dens Integral

$$W_1^d(R) = \iint_R F_1^d(P)\,dR \tag{8}$$

er en kontinuert Funktion af R. Man viser uden Vanskelighed, at denne Funktion, naar d konvergerer mod Nul, vil konvergere mod Funktionen

$$W_1(R) = \iint_R F_1(P)\,dR.$$

Betegner R_0 et Rektangel, som indeholder Σ_1, har vi for ethvert Rektangel R

$$|W_1^d(R) - W_1(R)| \leq |W_1^d(R_0) - W_1(R_0)|;$$

denne Ulighed viser, at *$W_1^d(R)$ konvergerer ligelig mod $W_1(R)$*.

31. Funktionen $F_1(P)$ er kontinuert i hele Planen undtagen paa Kurverne ω_i og ω_y, følgelig ligelig kontinuert og begrænset paa enhver afsluttet Punktmængde, som ikke indeholder noget Punkt af disse Kurver. Derimod er $F_1(P)$ ikke begrænset paa den aabne Punktmængde, der udgøres af samtlige Punkter, som ikke tilhører ω_i eller ω_y. Vi vil imidlertid vise, at Funktionen

$$F_1(P)\cdot\sqrt{d},$$

hvor d betegner den mindste Afstand fra P til et Punkt af ω_i eller ω_y, er en begrænset Funktion paa denne Mængde, *at der m. a. O. findes en positiv Konstant K_1, saaledes at*

(9) $$F_1(P) \leq \frac{K_1}{\sqrt{d}}$$

for ethvert Punkt P, der ikke falder paa ω_i eller ω_y.

For Punkter, som enten tilhører $I(\omega_i)$ eller $Y(\omega_y)$, er $F_1(P) = 0$. Vi kan derfor nøjes med at betragte $F_1(P)$ for Punkter, som samtidig tilhører $Y(\omega_i)$ og $I(\omega_y)$, d. v. s. for indre Punkter af Σ_1. For saadanne Punkter har vi

$$F_1(P)\sqrt{d} = f_0(P'_0)\,f_1(P'_1)\frac{\sqrt{d}}{\sin p'} + f_0(P''_0)\,f_1(P''_1)\frac{\sqrt{d}}{\sin p''}.$$

Nu er ifølge § 19

$$\frac{\sqrt{d}}{\sin p'} < k;\quad \frac{\sqrt{d}}{\sin p''} < k,$$

hvor $k = \sqrt{\dfrac{2r_{0,i}\,r_{1,y}}{r_{0,i} - r_{1,y}}}$ er uafhængig af Punktet P. Betegner φ_0 og φ_1 øvre Grænse for de kontinuerte Funktioner $f_0(P_0)$ og $f_1(P_1)$, har vi altsaa

$$F_1(P)\sqrt{d} < 2\cdot\varphi_0\cdot\varphi_1\cdot k,$$

hvormed Sætningen er bevist.

Konstruktion af $F_2(P)$.

32. Vi vender os nu til Betragtning af Punktmængden $\Sigma_2 = \Sigma_1 + \omega_2$. Betingelsen for, at et Punkt P i Planen tilhører Σ_2, er den, at Kurven $P - \omega_2$ indeholder Punkter af Σ_1. Lad d være et positivt Tal mindre end $r_{1,i}$. Vi betragter den ovenfor indførte Funktion $F_1^d(P)$, som er lig med $F_1(P)$ for alle Punkter hvis Afstand fra ω_i og ω_y er større end d, men som iøvrigt er lig med Nul. For ethvert Punkt P i Planen vil Funktionen

$$F_1^d(P - P_2),$$

hvor $P - P_2$ gennemløber Kurven $P - \omega_2$, være en begrænset og stykkevis kontinuert Funktion af Parameteren θ_2 for P_2 paa ω_2; dens Integral

(10) $$F_2^d(P) = \int_0^1 F_1^d(P - P_2)\,d\theta_2,$$

som er lig med Kurveintegralet

$$\int_{P-\omega_2} F_1^d(P - P_2)\,f_2(P_2)\,d(P - \omega_2),$$

vil fremstille en i hele Planen kontinuert Funktion af Punktet P.

Vi vil vise, at *Integralet*

(11) $$W_2^d(R) = \iint_R F_2^d(P)\,dR$$

af denne Funktion, naar d konvergerer mod Nul, *konvergerer ligelig mod Rektangelsandsynligheden $W_2(R)$.*

$W_2^d(R)$ bestemmes ved Integralet

$$\iint_R dR \int_0^1 F_1^d(P-P_2)\,d\theta_2.$$

Da Funktionen $F_1^d(P-P_2)$ er begrænset, indser man om dette Integral let, at det er lig med Integralet

$$\int_0^1 d\theta_2 \iint_R F_1^d(P-P_2)\,dR$$

af den kontinuerte Funktion

$$\iint_R F_1^d(P-P_2)\,dR = \iint_{R-P_2} F_1^d(P)\,d(R-P_2) = W_1^d(R-P_2).$$

Vi har altsaa

$$W_2^d(R) = \int_0^1 W_1^d(R-P_2)\,d\theta_2.$$

Naar d konvergerer mod Nul, konvergerer $W_1^d(R-P_2)$ som ovenfor vist ligelig mod Rektangelsandsynligheden $W_1(R-P_2)$. $W_2^d(R)$ konvergerer altsaa ligelig mod Integralet

$$\int_0^1 W_1(R-P_2)\,d\theta_2,$$

som netop fremstiller Rektangelsandsynligheden $W_2(R)$.

33. Naar d konvergerer (monotont) mod Nul, vil Funktionen $F_2^d(P)$ konvergere monotont mod en Funktion $F_2(P)$ fremstillet ved det Cauchy'ske Integral

$$(12) \qquad F_2(P) = \int_0^1 F_1(P-P_2)\,d\theta_2$$

af den stykkevis kontinuerte, men ikke altid begrænsede Funktion $F_1(P-P_2)$. Funktionen $F_2(P)$ fremstilles ogsaa ved Kurveintegralet

$$\int_{P-\omega_2} F_1(P-P_2)\,f_2(P_2)\,d(P-\omega_2).$$

Vi vil vise, at *$F_2^d(P)$ konvergerer ligelig mod $F_2(P)$ paa enhver afsluttet Punktmængde, hvis Afstand fra Kurverne ω_{ii}, ω_{iy}, ω_{yi}, ω_{yy} er større end Nul.*

Lad A være en afsluttet Punktmængde, hvis Afstand a fra Kurverne ω_{ii}, ω_{iy}, ω_{yi}, ω_{yy} er større end Nul. Vi skal vise, at Funktionen

$$F_2(P) - F_2^d(P)$$

konvergerer ligelig mod Nul paa Mængden A. For Punkter P, som tilhører enten $I(\omega_{ii})$ eller $Y(\omega_{yy})$, er for alle d

$$F_2(P) = F_2^d(P) = 0;$$

$F_2(P) - F_2^d(P)$ er altsaa konstant lig med Nul. Vi kan derfor i det følgende nøjes med at betragte $F_2(P) - F_2^d(P)$ for Punkter af A, som tilhører Σ_2.

Lad P være et Punkt af A, som tilhører Σ_2; Kurven $P-\omega_2$ tilhører helt eller delvis Σ_1. Enhver Forskydning, som bringer den til at røre ω_i eller ω_y, er større end eller lig med a. Falder P mellem Kurverne ω_{iy} og ω_{yi}, tilhører $P-\omega_2$ helt Σ_1; dens Afstand fra ω_i og ω_y er mindst a; for ethvert $d < a$ er følgelig

$$F_2(P) - F_2^d(P) = 0.$$

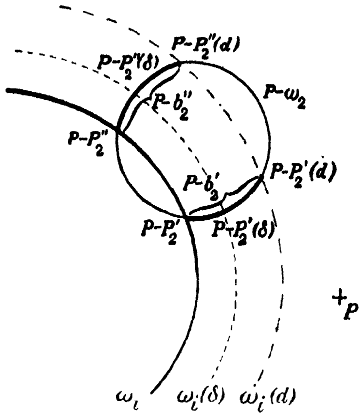


Fig. 27.

Falder P mellem ω_{ii} og ω_{iy} eller mellem ω_{yi} og ω_{yy}, tilhører Kurven $P-\omega_2$ kun delvis Σ_1; i det første Tilfælde skærer den ω_i, i det andet Tilfælde ω_y. Vi vil nøjes med at betragte det første Tilfælde (Fig. 27). $F_2(P) - F_2^d(P)$ kan bestemmes som Integralet af Funktionen $F_2(P-P_2) f_2(P_2)$ over den Del af $P-\omega_2$, som falder mellem Kurven ω_i og dens ydre Parallelkurve $\omega_i(d)$ i Afstanden d. For ethvert $d < a$ skærer $\omega_i(d)$ Kurven $P-\omega_2$ i to Punkter. Mellem Kurverne ω_i og $\omega_i(d)$ afskæres derfor to Buer $P-b_2'$, og $P-b_2''$ af $P-\omega_2$. En Parallelkurve $\omega_i(\delta)$ til ω_i vil, naar $0 \leq \delta < d$, skære begge disse Buer hver i et Punkt; betegner vi Skæringspunkterne $P-P_2'(\delta)$ og $P-P_2''(\delta)$, Skæringsvinklerne $p'(\delta)$ og $p''(\delta)$, faar vi

$$F_2(P) - F_2^d(P) = \int_0^d \frac{F_1\big(P-P_2'(\delta)\big) f_2\big(P_2'(\delta)\big)}{\sin p'(\delta)}\, d\delta + \int_0^d \frac{F_1\big(P-P_2''(\delta)\big) f_2\big(P_2''(\delta)\big)}{\sin p''(\delta)}\, d\delta.$$

Vi vil vise, at de to Integraler hver for sig konvergerer ligelig mod Nul med d paa den (afsluttede) Delmængde A_i af A, som falder mellem Kurverne ω_{ii} og ω_{iy}.

Lad os f. Eks. betragte det første Integral. Lad φ_2 betegne øvre Grænse for den kontinuerte Funktion $f_2(P_2)$ paa ω_2, og lad g' betegne den positive nedre Grænse for den ligeledes kontinuerte Funktion $\sin p'(\delta)$ af P og δ, naar P gennemløber Mængden A_i og δ Intervallet $0 \leq \delta \leq \frac{a}{2}$. For ethvert δ, som tilhører Intervallet $0 \leq \delta \leq \frac{a}{2}$, og ethvert Punkt P af A_i har vi

$$\frac{F_1\big(P-P_2'(\delta)\big) f_2\big(P_2'(\delta)\big)}{\sin p'(\delta)} < \frac{K_1 \cdot \varphi_2}{\sqrt{\delta} \cdot g'}.$$

For ethvert Punkt P af A_i og ethvert $d \leq \frac{a}{2}$ har vi altsaa

$$\int_0^d \frac{F_1\big(P-P_2'(\delta)\big) f_2\big(P_2'(\delta)\big)}{\sin p'(\delta)}\, d\delta \leq \frac{K_1 \varphi_2}{g'} \int_0^d \frac{d\delta}{\sqrt{\delta}} = \frac{K_1 \varphi_2}{g'} \cdot 2\sqrt{d};$$

men heraf følger straks, at Integralet konvergerer ligelig mod Nul med d.

34. Af den hermed beviste Sætning følger specielt, at *Funktionen $F_2(P)$ er kontinuert i ethvert Punkt P, som ikke falder paa Kurverne ω_{ii}, ω_{iy}, ω_{yi}, ω_{yy}.* Har Rektanglet R en positiv Afstand fra disse Kurver, bestemmes Integralet

$$\iint_R F_2(P)\,dR$$

af $F_2(P)$ over R som Grænseværdi for Integralet

$$W_2^d(R) = \iint_R F_2^d(P)\,dR,$$

naar d konvergerer mod Nul. Dette Integral konvergerer imidlertid som ovenfor vist mod Rektangelsandsynligheden $W_2(R)$. Vi har altsaa

$$W_2(R) = \iint_R F_2(P)\,dR.$$

Denne Relation kan udvides til at gælde for vilkaarlige Rektangler i Planen, naar man benytter en Integraldefinition analog med den ovenfor ved Betragtningen af $F_1(P)$ anvendte. *Funktionen $F_2(P)$ bestemmer saaledes fuldstændig Rektangelsandsynligheden $W_2(R)$.* Dette kommer vi dog ikke til at benytte i det følgende.

35. Funktionen $F_2(P)$ er ikke begrænset paa den aabne Punktmængde, der udgøres af samtlige Punkter, som ikke tilhører ω_{ii}, ω_{iy}, ω_{yi}, ω_{yy}; den er derfor ingen kontinuert Funktion af P i hele Planen. Vi vil imidlertid vise, at den er væsentlig svagere diskontinuert end Funktionen $F_1(P)$.

Paa Mængderne $I(\omega_{ii})$ og $Y(\omega_{yy})$ er $F_2(P)$ lig med Nul. Vi behøver derfor blot at betragte Funktionen paa Mængden Σ_2.

Lad a være et *fast* positivt Tal mindre end $\frac{r_{2,i}}{2}$. Vi betragter Mængden A_2 af Punkter af Σ_2, hvis Afstand fra Kurverne ω_{ii}, ω_{iy}, ω_{yi}, ω_{yy} er større end a. Paa denne Mængde er $F_2(P)$ begrænset. Ethvert Punkt af Σ_2, som ikke tilhører A_2, har fra en og, da Kurverne, som vist i § 21, to og to mindst har Afstanden $2r_{2,i}$, kun fra en af Kurverne ω_{ii}, ω_{iy}, ω_{yi}, ω_{yy} en Afstand mindre end a; det tilhører derfor en og kun en af de afsluttede Kurveringe Ω_{ii}^a, Ω_{iy}^a, Ω_{yi}^a, Ω_{yy}^a, som begrænses henholdsvis af Kurverne ω_{ii} og $\omega_{ii}(a)$, $\omega_{iy}(-a)$ og $\omega_{iy}(a)$, $\omega_{yi}(-a)$ og $\omega_{yi}(a)$, $\omega_{yy}(-a)$ og ω_{yy} (se Fig. 28). Vi vil vise, at *$F_2(P)$ er begrænset i Kurveringene Ω_{ii}^a og Ω_{yy}^a, medens den i Kurveringene Ω_{iy}^a og Ω_{yi}^a tilfredsstiller en Relation af Formen*

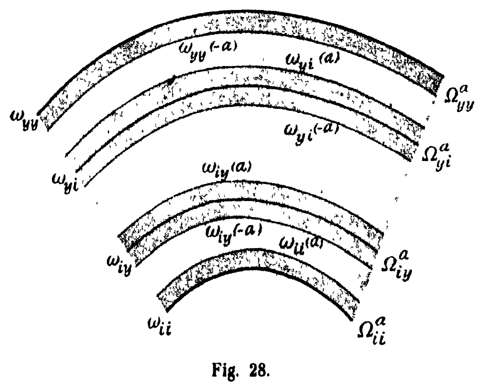


Fig. 28.

$$F_2(P) \leq K_2 + L_2 \log \frac{a}{d}, \tag{13}$$

hvor d betegner Afstanden fra P henholdsvis til ω_{iy} og ω_{yi}, og K_2 og L_2 er uafhængige af P.[1]

$F_2(P)$ bestemmes ved Integralet af Funktionen $F_1(P-P_2)\, f_2(P_2)$ over den Del af Kurven $P-\omega_2$, som tilhører Σ_1. Lad A_1 betegne Mængden af Punkter af Σ_1, hvis Afstand fra ω_i og ω_y er større end a. Paa Mængden A_1 er $F_1(P)$ begrænset; Integralet af $F_1(P-P_2)\, f_2(P_2)$ over den Del af $P-\omega_2$, som tilhører A_1, er følgelig ogsaa begrænset. Vi kan derfor nøjes med at betragte Integralet af $F_1(P-P_2)\, f_2(P_2)$ over den Del af $P-\omega_2$, som tilhører de afsluttede Kurveringe Ω_i^a og Ω_y^a, som begrænses henholdsvis af Kurverne ω_i og $\omega_i(a)$ og af Kurverne $\omega_y(-a)$ og ω_y; da Afstanden mellem Kurverne $\omega_i(a)$ og $\omega_y(-a)$ er mindst $2r_{1,i}-2a > 2r_{2,y}$, indeholder $P-\omega_2$ højst Punkter af den ene af disse Kurveringe. Tilhører P enten Ω_{ii}^a eller Ω_{iy}^a indeholder $P-\omega_2$ Punkter af Ω_i^a, tilhører P derimod Ω_{yi}^a eller Ω_{yy}^a indeholder $P-\omega_2$ Punkter af Ω_y^a. Vi vil nøjes med at betragte de Tilfælde, hvor $P-\omega_2$ indeholder Punkter af Ω_i^a.

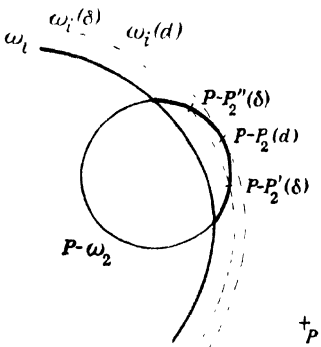


Fig. 29.

Vi vil først vise, at *$F_2(P)$ er begrænset i Kurveringen Ω_{ii}^a.* Lad (se Fig. 29) P være et Punkt af Ω_{ii}^a; dets Afstand d fra ω_{ii} er højst lig med a. Kurven $P-\omega_2$ indeholder ingen Punkter af A_1. Vi betragter Parallelkurverne $\omega_i(\delta)$ til ω_i for $0 \leq \delta \leq d$; den indre Radius $r_{1,i}+\delta$ i enhver af disse Kurver $\omega_i(\delta)$ er større end $r_{2,y}$. $P-\omega_2$ rører derfor $\omega_i(d)$ indvendig i et Punkt $P-P_2(d)$, medens den skærer hver af Kurverne $\omega_i(\delta)$, $0 < \delta < d$ i to Punkter $P-P_2'(\delta)$ og $P-P_2''(\delta)$; Skæringsvinklerne betegner vi $p'(\delta)$ og $p''(\delta)$. Punkterne $P-P_2'(\delta)$ og $P-P_2''(\delta)$ tænkes hver for sig at variere kontinuert med δ. $F_2(P)$ bestemmes da som Sum af de to Integraler

$$F_2'(P) = \int_0^d \frac{F_1(P-P_2'(\delta))\, f_2(P_2'(\delta))}{\sin p'(\delta)}\, d\delta \quad \text{og} \quad F_2''(P) = \int_0^d \frac{F_1(P-P_2''(\delta))\, f_2(P_2''(\delta))}{\sin p''(\delta)}\, d\delta.$$

Vi vil vise, at disse hver for sig er begrænsede. Lad os f. Eks. betragte det første.

Den mindste Forskydning, som bringer Kurven $P-\omega_2$ til at røre Parallelkurven $\omega_i(\delta)$ til ω_i, har Størrelsen $d-\delta$.

Ifølge § 19 er da for ethvert Punkt P af Ω_{ii}^a og ethvert $\delta < d$

[1] I det følgende anvendes uafbrudt de almindelige Sætninger i § 16 om Skæring mellem konvekse Kurver og i § 20 om Parallelkurver til konvekse Kurver. Vi overlader det til Læseren i hvert enkelt Tilfælde, udfra Relationerne (11) og (12) i § 21 og Relationen $a < \frac{r_{2,i}}{2}$, at konstatere, at Betingelserne for Anvendelsen af disse Sætninger er tilstede.

$$\frac{\sqrt{d-\delta}}{\sin p'(\delta)} < \sqrt{\frac{2(r_{i,i}+\delta)\, r_{2,y}}{r_{i,i}+\delta-r_{2,y}}} < l,$$

hvor $l = \sqrt{\dfrac{2(r_{i,i}+a)\, r_{2,y}}{r_{i,i}-r_{2,y}}}$ er uafhængig saavel af P som af δ; naar som ovenfor φ_2 betegner øvre Grænse for Funktionen $f_2(P_2)$, har vi da

$$F_2'(P) < K_1 \cdot \varphi_2 \cdot l \cdot \int_0^d \frac{d\delta}{\sqrt{\delta}\sqrt{d-\delta}} = K_1 \cdot \varphi_2 \cdot l \cdot \pi,$$

hvormed Sætningen er bevist.

Vi gaar nu over til Betragtning af Funktionen $F_2(P)$ i Kurveringen Ω_{iy}^a. Lad P være et Punkt af Ω_{iy}^a, som ikke falder paa Kurven ω_{iy}. Afstanden d fra P til ω_{iy} er højst lig med a. Vi antager først (se Fig. 30), at P falder indenfor ω_{iy}. Kurven $P-\omega_2$ rører den indre Parallelkurve $\omega_i(-d)$ til ω_i udvendig i et Punkt $P-P_2(-d)$. Da $d \leq a < \frac{r_{2,i}}{2}$ skærer $P-\omega_2$ samtlige Parallelkurver $\omega_i(\delta)$, $0 < \delta < a$, og den mindste Forskydning, som bringer den til at røre en af disse Kurver $\omega_i(\delta)$, har Størrelsen $d+\delta$. Idet vi iøvrigt anvender de samme Betegnelser som ovenfor, faar vi Integralet af $F_1(P-P_2) f_2(P_2)$ over den Del af $P-\omega_2$, som tilhører Ω_i^a, bestemt som Sum af de to Integraler

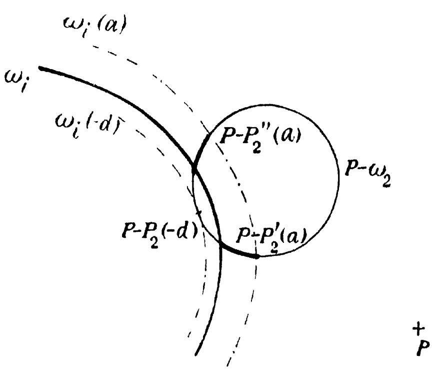


Fig. 30

$$F_2'(P) = \int_0^a \frac{F_1(P-P_2'(\delta))\, f_2(P_2'(\delta))}{\sin p'(\delta)}\, d\delta \quad \text{og} \quad F_2''(P) = \int_0^a \frac{F_1(P-P_2''(\delta))\, f_2(P_2''(\delta))}{\sin p''(\delta)}\, d\delta.$$

Vi vil vise, at disse hver for sig tilfredsstiller en Relation af Formen (13). Lad os f. Eks. betragte det første Integral. Man viser ganske som ovenfor, at for ethvert af de betragtede Punkter P og ethvert δ, $0 \leq \delta \leq a$,

$$\frac{\sqrt{d+\delta}}{\sin p'(\delta)} < l.$$

Heraf faas imidlertid straks

$$F_2'(P) < K_1 \cdot \varphi_2 \cdot l \cdot \int_0^a \frac{d\delta}{\sqrt{\delta}\sqrt{d+\delta}} =$$

$$K_1 \cdot \varphi_2 \cdot l \cdot \int_0^{\frac{a}{d}} \frac{dx}{\sqrt{x}\sqrt{1+x}} < K_1 \cdot \varphi_2 \cdot l \cdot \left\{ \int_0^1 \frac{dx}{\sqrt{x}} + \int_1^{\frac{a}{d}} \frac{dx}{x} \right\} = K_1 \cdot \varphi_2 \cdot l \cdot \left\{ 2 + \log \frac{a}{d} \right\},$$

hvormed Sætningen er bevist.

Falder P (se Fig. 31) udenfor ω_{iy}, rører $P-\omega_2$ den ydre Parallelkurve $\omega_i(d)$ til ω_i udvendig i et Punkt $P-P_2(d)$; den skærer Parallelkurverne $\omega_i(\delta)$ for $d<\delta\leq a$.

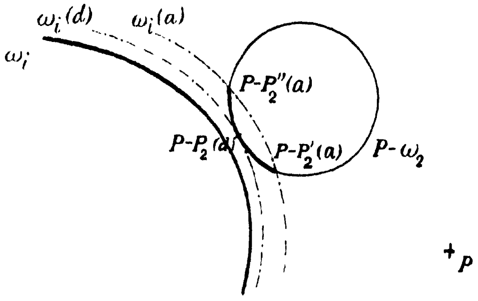


Fig. 31.

Integralet af $F_1(P-P_2)\,f_2(P_2)$ over den Del af $P-\omega_2$, som tilhører Ω_i^a, bestemmes som Sum af de to Integraler

$$F_2'(P)=\int_d^a\frac{F_1(P-P_2'(\delta))\,f_2(P_2'(\delta))}{\sin p'(\delta)}\,d\delta$$

og

$$F_2''(P)=\int_d^a\frac{F_1(P-P_2''(\delta))\,f_2(P_2''(\delta))}{\sin p''(\delta)}\,d\delta.$$

Nu er i dette Tilfælde

$$\frac{\sqrt{\delta-d}}{\sin p'(\delta)}<l;$$

følgelig er

$$F_2'(P)<K_1\cdot\varphi_2\cdot l\cdot\int_d^a\frac{d\delta}{\sqrt{\delta}\sqrt{\delta-d}}=K_1\cdot\varphi_2\cdot l\cdot\int_1^{\frac{a}{d}}\frac{dx}{\sqrt{x}\sqrt{x-1}}=$$

$$K_1\cdot\varphi_2\cdot l\cdot\int_0^{\frac{a}{d}-1}\frac{dx}{\sqrt{1+x}\sqrt{x}}<K_1\cdot\varphi_2\cdot l\cdot\left\{2+\log\frac{a}{d}\right\},$$

som er en Relation af Formen (13). Hermed er den opstillede Sætning fuldstændig bevist.

Konstruktion af $F_3(P)$.

36. Paa Grundlag af de foregaaende Resultater er vi nu i Stand til at vise Eksistensen af en kontinuert Funktion $F_3(P)$, hvis Integral over ethvert Rektangel R i Planen er lig med den til Rektanglet svarende Sandsynlighed $W_3(R)$.

Vi betragter den ovenfor indførte (kontinuerte) Funktion $F_2^d(P)$, hvis Integral

$$W_2^d(R)=\iint_R F_2^d(P)\,dR,$$

naar d konvergerer mod Nul, konvergerer ligelig mod Rektangelsandsynligheden $W_2(R)$. Funktionen

$$F_3^d(P)=\int_0^1 F_2^d(P-P_3)\,d\theta_3, \tag{14}$$

som ogsaa bestemmes ved Kurveintegralet

$$\int_{P-\omega_3}F_2^d(P-P_3)\,f_3(P_3)\,d(P-\omega_3),$$

er en kontinuert Funktion af P i hele Planen.

Integralet

$$W_3^d(R) = \iint_R F_3^d(P)\,dR \tag{15}$$

af denne Funktion vil konvergere ligelig mod Rektangelsandsynligheden $W_3(R)$, naar d konvergerer mod Nul. Thi idet

$$W_3^d(R) = \iint_R dR \int_0^1 F_2^d(P-P_3)\,d\theta_3 = \int_0^1 d\theta_3 \iint_R F_2^d(P-P_3)\,dR =$$

$$\int_0^1 d\theta_3 \iint_{R-P_3} F_2^d(P)\,d(R-P_3) = \int_0^1 W_2^d(R-P_3)\,d\theta_3,$$

vil $W_3^d(R)$, naar d konvergerer mod Nul, konvergere ligelig mod Integralet

$$\int_0^1 W_2(R-P_3)\,d\theta_3;$$

men dette Integral fremstiller netop Funktionen $W_3(R)$.

37. Naar d konvergerer (monotont) mod Nul, konvergerer $F_3^d(P)$ monotont mod en Funktion $F_3(P)$, der, som man let ser, fremstilles ved Integralet

$$F_3(P) = \int_0^1 F_2(P-P_3)\,d\theta_3 \tag{16}$$

af den stykkevis kontinuerte, men ikke altid begrænsede Funktion $F_2(P-P_3)$. Vi vil vise, at $F_3^d(P)$ i hele Planen konvergerer ligelig mod $F_3(P)$. Heraf vil straks følge, at $F_3(P)$ *er en i hele Planen kontinuert Funktion af Punktet P, hvis Integral*

$$\iint_R F_3(P)\,dR$$

over ethvert Rektangel R i Planen, som Grænseværdi for Integralet

$$\iint_R F_3^d(P)\,dR,$$

er lig med den tilsvarende Sandsynlighed $W_3(R)$.

Fremstiller vi $F_3(P)$ ved Kurveintegralet

$$\int_{P-\omega_3} F_2(P-P_3)\,f_3(P_3)\,d(P-\omega_3),$$

faar vi for Differensen $F_3(P)-F_3^d(P)$ Udtrykket

$$F_3(P)-F_3^d(P) = \int_{P-\omega_3} \{F_2(P-P_3)-F_2^d(P-P_3)\}\,f_3(P_3)\,d(P-\omega_3).$$

Da saavel $F_2(P)$ som $F_2^d(P)$ er Nul i Omraaderne $I(\omega_{ii})$ og $Y(\omega_{yy})$, kan vi nøjes med at integrere over den Del af $P-\omega_3$, som tilhører Σ_2. Hvad vi skal vise er, at den angivne Differens konvergerer ligelig mod Nul med d.

Lad φ_3 betegne øvre Grænse for Funktionen $f_3(P_3)$, og lad α være et (variabelt) positivt Tal mindre end det i § 35 indførte Tal a. Lad A_2 betegne Mængden af Punkter af Σ_2, hvis Afstand fra Kurverne ω_{ii}, ω_{iy}, ω_{yi}, ω_{yy} er større end α; et Punkt af Σ_2, som ikke tilhører A_2, tilhører en og kun en af de fire Kurveringe Ω^{α}_{ii}, Ω^{α}_{iy}, Ω^{α}_{yi}, Ω^{α}_{yy}, som begrænses henholdsvis af Kurverne ω_{ii} og $\omega_{ii}(\alpha)$, $\omega_{iy}(-\alpha)$ og $\omega_{iy}(\alpha)$, $\omega_{yi}(-\alpha)$ og $\omega_{yi}(\alpha)$, $\omega_{yy}(-\alpha)$ og ω_{yy}. Den Del af $P-\omega_3$, som tilhører disse Kurveringe, betegner vi $P-\omega_3(\alpha)$, den Del af $P-\omega_3$, som tilhører A_2, betegner vi $P-\omega_3^*(\alpha)$. Da er

$$F_3(P)-F_3^d(P) \leqq \varphi_3\left\{\int\limits_{P-\omega_3^*(\alpha)}\{F_2(P-P_3)-F_2^d(P-P_3)\}\,d\big(P-\omega_3^*(\alpha)\big)+\right.$$
$$\left.\int\limits_{P-\omega_3(\alpha)}F_2(P-P_3)\,d\big(P-\omega_3(\alpha)\big)\right\}.$$

Vi vil vise, at det sidste Integral, hvori d ikke indgaar, konvergerer ligelig mod Nul med α. Hermed vil den opstillede Sætning være bevist; thi er $\varepsilon>0$ et givet Tal, og bestemmer vi *først* α saaledes, at for alle P

$$\int\limits_{P-\omega_3(\alpha)}F_2(P-P_3)\,d\big(P-\omega_3(\alpha)\big)<\frac{\varepsilon}{2\varphi_3},$$

og *dernæst*, hvad der er muligt, da $F_2^d(P)$ konvergerer ligelig mod $F_2(P)$ paa Mængden A_2, d_0 saa lille, at, naar $d<d_0$,

$$\int\limits_{P-\omega_3^*(\alpha)}\{F_2(P-P_3)-F_2^d(P-P_3)\}\,d\big(P-\omega_3^*(\alpha)\big)<\frac{\varepsilon}{2\varphi_3},$$

er for ethvert Punkt P i Planen og ethvert $d<d_0$

$$F_3(P)-F_3^d(P)<\varepsilon;$$

men det betyder netop, at $F_3(P)-F_3^d(P)$ konvergerer ligelig mod Nul med d.

38. Kurven $P-\omega_3$ indeholder højst Punkter af en af Kurveringene Ω^{α}_{ii}, Ω^{α}_{iy}, Ω^{α}_{yi}, Ω^{α}_{yy}; $P-\omega_3(\alpha)$ tilhører altsaa helt en af disse Kurveringe. Ved Beviset for, at Integralet

$$\int\limits_{P-\omega_3(\alpha)}F_2(P-P_3)\,d\big(P-\omega_3(\alpha)\big)$$

konvergerer ligelig mod Nul med α, vil vi nøjes med at betragte de Tilfælde, hvor $P-\omega_3(\alpha)$ tilhører enten Ω^{α}_{ii} eller Ω^{α}_{iy}; de to andre Tilfælde behandles paa tilsvarende Maade.

Vi betragter først (Fig. 32) det Tilfælde, hvor $P-\omega_3(\alpha)$ tilhører Ω^{α}_{ii}. $F_2(P)$ er som ovenfor vist begrænset i Kurveringen Ω^{α}_{ii}. Hvad vi skal vise er derfor blot, at Integralet

$$\int\limits_{P-\omega_3(\alpha)}d\big(P-\omega_3(\alpha)\big),$$

som er lig med Længden af den eller de Buer af $P-\omega_3$, der tilhører Ω_{ii}^{α}, konvergerer ligelig mod Nul med α.

Lad δ_0 være den mindste, δ_1 den største Afstand fra et Punkt af $P-\omega_3(\alpha)$ til Kurven ω_{ii}. En Parallelkurve $\omega_{ii}(\delta)$ til ω_{ii} vil, naar $\delta_0 < \delta < \delta_1$, skære $P-\omega_3(\alpha)$ i to Punkter $P-P_3'(\delta)$ og $P-P_3''(\delta)$ under Vinkler $p'(\delta)$ og $p''(\delta)$. Betegnelserne tænkes valgt saaledes, at $P-P_3'(\delta)$ og $P-P_3''(\delta)$ hver for sig varierer kontinuert med δ. Da er

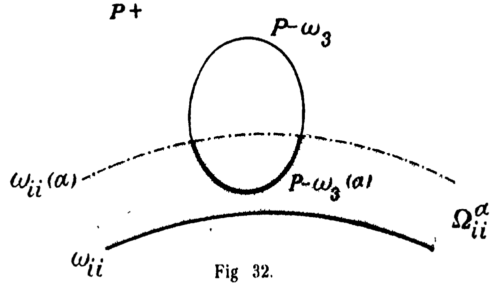


Fig 32.

$$\int_{P-\omega_3(\alpha)} d\left(P-\omega_3(\alpha)\right) = \int_{\delta_0}^{\delta_1} \frac{d\delta}{\sin p'(\delta)} + \int_{\delta_0}^{\delta_1} \frac{d\delta}{\sin p''(\delta)}.$$

De to sidste Integraler konvergerer ligelig mod Nul med α; lad os f. Eks. føre Beviset for det første. Den mindste Forskydning, som bringer Kurven $P-\omega_3$ til at røre $\omega_{ii}(\delta)$, hvor $\delta_0 < \delta < \delta_1$, er større end eller lig med det mindste af Tallene $\delta-\delta_0$ og $\delta_1-\delta$ og derfor ogsaa større end eller lig med

$$\frac{(\delta-\delta_0)(\delta_1-\delta)}{\delta_1-\delta_0}.$$

Følgelig er, som det fremgaar af Undersøgelsen ovenfor, for ethvert δ

$$\frac{\sqrt{\dfrac{(\delta-\delta_0)(\delta_1-\delta)}{\delta_1-\delta_0}}}{\sin p'(\delta)} < m_i,$$

hvor m_i er en Konstant, som hverken afhænger af α eller af P. Vi faar derfor

$$\int_{\delta_0}^{\delta_1} \frac{d\delta}{\sin p'(\delta)} < m_i \cdot \sqrt{\delta_1-\delta_0} \int_{\delta_0}^{\delta_1} \frac{d\delta}{\sqrt{(\delta-\delta_0)(\delta_1-\delta)}} \leq m_i \cdot \sqrt{\alpha} \cdot \pi,$$

hvoraf straks følger, at Integralet konvergerer ligelig mod Nul med α.

Vi gaar nu over til at betragte det Tilfælde, hvor $P-\omega_3(\alpha)$ tilhører Ω_{iy}^{α} (se Fig. 33). Vi bestemmer de Parallelkurver $\omega_{iy}(\delta)$ til ω_{iy}, som skærer $P-\omega_3(\alpha)$; nedre og øvre Grænse for de tilsvarende Værdier af δ betegner vi δ_0 og δ_1. Saavel δ_0 som δ_1 kan være negative Tal. Benytter vi de samme Betegnelser som ovenfor, faar vi

$$\int_{P-\omega_3(\alpha)} F_2(P-P_3)\, d\left(P-\omega_3(\alpha)\right) = \int_{\delta_0}^{\delta_1} \frac{F_2\left(P-P_3'(\delta)\right)}{\sin p'(\delta)}\, d\delta + \int_{\delta_0}^{\delta_1} \frac{F_2\left(P-P_3''(\delta)\right)}{\sin p''(\delta)}\, d\delta.$$

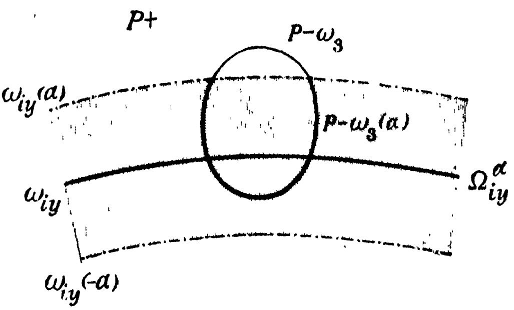


Fig. 33

De to sidste Integraler gaar ligelig mod Nul med α; vi vil nøjes med at betragte det første. Lad os foreløbig antage, at δ_0 og δ_1 har samme Fortegn, at de f. Eks. begge er positive, eller at en af dem, f. Eks. δ_0, er lig med Nul. For ethvert δ, $\delta_0 < \delta < \delta_1$, er

$$\frac{\sqrt{\dfrac{(\delta-\delta_0)(\delta_1-\delta)}{\delta_1-\delta_0}}}{\sin p'(\delta)} < m_y,$$

hvor m_y, som ovenfor m_i, er uafhængig saavel af α som af det betragtede Punkt P. Følgelig er

$$\int_{\delta_0}^{\delta_1} \frac{F_2(P-P_3'(\delta))}{\sin p'(\delta)}\, d\delta < m_y \sqrt{\delta_1-\delta_0} \int_{\delta_0}^{\delta_1} \frac{K_2 + L_2 \log \dfrac{a}{\delta}}{\sqrt{(\delta-\delta_0)(\delta_1-\delta)}}\, d\delta$$

eller, idet vi sætter $\delta = \delta_0 + (\delta_1-\delta_0)\sin^2 x$,

$$\int_{\delta_0}^{\delta_1} \frac{F_2(P-P_3'(\delta))}{\sin p'(\delta)}\, d\delta < m_y \sqrt{\delta_1-\delta_0} \int_0^{\frac{\pi}{2}} 2\left(K_2 + L_2 \log \frac{a}{\delta_0 + (\delta_1-\delta_0)\sin^2 x}\right) dx$$

$$\leq m_y \left\{ K_2 \cdot \pi \sqrt{\delta_1-\delta_0} + L_2 \cdot \pi \sqrt{\delta_1-\delta_0} \log \frac{a}{\delta_1-\delta_0} + 2 L_2 \sqrt{\delta_1-\delta_0}\, I \right\},$$

hvor I $(= \pi \log 2)$ betegner den endelige Værdi af Integralet $\int_0^{\frac{\pi}{2}} \log \frac{1}{\sin^2 x}\, dx$. Alle Leddene indenfor Parentesen konvergerer mod Nul med Størrelsen $\delta_1 - \delta_0$; følgelig konvergerer det betragtede Integral ligelig mod Nul med α. Har δ_0 og δ_1 modsat Fortegn viser Omskrivningen

$$\int_{\delta_0}^{\delta_1} \frac{F_2(P-P_3'(\delta))}{\sin p'(\delta)}\, d\delta = \int_{\delta_0}^{0} \frac{F_2(P-P_3'(\delta))}{\sin p'(\delta)}\, d\delta + \int_0^{\delta_1} \frac{F_2(P-P_3'(\delta))}{\sin p'(\delta)}\, d\delta,$$

at Integralet ogsaa i dette Tilfælde konvergerer ligelig mod Nul med α. Hermed er Beviset for den opstillede Sætning, og dermed Konstruktionen af en kontinuert Punktsandsynlighed i Planen, fuldført.

39. Vi gaar nu over til at betragte det almindelige Tilfælde, hvor vi om Radierne $r_{n,i}$ og $r_{n,y}$ i de konvekse Kurver ω_n kun forudsætter, at de konvergerer mod Nul, naar n vokser ud over alle Grænser. Hvad vi skal vise er, at der findes et

Tal $N = N_0$, som alene afhænger af de givne Kurver, og for hvilket Betingelsen

$$(4) \qquad W_N(R) = \iint_R F_N(P)\, dR$$

kan tilfredsstilles. Dette er imidlertid en umiddelbar Følge af det allerede opnaaede Resultat. Vi behøver blot at vælge N_0 saa stor, at der blandt Kurverne $\omega_0, \omega_1, \ldots, \omega_{N_0}$ findes fire $\omega'_0, \omega'_1, \omega'_2, \omega'_3$, hvis tilsvarende Radier tilfredsstiller Betingelserne

$$r'_{0,i} \geqq 2r'_{1,y}; \quad r'_{1,i} \geqq 2r'_{2,y}; \quad r'_{2,i} \geqq 2r'_{3,y}.$$

Adderer vi da Kurverne $\omega_0, \omega_1, \ldots, \omega_{N_0}$ i en saadan Orden $\omega'_0, \omega'_1, \ldots, \omega'_{N_0}$, at disse fire Kurver bliver de første, vil de tilsvarende Sandsynligheder $W'_n(R)$, $n = 0, 1, \ldots, N_0$, for $n \geqq 3$ kunne fremstilles som Integraler af kontinuerte Funktioner $F'_n(P)$. Nu er (§ 10) $W_{N_0}(R) = W'_{N_0}(R)$; $W_{N_0}(R)$ vil altsaa kunne fremstilles som Integral af den kontinuerte Funktion $F_{N_0}(P) = F'_{N_0}(P)$.

Punktsandsynligheder i Planen. Demonstration.

40. Vi har hidtil alene beskæftiget os med Opfyldelsen af Betingelsen

$$(4) \qquad W_N(R) = \iint_R F_N(P)\, dR,$$

d. v. s. med Bestemmelsen af kontinuerte Funktioner $F_N(P)$, hvis Integral over ethvert Rektangel i Planen er lig med den til Rektanglet svarende Sandsynlighed $W_N(R)$. Hvad vi har vist er, at der fra et vist Trin eksisterer saadanne kontinuerte Funktioner $F_N(P)$.

Vi vil nu vise, at en Funktion $F_N(P)$, som tilfredsstiller Betingelsen (4), af sig selv vil tilfredsstille den stærkere Betingelse

$$(3) \qquad W_N(M) = \iint_M F_N(P)\, dM,$$

at den m. a. O. vil være integrabel netop over de Punktmængder M i Planen, for hvilke den tilsvarende Sandsynlighed $W_N(M)$ er defineret og med denne Sandsynlighed til Integral. Dette vil betyde, at *de i det foregaaende Afsnit konstruerede Funktioner $F_N(P)$ er kontinuerte Punktsandsynligheder svarende til Mængdesandsynlighederne $W_N(M)$.*

At Eksistensen af Integralet

$$J(M) = \iint_M F_N(P)\, dM$$

af Funktionen $F_N(P)$ over en Mængde M i Σ_N-Planen medfører Eksistensen af en ligesaastor Sandsynlighed $W_N(M)$, er en simpel Følge af Integralets Definition i For-

bindelse med Afbildningen af Mængden Σ_N paa Enhedsterningen Q_N i det tilsvarende $\theta_0, \theta_1, \ldots, \theta_N$-Rum. Thi $J(M)$ er bestemt som øvre Grænse for Integralerne J_i af $F_N(P)$ over saadanne Punktmængder M_i, sammensat af et endeligt Antal Rektangler, som tilhører M; til disse Mængder svarer Delmængder Ω_i af Q_N, som tilhører den til M svarende Mængde Ω, og som er maalelige med Maalet J_i; det indre Maal for Ω er saaledes mindst lig med $J(M)$. Paa den anden Side ser man ved Anvendelse af den samme Betragtning paa Komplementærmængderne til M og Ω, at det ydre Maal for Ω højst er lig med $J(M)$ (smlgn. § 24). Men dette i Forbindelse med det foregaaende Resultat viser, at Ω er maalelig og har Maalet $J(M)$.

Beviset for, at ogsaa omvendt Eksistensen af Sandsynligheden $W_N(M)$ svarende til Mængden M medfører Eksistensen af et ligesaastort Integral, forløber ikke slet saa simpelt. Lad den til M svarende Punktmængde i Q_N være Ω; dens Maal $m(\Omega)$ bestemmer Sandsynligheden $W_N(M)$. $m(\Omega)$ bestemmes som øvre Grænse for Maalene $m(\Omega_i')$ af de Delmængder Ω_i' af Ω, som lader sig sammensætte af et endeligt Antal akseparallelle Parallellepipeder. Ved den entydige Afbildning af Q_N paa Σ_N afbildes hvert af disse Parallellepipeder paa en Mængde fremkommen ved Addition af $N+1$ konvekse Buer tilhørende Kurverne $\omega_0, \omega_1, \ldots, \omega_N$, altsaa (§ 4) sikkert paa en begrænset maalelig Punktmængde. Mængderne Ω_i' afbildes derfor paa maalelige Delmængder M_i af M. Funktionen $F_N(P)$, som er kontinuert og følgelig ogsaa begrænset, er sikkert integrabel over enhver maalelig Mængde i Planen, specielt over Mængderne M_i. De tilsvarende Integraler J_i er ifølge den ovenfor beviste Sætning lig med Maalene $m(\Omega_i)$ for de til Mængderne M_i svarende Delmængder Ω_i af Ω. Hver af disse Mængder Ω_i indeholder den tilsvarende Mængde Ω_i'. $m(\Omega)$ kan altsaa bestemmes som øvre Grænse for alle $m(\Omega_i)$, d. v. s. for alle J_i. Det indre Integral af $F_N(P)$ over Mængden M er derfor mindst lig med $m(\Omega)$. Ved Anvendelse af den samme Betragtning paa Komplementærmængderne til M og Ω ses, at det ydre Integral af $F_N(P)$ over M højst er lig med $m(\Omega)$. Heraf følger imidlertid at $F_N(P)$ er integrabel over M med Integralet $m(\Omega)$.

Indfører man svarende til enhver Punktmængde M i Planen en *indre Sandsynlighed* $W_{N,i}(M)$ og en *ydre Sandsynlighed* $W_{N,y}(M)$ for at et vilkaarligt Punkt af Σ_N tilhører M, bestemt henholdsvis som det indre og det ydre Maal for den til Mængden svarende Delmængde af Q_N, vil disse Sandsynligheder, som man ved den ovenfor anvendte Betragtning let viser, kunne fremstilles henholdsvis som det indre og det ydre Integral af $F_N(P)$ over Mængden M. Denne Bemærkning, som vi senere kommer til at anvende, viser os den egentlige »Grund« til Identiteten af de to Funktioner $W_N(M)$ og $\iint_M F_N(P)\,dM$.

Af den beviste Sætning i Forbindelse med den tidligere Bemærkning (§ 25), at Sandsynlighederne $F_N(P)$ for alle $N \geq N_0+2$ er positive i det Indre af de tilsvarende Mængder Σ_N (og ikke blot positive eller Nul), følger uden Vanskelighed om Mængdesandsynlighederne $W_N(M)$, at de for disse Værdier af N vil være definerede for de og kun de Mængder M i Planen, som har en *maalelig* Mængde af Punkter fælles med Σ_N. Dette gælder forøvrigt ogsaa for $N = N_0$ og $N = N_0+1$ (og af

samme Grund), da, som man udfra det foregaaende Afsnits Definitioner let viser, allerede den der indførte Funktion $F_3(P)$ er positiv i det Indre af den tilsvarende Mængde Σ_3.

Idet vi sammenfatter de opnaaede Resultater, har vi bevist følgende Sætning: *Adderer man en Følge*

$$\omega_0, \omega_1, \ldots, \omega_N,$$

af konvekse Kurver ω_n *af Klassen* K, *hvis Radier* $r_{n,i}$ *og* $r_{n,y}$ *konvergerer mod Nul, naar* n *vokser ud over alle Grænser, og er der paa Kurverne givet kontinuerte Punktsandsynligheder* $f_n(P_n)$, *da vil de til Punktmængderne*

$$\Sigma_0, \Sigma_1, \ldots, \Sigma_N,$$

svarende Mængdesandsynligheder

$$W_0(M), W_1(M), \ldots, W_N(M),$$

i Planen fra et vist Trin (d. v. s. for alle $N > N_0$) *kunne fremstilles som Integraler af kontinuerte Punktsandsynligheder* $F_N(P)$. *Disse vil være positive for indre Punkter af de tilsvarende Mængder* Σ_N *og vil være bestemt udfra* $F_{N_0}(P)$ *ved Hjælp af Relationen*

$$F_{n-1}(P) = \int_0^1 F_n(P - P_{n-1})\, d\theta_{n-1}, \tag{5}$$

hvor P_{n-1} *betegner det Punkt af* ω_{n-1}, *som svarer til Parameterværdien* θ_{n-1}.

KAPITEL V.

Sandsynlighedsfordelinger ved Addition af uendelig mange konvekse Kurver.

Vi betragter paany en Følge $\omega_0, \omega_1, \ldots, \omega_N, \ldots$ af konvekse Kurver ω_n af Klassen K, hvis Radier $r_{n,i}$ og $r_{n,y}$ konvergerer mod Nul for $n \to \infty$. Lad der paa hver af disse Kurver være givet en kontinuert Punktsandsynlighed $f_n(P_n)$; de til Punktmængderne Σ_N svarende Mængdesandsynligheder $W_N(M)$ kan da, som vi har set, for alle N fra et vist Trin N_0 fremstilles som Integraler af kontinuerte Punktsandsynligheder $F_N(P)$ i Planen. Vi vil nu vise, hvorledes denne Fremstilling i særlige Tilfælde kan give Anledning til Indførelse af en Mængdesandsynlighed $W(M)$ og

en Punktsandsynlighed $F(P)$ bestemt ved de givne uendelig mange konvekse Kurvers Addition. Vi betragter først det særlig simple Tilfælde, hvor den uendelige Række

$$(1) \qquad \sum_{n=0}^{\infty} \omega_n$$

er konvergent; de opnaaede Resultater overføres dernæst paa et mindre overskueligt, men for Anvendelserne nok saa vigtigt Tilfælde, hvor Rækken (1), til Trods for at den ikke mere forudsættes konvergent, dog paa naturlig Maade kan tilordnes en Sum; endelig anvendes de opnaaede Resultater paa et specielt Eksempel.

Sandsynlighedsfordelinger paa en ved en konvergent Række fremstillet Punktmængde.

41. Lad Rækken $\sum_{n=0}^{\infty} \omega_n$ være konvergent, og lad den have Summen Σ. Punktsandsynlighederne $F_N(P)$ bestemmes udfra Sandsynligheden $F_{N_0}(P)$ ved Hjælp af Relationen

$$(2) \qquad F_{n+1}(P) = \int_0^1 F_n(P - P_{n+1})\, d\theta_{n+1}.$$

Vi vil vise, at de for $N \to \infty$ konvergerer ligelig mod en Grænsefunktion $F(P)$.

Vi benytter ved Beviset det almindelige Konvergensprincip, idet vi viser, at øvre Grænse for Størrelsen

$$|F_N(P) - F_{N+p}(P)|,$$

naar p gennemløber de hele positive Tal og P Planens Punkter, konvergerer mod Nul for $N \to \infty$.

Lad p være et bestemt positivt helt Tal. For ethvert Tal n, $N \leq n < N+p$, og ethvert Punkt P i Planen er ifølge (2)

$$F_{n+1}(P) = F_n(P - P^{\bullet}_{n+1}),$$

hvor $P^{\bullet}_{n+1} = P^{\bullet}_{n+1}(P)$ tilhører ω_{n+1}. Følgelig er for ethvert Punkt P i Planen

$$F_{N+p}(P) = F_N(P - P^{\bullet}_{N,N+p}),$$

hvor $P^{\bullet}_{N,N+p} = P^{\bullet}_{N,N+p}(P)$ tilhører Mængden $\Sigma_{N,N+p} = \sum_{n=N+1}^{N+p} \omega_n$, og vi har

$$F_N(P) - F_{N+p}(P) = F_N(P) - F_N(P - P^{\bullet}_{N,N+p});$$

$$\underset{P,p}{\text{Øvre Grænse}}\, |F_N(P) - F_{N+p}(P)| = \underset{P,p}{\text{Øvre Grænse}}\, |F_N(P) - F_N(P - P^{\bullet}_{N,N+p})|.$$

Den største Afstand fra Begyndelsespunktet O til et Punkt af Mængden $\Sigma_{N,N+p}$ betegner vi $\delta_{N,N+p}$; sætter vi

$$\delta_N = \text{Øvre Grænse}\ \delta_{N,N+p}$$
er da $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad p$

$$\underset{P,p}{\text{Øvre Grænse}} \,|F_N(P) - F_N(P - P^{*}_{N,N+p})| \leqq \underset{P}{\text{Øvre Grænse}} \,|F_N(P) - F_N(P')|,$$

hvor P' er et *vilkaarligt* Punkt, hvis Afstand fra P er højst δ_N. Nu er ifølge (2) for ethvert $n \geqq N_0$

$$\underset{PP' \leqq \delta_N}{\text{Øvre Grænse}} \,|F_{n+1}(P) - F_{n+1}(P')| \leqq \underset{PP' \leqq \delta_N}{\text{Øvre Grænse}} \,|F_n(P) - F_n(P')|.$$

Vi har derfor for ethvert $N > N_0$

$$\underset{PP' \leqq \delta_N}{\text{Øvre Grænse}} \,|F_N(P) - F_N(P')| \leqq \underset{PP' \leqq \delta_N}{\text{Øvre Grænse}} \,|F_{N_0}(P) - F_{N_0}(P')|$$

og endelig

$$\underset{P,p}{\text{Øvre Grænse}} \,|F_N(P) - F_{N+p}(P)| < \underset{PP' \leqq \delta_N}{\text{Øvre Grænse}} \,|F_{N_0}(P) - F_{N_0}(P')|.$$

Nu er den konvergente Række $\sum_{n=0}^{\infty} P_n$, hvor P_n, $n = 0, 1, \ldots, N, \ldots$ gennemløber Kurven ω_n, ifølge § 5 *ligelig konvergent;* det betyder, at Størrelsen δ_N konvergerer mod Nul for $N \to \infty$. Da Funktionen $F_{N_0}(P)$ er kontinuert i hele Planen og lig med Nul i alle Punkter udenfor den begrænsede Mængde Σ_{N_0}, er den ligelig kontinuert. Størrelsen

$$\underset{PP' \leqq \delta_N}{\text{Øvre Grænse}} \,|F_{N_0}(P) - F_{N_0}(P')|$$

konvergerer derfor mod Nul for $N \to \infty$. Det samme gælder da ogsaa Størrelsen

$$\underset{P,p}{\text{Øvre Grænse}} \,|F_N(P) - F_{N+p}(P)|,$$

og Beviset er fuldført.

42. Grænsefunktionen $F(P)$ for $F_N(P)$ er en i hele Planen kontinuert Funktion af Punktet P. Parallelforskyder vi Kurverne ω_n til nye Stillinger ω'_n paa en saadan Maade, at Forskydningerne danner en konvergent Række, vil Summen Σ' af den nye Række $\sum_{n=0}^{\infty} \omega'_n$ og den tilsvarende Funktion $F'(P)$ fremgaa af Σ og $F(P)$ ved en og samme Parallelforskydning. Dette er en Følge af den ensartede Ligelighed i Kontinuiteten af Funktionerne $F_N(P)$, som fremgaar af Relationen (2). Vælges Forskydningerne specielt saaledes, at Kurverne ω'_n kommer til at gaa gennem Begyndelsespunktet, indser man uden Vanskelighed om Funktionen $F(P)$, at den er lig med Nul for alle Punkter udenfor eller paa Randen af Mængden Σ, ogsaa i det Tilfælde, hvor Σ bestaar af et enkelt afsluttet konvekst Omraade, men (se § 6) har en »indre Rand«. Derimod er $F(P)$, som det let fremgaar af den tilsvarende Egenskab ved Funktionerne $F_N(P)$, sikkert positiv i ethvert indre Punkt af Σ. *Integralet af $F(P)$ over Mængden Σ er lig med 1;* thi bestemmer vi et Rektangel R saa stort, at det indeholder samtlige Mængder Σ_{N_0}, Σ_{N_0+1}, $\ldots$, Σ, har vi

$$\iint_{\Sigma} F(P)\, d\Sigma = \iint_{R} F(P)\, dR = \lim_{N\to\infty} \iint_{R} F_N(P)\, dR = \lim_{N\to\infty} \iint_{\Sigma_N} F_N(P)\, d\Sigma_N;$$

nu er for ethvert $N \geqq N_0$

$$\iint_{\Sigma_N} F_N(P)\, d\Sigma_N = 1;$$

altsaa er ogsaa

$$\iint_{\Sigma} F(P)\, d\Sigma = 1.$$

Funktionen $F(P)$ betegnes som en *kontinuert Punktsandsynlighed* svarende til Punktmængden Σ.

43. Svarende til en given Punktmængde M i Planen betragter vi for ethvert $N \geqq N_0$ den indre Sandsynlighed $W_{N,i}(M)$ og den ydre Sandsynlighed $W_{N,y}(M)$ for, at et vilkaarligt Punkt af Σ_N tilhører M (se § 40); disse Sandsynligheder bestemmes henholdsvis som det indre Integral $J_{N,i}(M)$ og det ydre Integral $J_{N,y}(M)$ af Funktionen $F_N(P)$ over Mængden M. Naar N vokser ud over alle Grænser konvergerer disse Integraler henholdsvis mod det indre Integral $J_i(M)$ og det ydre Integral $J_y(M)$ af Funktionen $F(P)$ over Mængden M. Thi betegner M_0 den fælles Del for Mængden M og det ovenfor betragtede Rektangel R, har man for ethvert N

$$J_{N,i}(M) = J_{N,i}(M_0) \quad \text{og} \quad J_i(M) = J_i(M_0)$$

og for de ydre Integraler

$$J_{N,y}(M) = J_{N,y}(M_0); \quad J_y(M) = J_y(M_0).$$

Nu er for ethvert $N \geqq N_0$

$$\left.\begin{array}{l} |J_{N,i}(M_0) - J_i(M_0)| \\ |J_{N,y}(M_0) - J_y(M_0)| \end{array}\right\} \leqq \text{Øvre Grænse } |F_N(P) - F(P)| \cdot m(R),$$

hvor $m(R)$ betegner Arealet af Rektanglet R. Men da følger det straks af den ligelige Konvergens, at $J_{N,i}(M_0)$ og $J_{N,y}(M_0)$ for $N \to \infty$ konvergerer henholdsvis mod $J_i(M_0)$ og $J_y(M_0)$. Grænseværdierne

$$W_i(M) = \lim_{N\to\infty} W_{N,i}(M) \quad \text{og} \quad W_y(M) = \lim_{N\to\infty} W_{N,y}(M) \tag{3}$$

for $W_{N,i}(M)$ og $W_{N,y}(M)$ for $N \to \infty$ betegnes henholdsvis som den indre og den ydre Sandsynlighed for, at et vilkaarligt Punkt af Σ tilhører M. Er de to Sandsynligheder ligestore betegner vi deres fælles Værdi $W(M)$ som Sandsynligheden for, at et vilkaarligt Punkt af Σ tilhører M. Af Integralfremstillingen

$$W_i(M) = J_i(M), \; W_y(M) = J_y(M)$$

for de to Sandsynligheder fremgaar umiddelbart, at Mængdefunktionen $W(M)$ bestemmes som

$$W(M) = \iint_M F(P)\, dM. \tag{4}$$

Vi betegner den som den til Punktmængden Σ svarende *Mængdesandsynlighed* i Planen. Dens Definitionsomraade er Mængden af maalelige Delmængder af Σ forøget med vilkaarlige Mængder udenfor Σ.

For ethvert Rektangel R er

$$W(R) = \lim_{N \to \infty} W_N(R);$$

derimod gælder den almindeligere Relation

$$W(M) = \lim_{N \to \infty} W_N(M)$$

kun, forsaavidt de indgaaende Sandsynligheder er definerede. Indførelsen af indre og ydre Sandsynlighed, som tillader Opstillingen af de for alle Mængder gyldige Relationer (3), er saaledes nødvendig for en naturlig Definition af Mængdesandsynligheden $W(M)$.

44. Den hermed opnaaede Indførelse af en Mængdesandsynlighed i Planen svarende til en ved Addition af uendelig mange konvekse Kurver fremkommen Punktmængde og Sammenhængen mellem denne og en tilsvarende Punktsandsynlighed beror paa den i foregaaende Kapitel paaviste Sammenhæng mellem de tilsvarende Sandsynligheder ved Addition af et endeligt Antal Kurver. Punktsandsynligheden er her blevet det afgørende Begreb, som betinger Mængdesandsynlighedens Indførelse. I Modsætning hertil kunde man ønske en Behandling, hvorved Mængdesandsynligheden blev fremhævet som det oprindelige Begreb, og hvor Muligheden af *i særlige Tilfælde* at fremstille denne som Integral af en Punktsandsynlighed i højere Grad end ovenfor kom til at fremtræde som Undersøgelsens *Resultat.* Dette opnaar man ved følgende almindeligere Betragtning:

Lad

$$\sum_{n=0}^{\infty} \omega_n \tag{1}$$

være en konvergent Række, hvis enkelte Led er konvekse Jordankurver, og lad der paa disse være givet kontinuerte Buesandsynligheder $w_n(b_n)$ bestemt gennem Afbildninger af Kurverne paa Parameterintervaller $0 \leqq \theta_n < 1$. Den naturlige Vej til Indførelse af en Mængdesandsynlighed paa den ved Rækken (1) fremstillede Punktmængde Σ er da den, at man gennem en Afbildning af Mængden Σ paa Enhedsterningen $Q\,(0 \leqq \theta_n < 1)$, i det uendeligdimensionale $\theta_0, \theta_1, \ldots, \theta_N, \ldots$ -Rum tilordner enhver Mængde M i Σ-Planen en Delmængde Ω af Q bestaaende af samtlige Punkter $(\theta_0, \theta_1, \ldots, \theta_N, \ldots)$, hvis tilsvarende Punkt af Σ tilhører M, og saa, saafremt denne Mængde Ω er maalelig, betegner dens Maal $m(\Omega)$ som Sandsynligheden $W(M)$ for, at et vilkaarligt Punkt af Σ tilhører M (smlgn. § 10).

Naar vi ikke straks fra Begyndelsen har anvendt denne Definition er Grunden den, at Begreberne »maalelig Mængde«, »Punktmængdes Maal« er saa lidet bearbej-

dede, ja, maaske næppe endda kan siges at have nogen præcis Betydning, hvor Talen er om Punktmængder i et uendeligdimensionalt Rum. Imidlertid giver, som vi nu skal vise, den ovenstaaende Undersøgelse Anledning til, at man indfører et saadant Maal.

Lad Ω betegne en vilkaarlig Delmængde af den uendeligdimensionale Enhedsterning Q; med Ω_N betegner vi Projektionen af denne Mængde paa den $N+1$-dimensionale Enhedsterning Q_N, d. v. s. Mængden af Punkter $(\theta_0, \theta_1, \ldots, \theta_N)$, hvor $\theta_0, \theta_1, \ldots, \theta_N$ er de $N+1$ første Koordinater til et Punkt af Ω. Lad $m_y(\Omega_N)$ betegne det sædvanlige ($N+1$-dimensionale) JORDAN'ske Maal for Mængden Ω_N; Talfølgen

$$m_y(\Omega_0),\ m_y(\Omega_1),\ \ldots,\ m_y(\Omega_N),\ \ldots$$

er da stadig aftagende; vi betegner dens Grænseværdi

$$m_y(\Omega) = \lim_{N \to \infty} m_y(\Omega_N)$$

som *det ydre* JORDAN'*ske Maal for Punktmængden* Ω. Det *indre* JORDAN'ske Maal $m_i(\Omega)$ for Ω defineres ved Relationen

$$m_i(\Omega) = 1 - m_y(Q - \Omega).$$

For enhver Punktmængde Ω er $m_y(\Omega) + m_y(Q-\Omega) \geqq 1$, thi Projektionerne af Ω og $Q-\Omega$ paa Q_N (som godt kan have Punkter fælles) udfylder tilsammen hele Q_N; heraf følger Relationen $m_y(\Omega) \geq m_i(\Omega)$. Er $m_y(\Omega) = m_i(\Omega)$ betegner vi Punktmængden Ω som *maalelig* i JORDAN'sk Forstand; den fælles Værdi $m(\Omega)$ for de to Maal betegner vi da som *Punktmængdens Maal*[1].

Er nu (1) en vilkaarlig konvergent Række, og er der paa de enkelte Kurver ω_n givet kontinuerte Buesandsynligheder, giver den indførte Maaldefinition straks en Sandsynlighedsfordeling $W(M)$ svarende til Rækkens Sum Σ. Der gælder da, som vi vil vise, den Sætning, at *naar Kurverne ω_n er Kurver af Klassen K, hvis Radier konvergerer mod Nul for $n \to \infty$, og naar Sandsynlighederne $w_n(b_n)$ kan fremstilles som Integraler af kontinuerte Punktsandsynligheder $f_n(P_n)$, da vil ogsaa Mængdesandsynligheden $W(M)$, saaledes som den nu er defineret, kunne fremstilles som Integral af*

[1] Det indførte indre og ydre Maal for Delmængder af den uendeligdimensionale Enhedsterning Q kan defineres paa en anden Maade, som slutter sig nærmere til den sædvanlige Definition for et Rum af endelig mange Dimensioner. Betegner vi som et uendeligdimensionalt Interval enhver Punktmængde, som defineres ved et endeligt Antal Uligheder af Formen $\alpha_n \leqq \theta_n < \beta_n$ suppleret med Ulighederne $0 \leqq \theta_n < 1$ for de øvrige Koordinaters Vedkommende, vil nemlig, som man uden Vanskelighed viser, indre og ydre Maal for en vilkaarlig Delmængde Ω af Q være bestemt henholdsvis som øvre og nedre Grænse for Maalet af alle Mængder, sammensat af et endeligt Antal Delintervaller af Q, som henholdsvis tilhører og indeholder Ω; Maalet for en saadan Mængde beregnes imidlertid simpelthen som Summen af Maalene for de enkelte Intervaller, hvoraf den er sammensat, og disse er igen hver for sig bestemt som Produktet af de tilsvarende Differenser $\beta_n - \alpha_n$. I denne Formulering viser vor Definition af indre og ydre Maal sig som den svagest mulige; men samtidig som den, hvori Endeligheden i de JORDAN'ske Bestemmelser er stærkest fremhævet, idet den er udstrakt ogsaa til Intervallets Definition.

en kontinuert Punktsandsynlighed, d. v. s. der vil eksistere en Funktion $F(P)$, kontinuert i hele Planen, som er integrabel netop over de Punktmængder M, for hvilke $W(M)$ er defineret og med denne Sandsynlighed til Integral; og disse Sandsynligheder vil være de samme som de i foregaaende Paragraf definerede.

Vi viser denne Sætning, idet vi viser, at for enhver Punktmængde M

$$W_l(M) = m_l(\Omega); \quad W_y(M) = m_y(\Omega),$$

naar $W_l(M)$ og $W_y(M)$ betegner de ved Relationerne (3) definerede Sandsynligheder, og Ω er den til M svarende Delmængde af Q. Det er tilstrækkeligt at bevise den anden af disse Relationer; den første henføres hertil ved Betragtning af Komplementærmængderne. Lad som ovenfor M_0 betegne den Del af M, som tilhører Rektanglet R; da er $W_y(M) = W_y(M_0)$; til M_0 som til M svarer i Q Mængden Ω; hvad vi skal vise er, at $W_y(M_0) = m_y(\Omega)$. Idet

$$W_y(M_0) = \lim_{N \to \infty} W_{N,y}(M_0); \quad m_y(\Omega) = \lim_{N \to \infty} m_y(\Omega_N)$$

er dette ensbetydende med at vise, at der til ethvert $\varepsilon > 0$ svarer et $N = N(\varepsilon)$ saa stort, at Differensen

$$m_y(\Omega_N) - W_{N,y}(M_0)$$

er numerisk mindre end ε.

Lad (for et vilkaarligt N) $\Sigma_{N,\infty}$ betegne Summen af den (konvergente) uendelige Række $\sum_{n=N+1}^{\infty} \omega_n$. Den største Afstand η_N fra Begyndelsespunktet til et Punkt af Mængden $\Sigma_{N,\infty}$ konvergerer mod Nul for $N \to \infty$. Er $\sum_{n=0}^{\infty} P_n$ et Punkt af Σ, som tilhører M_0, vil Punktet $\sum_{n=0}^{N} P_n$ af Σ_N tilhøre $M_0 - \Sigma_{N,\infty}$; er omvendt $\sum_{n=0}^{N} P_n$ et Punkt af Σ_N, som tilhører $M_0 - \Sigma_{N,\infty}$, vil det kunne skrives paa Formen $P - \sum_{n=N+1}^{\infty} P_n$, hvor Punktet $P = \sum_{n=0}^{\infty} P_n$ tilhører M_0. Til Punktmængden $M_0 - \Sigma_{N,\infty}$ i Σ_N-Planen svarer derfor ved Afbildningen af Σ_N paa Q_N Punktmængden Ω_N. Vi har altsaa $m_y(\Omega_N) = W_{N,y}(M_0 - \Sigma_{N,\infty})$ og faar, idet vi benytter Integralfremstillingen for $W_{N,y}(M)$, for Differensen $m_y(\Omega_N) - W_{N,y}(M_0)$ Udtrykket

$$J_{N,y}(M_0 - \Sigma_{N,\infty}) - J_{N,y}(M_0).$$

Lad S_N betegne et vilkaarligt Punkt af $\Sigma_{N,\infty}$; da er sikkert

$$J_{N,y}(M_0 - \Sigma_{N,\infty}) \geqq J_{N,y}(M_0 - S_N).$$

Lad endvidere $\eta' > 0$ være valgt saa lille, at for hvilkesomhelst to Punkter P og P', hvis Afstand er mindre end η',

$$|F_{N_0}(P) - F_{N_0}(P')| < \frac{\varepsilon}{m(R)},$$

hvor $m(R)$ som ovenfor betegner Arealet af Rektanglet R. Da er ogsaa for ethvert $N > N_0$

$$|F_N(P)-F_N(P')| < \frac{\varepsilon}{m(R)}$$

for hvilkesomhelst to saadanne Punkter. Vælger vi nu $N_1 \geqq N_0$ saa stor, at $\eta_N < \eta'$ naar blot $N \geqq N_1$, er for ethvert saadant N

$$J_{N,y}(M_0 - S_N) > J_{N,y}(M_0) - \varepsilon$$

og følgelig

$$J_{N,y}(M_0 - \Sigma_{N,\infty}) - J_{N,y}(M_0) > -\varepsilon.$$

Vi vil nu vise, at ogsaa omvendt

$$J_{N,y}(M_0 - \Sigma_{N,\infty}) - J_{N,y}(M_0) < \varepsilon$$

for alle tilstrækkeligt store N.

Funktionsfølgen $F_{N_0}(P)$, $F_{N_0+1}(P)$, ..., $F_N(P)$, . . . er ligelig begrænset; dens øvre Grænse, som forøvrig er lig med Maksimum for Funktionen $F_{N_0}(P)$, betegner vi K. Vi bestemmer nu en Punktmængde $M_{0,y}$, sammensat af et endeligt Antal Rektangler, som indeholder M_0, samtidig med at den indeholder Begrænsningen for M_0 i sit Indre, og for hvilken

$$m(M_{0,y}) - m_y(M_0) < \frac{\varepsilon}{K}.$$

For at indse at dette er muligt, kan man gaa saaledes frem, at man først bestemmer en Mængde $M'_{0,y}$, sammensat af et endeligt Antal Rektangler, som indeholder M_0, og for hvilken

$$m(M'_{0,y}) - m_y(M_0) < \frac{\varepsilon}{2K}.$$

og dernæst, hvis Rektanglernes Antal er A, erstatter hvert af dem med et ligedannet og koncentrisk, hvis Areal er $\frac{\varepsilon}{2KA}$ større. Disse nye Rektangler vil da danne en Mængde $M_{0,y}$ af den ønskede Art. Vi har øjensynlig for ethvert $N > N_0$

$$J_N(M_{0,y}) - J_{N,y}(M_0) < \varepsilon.$$

Lad nu η'' betegne den positive nedre Grænse for Afstanden fra et Punkt af M_0 til Begrænsningen for $M_{0,y}$, og lad $N_2 \geqq N_0$ være valgt saa stor, at $\eta_N < \eta''$ for alle $N \geqq N_2$. Da tilhører for ethvert saadant N Punktmængden $M_0 - \Sigma_{N,\infty}$ Mængden $M_{0,y}$, og vi har

$$J_{N,y}(M_0 - \Sigma_{N,\infty}) \leqq J_N(M_{0,y})$$

og

$$J_{N,y}(M_0 - \Sigma_{N,\infty}) - J_{N,y}(M_0) < \varepsilon.$$

Vælger vi altsaa N lig med det største af Tallene N_1 og N_2, har vi

$$|m_y(\Omega_N) - W_{N,y}(M_0)| = |J_{N,y}(M_0 - \Sigma_{N,\infty}) - J_{N,y}(M_0)| < \varepsilon,$$

og Beviset er fuldført[1].

Almindeliggørelse til visse Tilfælde af divergente Rækker.

45. Behandlingen af Sandsynlighedsfordelinger ved Addition af konvekse Kurver kan siges at have faaet sin naturlige Afslutning gennem den foregaaende Undersøgelse, forsaavidt som Forudsætningen om, at den uendelige Række

$$(1) \qquad \sum_{n=0}^{\infty} \omega_n$$

er konvergent, frembyder sig af sig selv. Vi vil imidlertid vise, at allerede en væsentlig svagere Forudsætning er tilstrækkelig til at sikre den ligelige Konvergens af Funktionsfølgen $F_{N_0}(P),\ F_{N_0+1}(P), \ldots, F_N(P), \ldots$.

Vi beviser først en Hjælpesætning om *Sammensætning af Punktsandsynligheder.* For ethvert fast N og ethvert positivt helt Tal p bestemmes Punktmængden $\Sigma_{N+p} = \sum_{n=0}^{N+p} \omega_n$ som Sum af de to Punktmængder $\Sigma_N = \sum_{n=0}^{N} \omega_n$ og $\Sigma_{N,N+p} = \sum_{n=N+1}^{N+p} \omega_n$,

$$(5) \qquad \Sigma_{N+p} = \Sigma_N + \Sigma_{N,N+p}.$$

Antager vi $N \geqq N_0$, vil der paa Punktmængderne Σ_N og Σ_{N+p} være givet bestemte Punktsandsynligheder $F_N(P)$ og $F_{N+p}(P)$. For alle p fra et vist Trin p_0 (som afhænger af N) vil den ved Afbildningen af Mængden $\Sigma_{N,N+p}$ paa Enhedsterningen $Q_{N,N+p}$, $0 \leqq \theta_n < 1$, i det p-dimensionale $\theta_{N+1}, \ldots, \theta_{N+p}$-Rum definerede Mængdesandsynlighed $W_{N,N+p}(M)$ i $\Sigma_{N,N+p}$-Planen kunne fremstilles som Integral af en kontinuert Punktsandsynlighed $F_{N,N+p}(P)$; vi vil vise, at der for ethvert saadant $p \geqq p_0$ gælder Relationen

$$(6) \qquad \underline{F_{N+p}(P) = \iint_{\Sigma_{N,N+p}} F_N(P - P_{N,N+p})\, F_{N,N+p}(P_{N,N+p})\, d\Sigma_{N,N+p},}$$

hvor $P_{N,N+p}$ betegner det Punkt af $\Sigma_{N,N+p}$, der bestemmes som Sum af de til Parameterværdierne $\theta_{N+1}, \ldots, \theta_{N+p}$ svarende Punkter af Kurverne $\omega_{N+1}, \ldots, \omega_{N+p}$.

---

[1] I det almindelige Tilfælde, hvor vi paa Kurverne ω_n kun tænker os givet kontinuerte Buesandsynligheder $w_n(b_n)$, indser man ved en lignende Betragtning som den ovenfor anvendte om Mængdesandsynligheden $W(M)$, at den i alle Tilfælde, hvor blot Kurverne ω_n ikke indeholder rette Liniestykker, nøjagtig som Mængdesandsynlighederne $W_N(M)$ (se § 11) vil være defineret for ethvert Rektangel R i Planen, og at Rektangelsandsynligheden $W(R)$ (der bestemmes som $\lim_{N \to \infty} W_N(R)$) vil være en i hele Planen kontinuert Funktion af R, som da og kun da er positiv, naar R i sit Indre indeholder Punkter af Σ. Denne Bemærkning (som vi dog ikke skal udføre nærmere) viser Anvendeligheden af det indførte Maal ogsaa i Tilfælde, hvor der ikke er Tale om nogen Punktsandsynlighed.

Den ved Integralet fremstillede Funktion af P er kontinuert i hele Planen. For at vise, at den er lig med $F_{N+p}(P)$, er det tilstrækkeligt at vise, at dens Integral over ethvert Rektangel R er lig med Rektangelsandsynligheden $W_{N+p}(R)$. Nu er

$$\iint_R dR \iint_{\Sigma_{N,N+p}} F_N(P-P_{N,N+p})\, F_{N,N+p}(P_{N,N+p})\, d\Sigma_{N,N+p} =$$

$$\iint_{\Sigma_{N,N+p}} d\Sigma_{N,N+p} \iint_R F_N(P-P_{N,N+p})\, F_{N,N+p}(P_{N,N+p})\, dR =$$

$$\iint_{\Sigma_{N,N+p}} W_N(R-P_{N,N+p})\, F_{N,N+p}(P_{N,N+p})\, d\Sigma_{N,N+p}.$$

Hvad vi skal vise er derfor blot, at dette sidste Integral

$$I = \iint_{\Sigma_{N,N+p}} W_N(R-P_{N,N+p})\, F_{N,N+p}(P_{N,N+p})\, d\Sigma_{N,N+p}$$

er lig med $W_{N+p}(R)$, hvilket er ensbetydende med, at der for ethvert positivt Tal ε skal gælde Uligheden

$$|I - W_{N+p}(R)| < \varepsilon.$$

Funktionen $F_{N,N+p}(P)$ er Nul for alle Punkter, som ikke tilhører $\Sigma_{N,N+p}$. Vi har derfor

$$I = \iint_{R_0} W_N(R-P)\, F_{N,N+p}(P)\, dR_0,$$

hvor R_0 er et Rektangel, som indeholder $\Sigma_{N,N+p}$. Da Funktionen $W_N(R-P)$ er kontinuert, er det muligt at dele R_0 i et endeligt Antal smaa Rektangler $R_0^1, R_0^2, \ldots, R_0^m$ med Midtpunkter $P^1, P^2, \ldots, P^m$ saaledes, at for ethvert Punkt P, som tilhører et af disse Rektangler R_0^μ,

$$|W_N(R-P^\mu) - W_N(R-P)| < \frac{\varepsilon}{2}.$$

Da er øjensynlig

$$\left| I - \sum_{\mu=1}^{m} \left\{ W_N(R-P^\mu) \iint_{R_0^\mu} F_{N,N+p}(P)\, dR_0^\mu \right\} \right| < \frac{\varepsilon}{2}$$

eller

$$\left| I - \sum_{\mu=1}^{m} \left\{ W_N(R-P^\mu)\, W_{N,N+p}(R_0^\mu) \right\} \right| < \frac{\varepsilon}{2}. \tag{7}$$

For Rektangelsandsynligheden $W_{N+p}(R)$ har vi, som man ser ved gentagen Anvendelse af Formlen

$$W_{n+1}(R) = \int_0^1 W_n(R-P_{n+1})\,d\theta_{n+1},$$

Fremstillingen

(8) $$W_{N+p}(R) = \int_0^1 \cdots \int_0^1 W_N(R-P_{N,N+p})\,d\theta_{N+1}\ldots d\theta_{N+p} =$$

$$\int \cdots \int_{Q_{N,N+p}} W_N(R-P_{N,N+p})\,dQ_{N,N+p}.$$

De til Rektanglerne $R_0^\mu (\mu = 1, 2, \ldots, m)$ svarende Delmængder Ω^μ af $Q_{N,N+p}$ er maalelige Mængder med Maalet $W_{N,N+p}(R_0^\mu)$; de har to og to intet Punkt fælles og udfylder tilsammen $Q_{N,N+p}$; for ethvert Punkt $(\theta_{N+1}, \ldots, \theta_{N+p})$ af Ω^μ er

$$|W_N(R-P^\mu) - W_N(R-P_{N,N+p})| < \frac{\varepsilon}{2}.$$

Nu er ifølge (8)

$$W_{N+p}(R) = \sum_{\mu=1}^{m} \int \cdots \int_{\Omega^\mu} W_N(R-P_{N,N+p})\,d\Omega^\mu;$$

ved Anvendelse af Middelværdisætningen faas heraf

$$\left| W_{N+p}(R) - \sum_{\mu=1}^{m} \left\{ W_N(R-P^\mu)\, W_{N,N+p}(R_0^\mu) \right\} \right| < \frac{\varepsilon}{2},$$

som i Forbindelse med (7) giver os den søgte Relation

$$|I - W_{N+p}(R)| < \varepsilon.$$

Hermed er den opstillede Hjælpesætning bevist.

46. Ved Hjælp af Formlen (6) faar vi for Størrelsen

$$F_N(P) - F_{N+p}(P),$$

hvor vi antager $N \geqq N_0$, $p \geqq p_0 = p_0(N)$, Udtrykket

$$F_N(P) - F_{N+p}(P) = \iint_{\Sigma_{N,N+p}} \left\{ F_N(P) - F_N(P-P_{N,N+p}) \right\} F_{N,N+p}(P_{N,N+p})\,d\Sigma_{N,N+p}.$$

Lad ϱ_N være et vilkaarligt positivt Tal, og lad Γ_N betegne den afsluttede Cirkelskive med Centrum i Begyndelsespunktet og Radius ϱ_N. Integralet af $F_{N,N+p}(P_{N,N+p})$ over den Del af $\Sigma_{N,N+p}$, som tilhører Γ_N, er lig med Sandsynligheden $W_{N,N+p}(\Gamma_N)$

for, at et vilkaarligt Punkt af $\Sigma_{N,N+p}$ tilhører Γ_N. Integralet af $F_{N,N+p}(P_{N,N+p})$ over den Del af $\Sigma_{N,N+p}$, som ikke tilhører Γ_N, er lig med $1-W_{N,N+p}(\Gamma_N)$. Vi har derfor

$$\underset{P}{\text{Øvre Grænse}}\,|F_N(P)-F_{N+p}(P)| \leqq$$
$$\underset{PP' \leqq \varrho_N}{\text{Øvre Grænse}}\,|F_N(P)-F_N(P')|\,W_{N,N+p}(\Gamma_N)+$$
$$\underset{PP' > \varrho_N}{\text{Øvre Grænse}}\,|F_N(P)-F_N(P')|\,\big(1-W_{N,N+p}(\Gamma_N)\big).$$

Nu er, som Følge af Formlen

$$F_{n+1}(P) = \int_0^1 F_n(P-P_{n+1})\,d\theta_{n+1},$$

$$\underset{PP' \leqq \varrho_N}{\text{Øvre Grænse}}\,|F_N(P)-F_N(P')| \leqq \underset{PP' \leqq \varrho_N}{\text{Øvre Grænse}}\,|F_{N_0}(P)-F_{N_0}(P')|.$$

Endvidere er

$$\underset{PP' > \varrho_N}{\text{Øvre Grænse}}\,|F_N(P)-F_N(P')| \leqq \underset{P}{\text{Øvre Grænse}}\,F_N(P) \leqq \underset{P}{\text{Øvre Grænse}}\,F_{N_0}(P) = K.$$

Idet $W_{N,N+p}(\Gamma_N) \leqq 1$, faar vi da

$$\underset{P}{\text{Øvre Grænse}}\,|F_N(P)-F_{N+p}(P)| \leqq \underset{PP' \leqq \varrho_N}{\text{Øvre Grænse}}\,|F_{N_0}(P)-F_{N_0}(P')|+K\big(1-W_{N,N+p}(\Gamma_N)\big).$$

Heraf følger umiddelbart, at det for at sikre den ligelige Konvergens af Funktionsfølgen $F_{N_0}(P)$, $F_{N_0+1}(P)$, $\ldots$, $F_N(P)$, $\ldots$ er tilstrækkeligt at forlange, at *der skal kunne tilordnes ethvert Tal $N \geqq N_0$ to positive Tal ϱ_N og η_N, som begge konvergerer mod Nul for $N \to \infty$, saaledes at for ethvert $p \geqq p_0 = p_0(N)$*

$$\underline{1-W_{N,N+p}(\Gamma_N) < \eta_N,} \tag{9}$$

hvor Γ_N betegner den afsluttede Cirkelskive med Centrum i Begyndelsespunktet og Radius ϱ_N.

Thi antager vi denne Betingelse opfyldt, og bestemmer vi svarende til et vilkaarligt Tal $\varepsilon > 0$ et Tal $N = N(\varepsilon)$ saa stort, at samtidig

$$\underset{PP' \leqq \varrho_N}{\text{Øvre Grænse}}\,|F_{N_0}(P)-F_{N_0}(P')| < \frac{\varepsilon}{4}$$

og

$$\eta_N < \frac{\varepsilon}{4K},$$

hvad der er muligt som Følge af den ligelige Kontinuitet af Funktionen $F_{N_0}(P)$, er for ethvert $p \geqq p_0 = p_0(N)$

$$\underset{P}{\text{Øvre Grænse}}\,|F_N(P)-F_{N+p}(P)| < \frac{\varepsilon}{2}.$$

For hvilkesomhelst to Tal N_1 og N_2 større end eller lig med $N+p_0$ og for ethvert Punkt P i Planen er da

$$|F_{N_1}(P) - F_{N_2}(P)| < \varepsilon;$$

men det er netop Betingelsen for den ligelige Konvergens af Funktionsfølgen $F_{N_0}(P)$, $F_{N_0+1}(P), \ldots, F_N(P), \ldots$.

Den angivne Betingelse er specielt opfyldt, naar den uendelige Række $\sum_{n=0}^{\infty} \omega_n$ er konvergent. Thi sætter vi

$$\varrho_N = \delta_N = \text{Øvre Grænse}_{p}\ \delta_{N,N+p},$$

hvor $\delta_{N,N+p}$ som ovenfor betegner øvre Grænse for Afstanden fra Begyndelsespunktet til et Punkt af Mængden $\Sigma_{N,N+p}$, vil ϱ_N konvergere mod Nul for $N \to \infty$ og

$$1 - W_{N,N+p}(\Gamma_N)$$

vil være konstant lig med Nul. Vi skal imidlertid senere ved et Eksempel vise, at Betingelsen ogsaa kan være opfyldt i Tilfælde, hvor Rækken $\sum_{n=0}^{\infty} \omega_n$ er divergent.

47. Lad os nu antage den fundne Betingelse opfyldt. Da $\eta_N \to 0$ for $N \to \infty$, kan vi bestemme et Tal N_1 saa stort, at for alle $N \geqq N_1$, $\eta_N < 1$. For ethvert saadant N og ethvert $p \geqq p_0 = p_0(N)$ er da ifølge (9)

$$W_{N,N+p}(\Gamma_N) > 0$$

og Punktmængden $\Sigma_{N,N+p}$ indeholder sikkert Punkter af Γ_N. Rækken $\sum_{n=0}^{\infty} \omega_n$ falder altsaa ind under de i § 7 betragtede uendelige Rækker, som det var muligt paa naturlig Maade at tilordne en bestemt Sum; denne Sum betegner vi som ovenfor Σ.

Grænsefunktionen $F(P)$ for Funktionsfølgen $F_{N_0}(P), F_{N_0+1}(P), \ldots, F_N(P), \ldots$ er en i hele Planen kontinuert og begrænset Funktion af Punktet P; den er aldrig negativ. Vi vil vise, at den er positiv netop for de indre Punkter af Mængden Σ, d. v. s. for de og kun de Punkter, som tillige med en vis Omegn tilhører det Indre for samtlige Mængder Σ_N fra et vist Trin. At ethvert Punkt Q, for hvilket $F(Q) > 0$, er indre Punkt for Σ er klart; thi for alle Punkter P i en vis Omegn af Q er som Følge af Kontinuiteten $F(P) > \frac{1}{2} F(Q)$, altsaa som Følge af den ligelige Konvergens for alle N fra et vist Trin $F_N(P) > 0$, hvad der netop er ensbetydende med, at den betragtede Omegn tilhører det Indre af Σ_N. At ogsaa omvendt $F(Q) > 0$ for ethvert indre Punkt Q af Σ følger af Betingelsen fra § 46; vælger vi nemlig N saa stor, at den afsluttede Cirkelskive $Q + \Gamma_N$ med Centrum Q og Radius ϱ_N tilhører det Indre for Mængden Σ_N samtidig med at $\eta_N < \frac{1}{2}$, er for ethvert $p \geqq p_0 = p_0(N)$ ifølge (9) $W_{N,N+p}(\Gamma_N) > \frac{1}{2}$; altsaa er ifølge (6) $F_{N+p}(Q) > \frac{1}{2} g$, naar vi med g betegner den positive nedre Grænse for $F_N(P)$ paa $Q + \Gamma_N$. Men heraf følger ved Grænseovergangen $p \to \infty$ at $F(Q) \geqq \frac{1}{2} g$, hvormed Beviset er fuldført.

Vi vil vise, at *Integralet af Funktionen $F(P)$ udstrakt over Punktmængden Σ*, som er lig med Integralet udstrakt over hele Planen, *er lig med 1.*

Lad C med Radius r være en afsluttet Cirkelskive med Centrum i Begyndelsespunktet. Vi har

$$\iint_C F(P)\,dC = \lim_{N\to\infty}\iint_C F_N(P)\,dC \leqq 1.$$

Hvad vi skal vise er, at Integralet $\iint_C F(P)\,dC$ for $r \to \infty$ konvergerer mod 1, at der m. a. O. til ethvert selv nok saa lille positivt Tal η svarer et $r = r(\eta)$ saa stort, at

$$\iint_C F(P)\,dC \geqq 1-\eta, \tag{10}$$

hvor $C = C(\eta)$ er Cirklen med Radius $r(\eta)$.

Som Følge af vor Forudsætning kan vi bestemme et Tal N saa stort, at for ethvert $p \geqq p_0 = p_0(N)$

$$1 - W_{N,N+p}(\Gamma_N) < \eta,$$

hvor Γ_N er den afsluttede Cirkelskive med Centrum O og Radius ϱ_N. Lad d_N betegne den største Afstand fra Begyndelsespunktet til et Punkt af Mængden Σ_N; vi vil vise, at Betingelsen (10) er opfyldt, naar vi sætter $r = d_N + \varrho_N$. For ethvert Punkt Q af Γ_N er da Punktmængden $\Sigma_N + Q$ en Delmængde af C, og vi har

$$\iint_C F_N(P-Q)\,dC = 1.$$

Lad $\Sigma^*_{N,N+p}$ betegne den Del af $\Sigma_{N,N+p}$, som tilhører Γ_N; som Følge af Relationen (6) i § 45 er for ethvert $p \geqq p_0$

$$\begin{aligned}\iint_C F_{N+p}(P)\,dC &= \iint_C dC \iint_{\Sigma_{N,N+p}} F_N(P-P_{N,N+p})\,F_{N,N+p}(P_{N,N+p})\,d\Sigma_{N,N+p} = \\ &\iint_{\Sigma_{N,N+p}} d\Sigma_{N,N+p}\iint_C F_N(P-P_{N,N+p})\,F_{N,N+p}(P_{N,N+p})\,dC \geqq \\ &\iint_{\Sigma^*_{N,N+p}} F_{N,N+p}(P_{N,N+p})\,d\Sigma^*_{N,N+p} = W_{N,N+p}(\Gamma_N) > 1-\eta.\end{aligned}$$

Heraf følger imidlertid umiddelbart ved Grænseovergangen $p \to \infty$, at

$$\iint_C F(P)\,dC \geqq 1-\eta, \tag{10}$$

hvormed den opstillede Sætning er bevist.

Funktionen $F(P)$ betegnes som en *kontinuert Punktsandsynlighed* svarende til Punktmængden Σ.

48. Lad M betegne en vilkaarlig Punktmængde i Planen; vi vil vise, at Sandsynlighederne $W_{N,i}(M)$ og $W_{N,y}(M)$ som ovenfor for $N \to \infty$ konvergerer mod bestemte Grænseværdier

$$W_i(M) = \lim_{N \to \infty} W_{N,i}(M) \quad \text{og} \quad W_y(M) = \lim_{N \to \infty} W_{N,y}(M)$$

bestemt henholdsvis som det indre Integral $J_i(M)$ og det ydre Integral $J_y(M)$ af Funktionen $F(P)$ over Mængden M. Er M begrænset, følger dette straks af Ulighederne

$$\left.\begin{matrix} |J_{N,i}(M) - J_i(M)| \\ |J_{N,y}(M) - J_y(M)| \end{matrix}\right\} \leqq \text{Øvre Grænse}_{P} \, |F_N(P - F(P)| \cdot m_y(M),$$

hvor $J_{N,i}(M) = W_{N,i}(M)$ og $J_{N,y}(M) = W_{N,y}(M)$ betegner det indre og det ydre Integral af $F_N(P)$ over Mængden M og $m_y(M)$ det ydre Maal for M. Er M ubegrænset, kan vi gaa saaledes frem: Lad η betegne et vilkaarligt lille positivt Tal; svarende til dette bestemmer vi som ovenfor et positivt helt Tal N og en Cirkel C med Centrum i Begyndelsespunktet saa stor, at for ethvert $p \geqq p_0 = p_0(N)$

$$\iint_C F_{N+p}(P)\, dC > 1 - \eta.$$

Betegner vi med M_0 den fælles Del for M og C, er da

$$(11) \qquad J_{N+p,i}(M) - J_{N+p,i}(M_0) < \eta, \quad J_{N+p,y}(M) - J_{N+p,y}(M_0) < \eta.$$

Samtidig er ifølge (10)

$$(12) \qquad J_i(M) - J_i(M_0) \leqq \eta, \quad J_y(M) - J_y(M_0) \leqq \eta.$$

Nu er M_0 begrænset; bestemmer vi derfor et Tal $p_1 \geqq p_0$ saa stort, at for ethvert $p \geqq p_1$

$$|J_{N+p,i}(M_0) - J_i(M_0)| < \eta, \quad |J_{N+p,y}(M_0) - J_y(M_0)| < \eta,$$

faas ved Hjælp af (11) og (12) for enhver saadan Værdi af p

$$|J_{N+p,i}(M) - J_i(M)| < 3\eta; \quad |J_{N+p,y}(M) - J_y(M)| < 3\eta.$$

Disse Uligheder viser, at $J_{N+p,i}(M)$ og $J_{N+p,y}(M)$ for $p \to \infty$ konvergerer henholdsvis mod $J_i(M)$ og $J_y(M)$.

Størrelserne $W_i(M)$ og $W_y(M)$ betegnes henholdsvis som den indre og den ydre Sandsynlighed for, at et vilkaarligt Punkt af Σ tilhører M. Er de to Sandsynligheder ligestore, betegner vi deres fælles Værdi $W(M)$ som Sandsynligheden for, at et vilkaarligt Punkt af Σ tilhører M; Mængdefunktionen $W(M)$, der betegnes som den til Mængden Σ svarende *Mængdesandsynlighed* i Planen bestemmes paany ved Integralfremstillingen

$$(4) \qquad W(M) = \iint_M F(P)\, dM,$$

og er som den tidligere Mængdesandsynlighed defineret for de og kun de Mængder M, som af Σ indeholder en maalelig Mængde. Til en Fremstilling af $W(M)$ ved Hjælp af et uendeligdimensionalt Maal, saaledes som det i § 44 er sket for de konvergente Rækker, er der derimod ingen Anledning i det foreliggende Tilfælde, hvor Punktmængden Σ ikke umiddelbart fremtræder som Billede af den uendeligdimensionale Enhedsterning.

Eksempel.

49. Vi vil anvende de foregaaende Betragtninger paa et specielt Tilfælde, hvor de givne Kurver er definerede ad analytisk Vej.

Den analytiske Funktion

$$Y = \operatorname{Log} X \tag{13}$$

af den komplekse Variable X antager for $X = 0$ og for $X = \infty$ Værdien $Y = \infty$. Sætter vi iøvrigt

$$Y = U + iV; \; X = Re^{2\pi i \Theta}, \left(-\frac{1}{2} \leqq \Theta < \frac{1}{2}\right),$$

faar vi

$$U = \log R; \; V = 2\pi\Theta.$$

Funktionen (13) afbilder derfor den langs den negative Halvakse opskaarne X-Plan paa en Parallelstrimmel $-\pi \leqq V < \pi$ i Y-Planen; ved Afbildningen gaar Cirkler $R = R_0$ over i rette Liniestykker $U = U_0$, medens Halvlinier $\Theta = \Theta_0$ gaar over i rette Linier $V = V_0$; Afbildningen er konform.

En Cirkel

$$X = 1 - re^{2\pi i \theta}, \; 0 \leqq \theta < 1 \tag{14}$$

i X-Planen gaar ved Afbildningen over i en Kurve

$$Y = \operatorname{Log}\,(1 - re^{2\pi i \theta}) \tag{15}$$

i Y-Planen uden Dobbeltpunkter. Er $r < 1$ er denne Kurve lukket og tilhører den af Linierne $V = -\frac{\pi}{2}$ og $V = \frac{\pi}{2}$ begrænsede Parallelstrimmel; er $r = 1$ har Kurven disse Linier til Asymptoter, men lukker sig i det uendelig fjerne Punkt; er endelig $r > 1$ er Kurven en aaben Bue, hvis Endepunkter falder paa Linierne $V = -\pi$ og $V = \pi$.

Lad os nu antage $r < 1$. Ligningen (15) fremstiller da (se Fig. 34) en *konveks* Jordankurve ω; thi som Følge af den konforme Afbildning har Kurven i det vilkaarlige Punkt Y en bestemt Tangent, hvis Vinkel med den med U-Aksen parallele Linie gennem Punktet er lig med Vinklen mellem Cirkeltangenten i det tilsvarende Punkt X og dette Punkts Radiusvektor. Denne Vinkel, hvis Gradantal er det halve

af Gradantallet for den af Radiusvektor paa Cirklen afskaarne Bue, varierer monotont, naar X gennemløber Cirklen (14). Det samme gælder derfor Tangenten til Jordankurven (15), som altsaa er konveks. Foruden at være symmetrisk omkring Linien $V = 0$ er Kurven ω symmetrisk om den derpaa vinkelrette Linie $U = \log \sqrt{1-r^2}$; thi til en Spejling af Y-Planen i denne Linie svarer en Spejling af X-Planen i Cirklen $R = \sqrt{1-r^2}$; men ved denne Spejling gaar Cirklen (14) over i sig selv.

Til voksende Værdier af θ svarer en bestemt Omløbsretning paa Kurven ω. Vinklen mellem den positive U-Akse og den i Overensstemmelse med denne Omløbsretning orienterede Tangent til ω i det vilkaarlige Punkt $Y = Y(\theta)$ har Størrelsen $\arg \frac{dY}{d\theta}$. Nu er som Følge af Relationerne (14) og (15) ikke alene Funktionerne X og Y, men ogsaa R og Θ, $\frac{dY}{d\theta}$, $\arg \frac{dY}{d\theta}$ o. s. v. differentiable Funktioner af den reelle Variable θ. Kurven ω har derfor i Punktet Y en bestemt Krumningsradius (§ 14) bestemt som Grænseværdi for Forholdet mellem Korden $|\Delta Y|$ og Totalkrumningen $\Delta \arg \frac{dY}{d\theta}$ for den til Parameterintervallet $(\theta, \theta + \Delta\theta)$ svarende forsvindende Bue Y, $Y + \Delta Y$ af ω. Thi for $\Delta\theta \to 0$ vil dette Forhold konvergere mod Forholdet mellem den numeriske Værdi $\left|\frac{dY}{d\theta}\right|$ af Differentialkvotienten $\frac{dY}{d\theta}$ og Differentialkvotienten $\frac{d \arg \frac{dY}{d\theta}}{d\theta}$ i Punktet θ. Idet $\frac{dY}{dX} = \frac{1}{X}$, har man $\frac{dY}{d\theta} = \frac{dX}{d\theta} \cdot \frac{1}{X} = -r \cdot 2\pi i \cdot e^{2\pi i \theta} \cdot \frac{1}{X}$, hvoraf dels $\left|\frac{dY}{d\theta}\right| = \frac{r \cdot 2\pi}{R}$ dels $\arg \frac{dY}{d\theta} = -\frac{\pi}{2} + 2\pi\theta - 2\pi\Theta$ og altsaa $\frac{d \arg \frac{dY}{d\theta}}{d\theta} = 2\pi - 2\pi \frac{d\Theta}{d\theta}$. Vi faar derfor for Krumningsradius Udtrykket

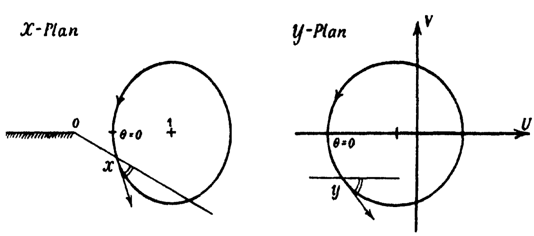

X-Plan
Y-Plan

Fig. 34.

$$\lim_{\Delta\theta \to 0} \frac{|\Delta Y|}{\Delta \arg \frac{dY}{d\theta}} = \frac{r}{R\left(1 - \frac{d\Theta}{d\theta}\right)}. \tag{16}$$

Nu er ifølge (14)

$$R \cos 2\pi\Theta = 1 - r \cos 2\pi\theta$$
$$R \sin 2\pi\Theta = \quad -r \sin 2\pi\theta.$$

Ved Differentiation faas heraf to lineære Ligninger til Bestemmelse af $\frac{dR}{d\theta}$ og $\frac{d\Theta}{d\theta}$; man faar

$$\frac{d\Theta}{d\theta} = -\frac{r \cos 2\pi\theta \cos 2\pi\Theta + r \sin 2\pi\theta \sin 2\pi\Theta}{R} = -\frac{\cos 2\pi\Theta - R}{R}.$$

Indsættes dette i (16), faar man for Krumningsradius Udtrykket

$$\frac{r}{\cos 2\pi\Theta}.$$

Indre og ydre Radius i den givne Kurve, der (se § 14) bestemmes som nedre og øvre Grænse for samtlige Krumningscirklers Radier, bliver derefter henholdsvis

$$r_i = r \text{ og } r_y = \frac{r}{\sqrt{1-r^2}}; \tag{17}$$

for enhver Værdi af r er $0 < r_i$; $r_y < \infty$; Kurven tilhører derfor den i § 12 afgrænsede Kurveklasse.

Ved Hjælp af Parameterfremstillingen (15) er der defineret en kontinuert Buesandsynlighed paa Kurven ω; som man ser, er denne Sandsynlighed differentiabel; dens Differentialkvotient $\left|\frac{d\theta}{dY}\right| = \frac{R}{r\cdot 2\pi}$ fremstiller en kontinuert Punktsandsynlighed paa Kurven.

50. Lad der nu være givet *en Følge*

$$r_0, r_1, \ldots, r_N, \ldots \tag{18}$$

af positive Tal r_n *mindre end 1, som konvergerer mod Nul for* $n \to \infty$. *De tilsvarende Kurver* ω_n, hvis Parameterfremstillinger er

$$Y_n = \operatorname{Log}(1 - r_n e^{2\pi i \theta_n}),\ 0 \leq \theta_n < 1, \tag{19}$$

er konvekse Kurver af Klassen K; *deres Radier*

$$r_{n,i} = r_n \text{ og } r_{n,y} = \frac{r_n}{\sqrt{1-r_n^2}}$$

konvergerer mod Nul for $n \to \infty$. Betingelserne for Anvendelsen af Betragtningerne i forrige Kapitel er saaledes tilstede.

For enhver Værdi af N fremkommer Punktmængden $\Sigma_N = \sum_{n=0}^{N}\omega_n$ som Værdiforraad for Funktionen

$$S_N(\theta_0, \theta_1, \ldots, \theta_N) = \sum_{n=0}^{N} \operatorname{Log}(1 - r_n e^{2\pi i \theta_n}),$$

naar de Variable $\theta_0, \theta_1, \ldots, \theta_N$ uafhængig af hinanden gennemløber Intervallet (0, 1), eller, hvad der kommer ud paa det samme, naar Punktet $(\theta_0, \theta_1, \ldots, \theta_N)$ gennemløber Enhedsterningen Q_N i det $N+1$-dimensionale $\theta_0, \theta_1, \ldots, \theta_N$-Rum. Den indre og ydre Sandsynlighed $W_{N,i}(M)$ og $W_{N,y}(M)$ for at et Punkt af Σ_N tilhører en given Punktmængde M i Planen bestemmes henholdsvis som det indre og ydre Jordan'ske Maal for Mængden Ω af Punkter $(\theta_0, \theta_1, \ldots, \theta_N)$ af Q_N, for hvilke $S_N(\theta_0, \theta_1, \ldots, \theta_N)$ tilhører M. For alle N fra et vist Trin N_0 fremstilles disse Sandsynligheder hen-

holdsvis som det indre og ydre Integral over Mængden M af en kontinuert Punktsandsynlighed $F_N(Y)$ i Planen.

Er den uendelige Række

$$\sum_{n=0}^{\infty} \operatorname{Log}(1 - r_n e^{2\pi i \theta_n}) \tag{20}$$

konvergent, d. v. s. *er Rækken*

$$\sum_{n=0}^{\infty} r_n \tag{21}$$

konvergent, vil Punktsandsynlighederne $F_N(Y)$, som det fremgaar af den almindelige Undersøgelse i §§ 41—43, for $N \to \infty$ konvergere mod en kontinuert Punktsandsynlighed $F(Y)$ paa den ved Rækken (20) fremstillede Mængde $\Sigma = \sum_{n=0}^{\infty} \omega_n$.

Er Rækken (21) divergent, er ogsaa den uendelige Række (20) divergent; vi kan derfor ikke umiddelbart udsige noget om Eksistensen af en Grænsefunktion for Funktionsfølgen $F_{N_0}(Y)$, $F_{N_0+1}(Y)$, ..., $F_N(Y)$, ... Vi vil imidlertid vise, at der ogsaa i dette Tilfælde, *naar blot den uendelige Række*

$$\sum_{n=0}^{\infty} r_n^2, \tag{22}$$

hvis Led er *Kvadraterne* paa de givne Radier, *er konvergent*, i den i §§ 45—48 angivne Betydning eksisterer en kontinuert Punktsandsynlighed $F(Y)$ bestemt ved de givne, uendelig mange, konvekse Kurvers Addition[1].

For ethvert Tal $N \geq N_0$ og ethvert positivt Tal p fremkommer den i § 45 indførte Punktmængde $\Sigma_{N,N+p} = \sum_{n=N+1}^{N+p} \omega_n$ som Værdiforraad for Funktionen

$$S_{N,N+p}(\theta_{N+1}, \ldots, \theta_{N+p}) = \sum_{n=N+1}^{N+p} \operatorname{Log}(1 - r_n e^{2\pi i \theta_n}),$$

naar Punktet $(\theta_{N+1}, \ldots, \theta_{N+p})$ gennemløber Enhedsterningen $Q_{N,N+p}$ i det p-dimensionale $\theta_{N+1}, \ldots, \theta_{N+p}$-Rum. Afbildningen af $\Sigma_{N,N+p}$ paa $Q_{N,N+p}$ bestemmer en Mængdesandsynlighed $W_{N,N+p}(M)$ i $\Sigma_{N,N+p}$-Planen. Er $p \geq p_0 = p_0(N)$, kan denne

[1] I alle Tilfælde, hvor Rækken (21) er divergent, falder Rækken (20) ind under de i § 7 betragtede divergente Rækker $\Sigma\omega_n$, som det er muligt paa naturlig Maade at tillægge en bestemt Sum. Summen Σ af Rækken (20) eksisterer derfor uafhængigt af, om Rækken (22) er konvergent, og udgør, som man ved Betragtning af Mængderne Σ_N let viser, i alle Tilfælde selve den komplekse Plan. Medens for konvergente Rækker (20) Punktmængden Σ godt kan falde ind under det i § 6 nævnte Tilfælde af en afsluttet konveks Mængde med en udartet Rand i det Indre, er Tilstedeværelsen af en saadan Rand udelukket, naar Rækken (20) er divergent. Funktionen $F(Y)$ bliver derfor i dette Tilfælde (Konvergensen af Rækken (22) forudsat) altid en i hele den komplekse Plan positiv Funktion.

Mængdesandsynlighed fremstilles som Integral af en kontinuert Punktsandsynlighed paa Mængden $\Sigma_{N,N+p}$.

Vi betragter det over den p-dimensionale Enhedsterning $Q_{N,N+p}$ udstrakte Integral

$$\int \cdots \int_{Q_{N,N+p}} |S_{N,N+p}(\theta_{N+1}, \ldots, \theta_{N+p})|^2 \, dQ_{N,N+p} =$$
$$\int_0^1 \cdots \int_0^1 |S_{N,N+p}(\theta_{N+1}, \ldots, \theta_{N+p})|^2 \, d\theta_{N+1} \ldots d\theta_{N+p}$$

af Kvadratet paa den absolute Værdi af Funktionen $S_{N,N+p}(\theta_{N+1}, \ldots, \theta_{N+p})$. Da

$$|S_{N,N+p}(\theta_{N+1}, \ldots, \theta_{N+p})|^2 = S_{N,N+p}(\theta_{N+1}, \ldots, \theta_{N+p}) \cdot \bar{S}_{N,N+p}(\theta_{N+1}, \ldots, \theta_{N+p}) =$$
$$\left(\sum_{n=N+1}^{N+p} \operatorname{Log}\left(1 - r_n e^{2\pi i \theta_n}\right)\right)\left(\sum_{n=N+1}^{N+p} \operatorname{Log}\left(1 - r_n e^{-2\pi i \theta_n}\right)\right) =$$
$$\left(\sum_{n=N+1}^{N+p} \sum_{m=1}^{\infty} \frac{r_n^m e^{2\pi i \theta_n \cdot m}}{m}\right)\left(\sum_{n=N+1}^{N+p} \sum_{m=1}^{\infty} \frac{r_n^m e^{-2\pi i \theta_n \cdot m}}{m}\right),$$

faas paa bekendt Maade

$$\int \cdots \int_{Q_{N,N+p}} |S_{N,N+p}(\theta_{N+1}, \ldots, \theta_{N+p})|^2 \, dQ_{N,N+p} = \sum_{n=N+1}^{N+p} \sum_{m=1}^{\infty} \frac{r_n^{2m}}{m^2},$$

hvoraf, da

$$\sum_{m=1}^{\infty} \frac{r_n^{2m}}{m^2} < \sum_{m=1}^{\infty} \frac{r_n^2}{m^2} = \frac{\pi^2}{6} r_n^2,$$

$$\int \cdots \int_{Q_{N,N+p}} |S_{N,N+p}(\theta_{N+1}, \ldots, \theta_{N+p})|^2 \, dQ_{N,N+p} < \frac{\pi^2}{6} \sum_{n=N+1}^{N+p} r_n^2. \tag{23}$$

Under Forudsætning af, at den uendelige Række

$$\sum_{n=0}^{\infty} r_n^2 \tag{22}$$

er konvergent, kan vi nu vise, at den i § 46 angivne Betingelse for den ligelige Konvergens af Funktionsfølgen $F_{N_0}(Y), F_{N_0+1}(Y), \ldots, F_N(Y), \ldots$ er tilfredsstillet, at der m. a. O. til ethvert $N \geqq N_0$ svarer to positive Tal ϱ_N og η_N, som konvergerer mod Nul for $N \to \infty$, saaledes at for ethvert $p \geqq p_0 = p_0(N)$

$$1 - W_{N,N+p}(\Gamma_N) < \eta_N,$$

hvor Γ_N betegner den afsluttede Cirkelskive med Radius ϱ_N, som har sit Centrum i Punktet $Y = 0$.

For enhver Værdi af ϱ_N svarer der til den Del af $\Sigma_{N,N+p}$, som ikke tilhører Γ_N, en maalelig Delmængde Ω af $Q_{N,N+p}$ med Maalet $1-W_{N,N+p}(\Gamma_N)$; følgelig er

$$\varrho_N^2(1-W_{N,N+p}(\Gamma_N)) = \int\cdots\int_\Omega \varrho_N^2\, d\Omega < \int\cdots\int_\Omega |S_{N,N+p}(\theta_{N+1},\ldots,\theta_{N+p})|^2 d\Omega <$$

$$\frac{\pi^2}{6}\sum_{n=N+1}^{N+p} r_n^2 < R_N,$$

hvor

$$R_N = \frac{\pi^2}{6}\sum_{n=N+1}^{\infty} r_n^2$$

konvergerer mod Nul for $N \to \infty$. Sætter vi derfor

$$\varrho_N = \eta_N = \sqrt[3]{R_N},$$

vil der for ethvert $p \geqq p_0$ gælde Relationen

$$1-W_{N,N+p}(\Gamma_N) < \frac{R_N}{\varrho_N^2} = \eta_N,$$

og ϱ_N og η_N vil konvergere mod Nul for $N \to \infty$. Hermed er Beviset fuldført.

INDHOLD

OTHER PAPERS

RECHERCHES SUR LA MULTIPLICATION DE DEUX INTÉGRALES DÉFINIES PRISES ENTRE DES LIMITES INFINIES

PAR

H. BOHR

Introduction

Soient $u(x)$ et $v(x)$ deux fonctions réelles ou complexes de la variable réelle x, et soient d'une part $u(x)$ continue pour $0 \leqq x \leqq a$, et de l'autre $v(x)$ continue pour $0 \leqq x \leqq b$, a et b étant deux nombres positifs et finis quelconques; nous aurons, comme l'on sait, l'équation suivante:

$$\int_0^a u(x)\,dx \,.\int_0^b v(x)\,dx = \iint_{R_{a,b}} u(x)\,v(y)\,d\omega \tag{1}$$

où le champ $R_{a,b}$ de l'intégrale double du second membre sera un rectangle aux angles opposés $(0, 0)$, (a, b), ayant ses côtés parallèles aux axes du système de coordonnées rectangulaires X-Y, tandis que $d\omega$ représentera l'élément du plan X-Y.

Considérons maintenant deux intégrales convergentes

$$\int_0^\infty u(x)\,dx \quad \text{et} \quad \int_0^\infty v(x)\,dx$$

— dans ce qui suit nous entendrons toujours, à moins que le lecteur ne soit informé du contraire, par $u(x)$ et par $v(x)$ des fonctions continues pour $x \geqq 0$ —; l'équation (1) entraîne immédiatement l'équation suivante:

$$\int_0^\infty u(x)\,dx \,.\int_0^\infty v(x)\,dx = \lim_{a=\infty,\, b=\infty} \int_0^a u(x)\,dx \,.\int_0^b v(x)\,dx$$
$$= \lim_{a=\infty,\, b=\infty} \iint_{R_{a,b}} u(x)\,v(y)\,d\omega \tag{2}$$

G 1. *Overs. Dan. Vid. Selsk. Forh.* 1908.

où le passage à la limite du dernier membre devra être effectué de manière à faire croître le rectangle $R_{a,b}$ indéfiniment, a et b tendant tous les deux vers ∞.

On sait que dans le cas où les deux intégrales considérées $\int_0^\infty u(x)\,dx$ et $\int_0^\infty v(x)\,dx$ sont absolument convergentes, en d'autres termes: au cas où $\int_0^\infty |u(x)|\,dx$ et $\int_0^\infty |v(x)|\,dx$ sont convergentes, le théorème exprimé par l'équation (2) peut être généralisé comme il suit:

$$\int_0^\infty u(x)\,dx \,.\int_0^\infty v(x)\,dx = \lim_{S=\infty} \iint_S u(x)\,v(y)\,d\omega\,; \tag{3}$$

dans cette équation le champ S de l'intégrale double est une aire finie quelconque, cohérente ou incohérente, située dans le quart de plan positif, et le passage à la limite doit se faire de telle sorte que le champ S croissant indéfiniment contienne à un certain moment un carré $[(0, 0)\text{-}(x, x)]$ qu'on pourra d'ailleurs choisir aussi grand qu'on le voudra.

Ce théorème est d'une grande importance pour l'analyse. Rappelons ici l'application classique qu'en a faite Laplace[1]) pour trouver la valeur de l'intégrale absolument convergente $\int_0^\infty e^{-x^2}dx$. Laplace suppose que dans l'équation

$$\int_0^\infty e^{-x^2}dx \,.\int_0^\infty e^{-x^2}dx = \lim_{S=\infty} \iint_S e^{-(x^2+y^2)}\,d\omega$$

le champ d'intégration S est un secteur circulaire limité par les axes et par un cercle ayant son centre à l'origine. Cela posé, il trouve facilement la valeur de l'intégrale double. Et si, ensuite, on passe à la limite en faisant croître indéfiniment le rayon du cercle et qu'on extraie la racine carrée, on trouve immédiatement $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$.

Si, au contraire, l'une des deux intégrales convergentes en question $\int_0^\infty u(x)\,dx$ ou $\int_0^\infty v(x)\,dx$ n'est pas absolument convergente mais „semi-convergente" ($\lim_{x=\infty}\int_0^x |u(x)|\,dx = \infty$), ou bien, si les intégrales sont toutes les deux „semi-conver-

1) Mémoires de l'Académie Royale des Sciences (Paris, 1778).

gentes", l'équation (3) n'a pas lieu généralement, et même dans les cas où $\iint_S u(x)v(y)\,d\omega$ tend vers une valeur limite finie et déterminée, le domaine S croissant indéfiniment d'une manière déterminée, cette valeur limite n'est pas nécessairement égale au produit des deux intégrales $\int_0^\infty u(x)\,dx$ et $\int_0^\infty v(x)\,dx$.

Abstraction faite du théorème immédiatement évident qu'exprime l'équation (2), théorème qui ne permet probablement pas des applications nombreuses, on n'a pas, que je sache, étudié jusqu'ici les rapports qui auront lieu (dans le cas de deux intégrales dont l'une au moins est semi-convergente) entre le produit $\int_0^\infty u(x)\,dx \, . \int_0^\infty v(x)\,dx$ et l'intégrale double $\iint_S u(x)v(y)\,d\omega$, quand nous faisons croître le domaine S à l'infini autrement que par l'accroissement du rectangle $R_{a,b}$.

Dans ce qui suit je me propose de montrer que pour ces rapports il existe dans le cas où l'une des deux intégrales est absolument convergente, l'autre semi-convergente, et aussi dans le cas de deux intégrales semi-convergentes, une suite de théorèmes généraux. Ensuite nous donnerons quelques exemples des applications à faire des dits théorèmes dans le domaine de l'„analyse".

CHAPITRE I

Multiplication de deux intégrales dont l'une est absolument convergente, l'autre semi-convergente.

Théorème I. Soient $U = \int_0^\infty u(x)\,dx$ une intégrale absolument convergente et $V = \int_0^\infty v(x)\,dx$ une intégrale semi-convergente; on aura, en posant

$$w(x) = \int_0^x u(y)\,v(x-y)\,dy,$$

$\int_0^\infty w(x)\,dx$ convergente et de valeur $U.V$ ou bien, en réunissant: $\int_0^\infty u(x)\,dx \, . \int_0^\infty v(x)\,dx = \int_0^\infty dx \, . \int_0^x u(y)\,.\,v(x-y)\,dy.$

3

Démonstration: Posant $s(x) = \int_0^x u(y)\,dy$, $t(x) = \int_0^x v(y)\,dy$ et

$$W(x) = \int_0^x w(y)\,dy,$$

il viendra: $s(x) \,.\, t(x) = W(x) + \int_0^x u(y) \left(\int_{x-y}^x v(z)\,dz \right) dy$ [1]) $\quad$ (1)

ce qui donnera:

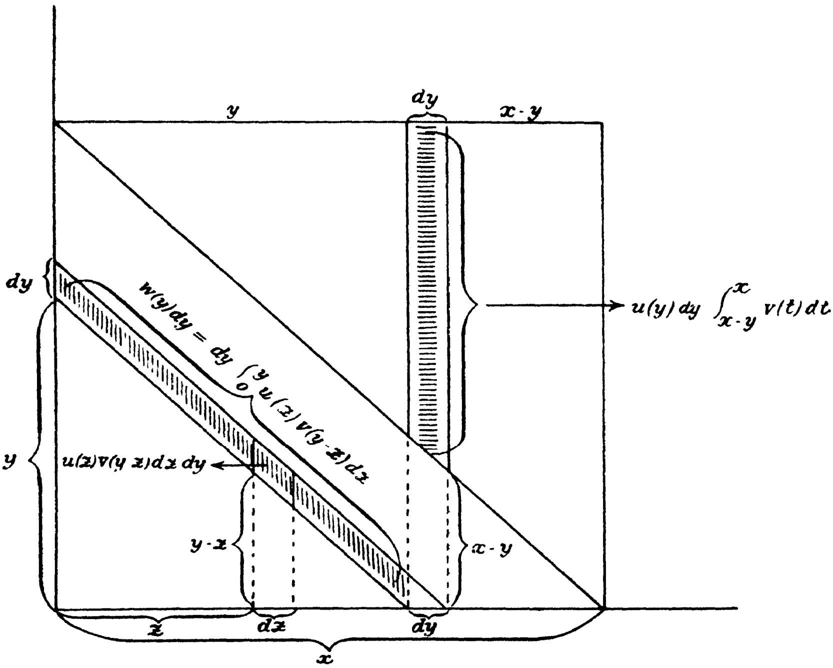


Fig. I.

[1]) L'équation (1), identique pour $x = 0$, se vérifie à l'aide d'une différentiation. Il vient: $s(x) \,.\, v(x) + u(x) \,.\, t(x) = w(x) + u(x) \,.\, t(x) + \int_0^x u(y)\,v(x)\,dy - \int_0^x u(y)\,v(x-y)\,dy = u(x)\,t(x) + v(x)\,s(x)$ (identité). La figure représentée ci-contre (fig. I) donne l'interprétation géométrique de l'équation (1).

$$|U.V - W(x)| \leqq |U.V - s(x).t(x)| + |s(x).t(x) - W(x)| \leqq$$
$$|U.V - s(x).t(x)| + \int_0^x |u(y)|.\left|\int_{x-y}^x v(z)\,dz\right| dy \leqq$$
$$|U.V - s(x).t(x)| + \int_0^x |u(y)|.\varepsilon(x-y)\,dy,$$ où $\varepsilon(x)$ désigne la limite supérieure de $\left|\int_{x+t}^{x+z} v(y)\,dy\right|$ pour $z \geqq t \geqq 0$, $\varepsilon(x)$ décroissant à mesure qu'augmente x: $\lim_{x=\infty} \varepsilon(x) = 0$, d'où

$$|U.V - W(x)| \leqq |U.V - s(x)t(x)| + \varepsilon\left(\frac{x}{2}\right)\int_0^{\frac{x}{2}} |u(y)|\,dy +$$
$$\varepsilon(0)\int_{\frac{x}{2}}^x |u(y)|\,dy.$$

En désignant d'autre part par $\varepsilon_1(x)$ la limite supérieure de $\int_x^{x+z} |u(y)|\,dy$, $\varepsilon_1(x)$ allant toujours en décroissant: $\lim_{x=\infty} \varepsilon_1(x) = 0$, il viendra:

$$|U.V - W(x)| \leqq$$
$$|U.V - s(x)t(x)| + \varepsilon\left(\frac{x}{2}\right)\varepsilon_1(0) + \varepsilon(0)\,\varepsilon_1\left(\frac{x}{2}\right)$$

et par suite:

$$\lim_{x=\infty} |U.V - W(x)| = 0; \quad \lim_{x=\infty} W(x) = U.V$$

c. q. f. d.

Il ressort de la fig. I que $\int_0^z w(t)\,dt = \iint u(x)\,v(y)\,d\omega$, l'intégrale double ayant pour champ la portion du plan X-Y que délimitent les axes des coordonnées et la droite $x + y = z$; le théorème peut donc s'énoncer comme il suit:

Il est permis de multiplier deux intégrales définies, dont l'une est absolument convergente et l'autre semi-convergente, en prenant pour courbe limite $x + y = z$, si nous entendons par là qu'en intégrant $f(x, y) = u(x)v(y)$ dans la partie du quart de plan positif que limite la droite $x + y = z$, le résultat obtenu par l'intégration aura, pour z croissant à l'infini, une valeur limite égale au produit $U.V$ des intégrales considérées.

Le théorème I peut se généraliser jusqu'à comprendre des cas où nous avons des courbes d'un type général déterminé limitant, avec les axes des coordonnées, le champ de l'intégrale double $\iint u(x)\,v(y)\,d\omega$ dans le plan X-Y. Pour établir de telles généralisations nous avons à démontrer le théorème suivant:

Théorème II. Soit $U = \int_0^\infty u(x)\,dx$ une intégrale absolument convergente et soit $V = \int_0^\infty v(x)\,dx$ une intégrale semi-convergente; l'intégrale de $f(x, y) = u(x)\,v(y)$, qui a pour champ la partie du plan que délimitent les axes et une courbe continue n'ayant avec une ordonnée quelconque qu'un seul point d'intersection au plus, aura une valeur limite égale à $U \cdot V$ lorsque la courbe s'éloigne à l'infini de sorte qu'elle renferme finalement un carré quelconque qu'on pourra d'ailleurs choisir aussi grand qu'on le voudra.

Voici la démonstration de ce théorème. Les notations employées sont celles dont nous nous sommes déjà servi en démontrant le théorème I.

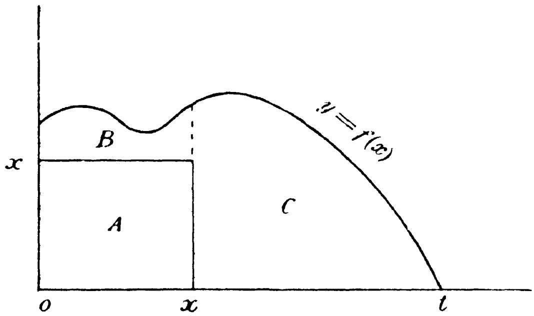


Fig. II.

$$A = s(x) \cdot t(x), \ \lim A = U \cdot V,$$

$$B = \int_0^x u(y) \cdot \left(\int_x^{f(y)} v(z)\,dz \right) dy,$$

$$|B| \leq \varepsilon(x) \int_0^x |u(y)|\,dy \leq \varepsilon(x)\,\varepsilon_1(0), \ \lim B = 0,$$

$$C = \int_x^t u(y) \left(\int_0^{f(y)} v(z)\, dz \right) dy,$$

$$| C | \leqq \varepsilon(0) . \int_x^t | u(y) | \, dy \leqq \varepsilon(0) . \varepsilon_1(x), \quad \lim C = 0,$$

$$\lim (A + B + C) = U . V, \quad \text{c. q. f. d.}$$

Il ressort avec évidence de cette démonstration que la restriction à laquelle sont assujetties les courbes considérées dans le théorème I par opposition à celles du cas de deux intégrales absolument convergentes — restriction exigeant que les courbes ne soient coupées par aucune ordonnée en plus d'un point —, peut être atténuée jusqu'à admettre que chacune des courbes soit au plus traversée par une ordonnée un nombre fini ($< N$ (Const.)) de fois. Il est clair que la restriction atténuée n'est pas essentiellement différente de la restriction première.

CHAPITRE II

Multiplication de deux intégrales semi-convergentes.

Théorème III. Soient $U = \int_0^\infty u(x)\, dx$ et $V = \int_0^\infty v(x)\, dx$ deux intégrales semi-convergentes et soit $w(x) = \int_0^x u(y)\, v(x-y)\, dy$; l'intégrale $\int_0^x w(x) dx$ aura la valeur $U . V$, pourvu qu'elle soit convergente[1]).

Au lieu de donner ici une démonstration particulière de ce théorème, nous allons énoncer un théorème beaucoup plus général (théorème IV, p. 220) dont on pourra déduire le théorème III comme un cas particulier.

En vue du développement qui va suivre nous aurons tout d'abord à généraliser la notion de convergence en introduisant celle de sommabilité. A cet effet, nous introduirons la définition que voici:

[1]) Nous donnerons plus loin (p. 224) un exemple où l'intégrale $\int_0^\infty w(x)\, dx$ est convergente et un autre exemple où elle ne l'est pas.

7

Posant $s(x) = \int_0^x u(y)\,dy$, on dira que l'intégrale $\int_0^\infty u(x)dx$ est sommable avec la valeur de sommabilité s toutes les fois que $\frac{1}{a}\int_0^a s(x)\,dx$ $(a > 0)$ aura, pour $a = \infty$, la valeur limite s.

Or il est bien connu que $\lim\limits_{a=\infty} \frac{1}{a}\int_0^a s(x)\,dx = s$ lorsque $\lim\limits_{x=\infty} s(x) = s$ tandis que $\frac{1}{a}\int_0^a s(x)\,dx$ peut avoir une valeur limite s sans que $s(x)$ ait pour $x = \infty$ une valeur limite finie et déterminée [1]); il s'ensuit que la notion de sommabilité représente une application généralisée de la notion de convergence, en d'autres termes: toute intégrale convergente de valeur s est sommable avec la valeur s, tandis que la réciproque n'a pas lieu.

Nous pouvons maintenant démontrer le théorème suivant:

Théorème IV. Soient $U = \int_0^\infty u(x)\,dx$ et $V = \int_0^\infty v(x)\,dx$ deux intégrales semi-convergentes; on aura, en posant $w(x) = \int_0^x u(y)\,v(x-y)\,dy$, $\int_0^\infty w(x)\,dx$ toujours sommable et de valeur $U.V$, c'est-à-dire que, en posant $W(x) = \int_0^x w(y)\,dy$, $\lim\limits_{a=\infty} \frac{1}{a}\int_0^a W(x)\,dx = U.V$ ou bien, en réunissant:

$$\int_0^\infty u(x)\,dx \,.\int_0^\infty v(x)\,dx = \lim_{a=\infty} \frac{1}{a}\int_0^a dx \,.\int_0^x dy \,.\int_0^y u(z)\,.\,v(y-z)\,dz.$$

Comme nous l'avons dit plus haut, le théorème III peut être déduit immédiatement du théorème IV dont il n'est qu'un cas particulier, car la valeur de sommabilité d'une intégrale convergente étant égale à la valeur de l'intégrale, $\int_0^\infty w(x)\,dx$ est évidemment égale à $U.V$ pourvu qu'elle soit convergente.

[1]) Soit par exemple $s(x)$ une fonction périodique continue de période p. Il est aisé de démontrer que $\lim\limits_{a=\infty} \frac{1}{a}\int_0^a s(x)\,dx$ existe toujours et que cette limite est égale à $\frac{1}{p}\int_x^{x+p} s(x)\,dx$, c.-à-d. à la fonction intégrée dans le champ d'une période et divisée ensuite par la longueur de la période p.

Avant de passer à la démonstration du théorème IV il nous faut énoncer un théorème auxiliaire:

Soient $\lim\limits_{x=\infty} s(x) = U$ et $\lim\limits_{x=\infty} t(x) = V$, nous aurons, $t(x)$ et $s(x)$ étant continues pour $x \geqq 0$, $\lim\limits_{x=\infty} \frac{1}{x} \cdot \int_0^x s(y) \,.\, t(x-y)\, dy = U \,.\, V.$

Démonstration:

$$\frac{1}{x}\int_0^x s(y) \,.\, t(x-y)\,dy = \frac{1}{x}\int_0^{\frac{x}{2}} s(y)\, t(x-y)\,dy + \frac{1}{x}\int_0^{\frac{x}{2}} t(y)\, s(x-y)\,dy$$

$$\frac{1}{x}\int_0^{\frac{x}{2}} s(y)\, t(x-y)\,dy = \frac{1}{2}\left[V \cdot \left(\frac{1}{\frac{x}{2}} \cdot \int_0^{\frac{x}{2}} s(y)\,dy\right) + \frac{1}{\frac{x}{2}}\int_0^{\frac{x}{2}} s(y)\left(t(x-y) - V\right) dy\right].$$

Dans cette expression la quantité $\frac{1}{\frac{x}{2}}\int_0^{\frac{x}{2}} s(y)\,dy$ a pour valeur limite U, $\lim\limits_{x=\infty} s(x)$ étant égale à U; la valeur limite de $\frac{1}{\frac{x}{2}}\int_0^{\frac{x}{2}} s(y)\left(t(x-y) - V\right) dy$ est égale à zéro, car $|s(y)|$ est moindre que Const. pour tous les y, et, en faisant x suffisamment grand, nous aurons, pour tous les y compris entre 0 et $\frac{x}{2}$, $|t(x-y) - V| < \varepsilon$, où ε désigne une quantité aussi petite qu'on le voudra. On aura donc

$$\lim_{x=\infty} \frac{1}{x} \cdot \int_0^{\frac{x}{2}} s(y)\, t(x-y)\,dy = \frac{1}{2}\, U \,.\, V$$

et, par un raisonnement tout à fait analogue:

$$\lim_{x=\infty} \frac{1}{x}\int_0^{\frac{x}{2}} t(y)\, s(x-y)\,dy = \frac{1}{2}\, U \,.\, V.$$

Donc:

$$\lim_{x=\infty} \frac{1}{x}\int_0^x s(y)\, t(x-y)\,dy = 2 \cdot \frac{1}{2}\, U \,.\, V = U \,.\, V$$

c. q. f. d.

Cela posé, nous pouvons démontrer le théorème IV:

$$U = \int_0^\infty u(x)\,dx\,, \quad V = \int_0^\infty v(x)\,dx\,, \quad s(x) = \int_0^x u(y)\,dy\,,$$

9

$$t(x) = \int_0^x v(y)\,dy, \quad w(x) = \int_0^x u(y)\,v(x-y)\,dy;$$

$$W(x) = \int_0^x w(y)\,dy = \int_0^x u(y)\,t(x-y)\,dy, \quad (1)^{1})$$

$$\int_0^a W(x)\,dx = \int_0^a t(x)\,s(a-x)\,dx \quad (2)^{1});$$

$$\lim_{a=\infty} \frac{1}{a}\int_0^a W(x)\,dx = \lim_{a=\infty} \frac{1}{a}\int_0^a t(x)\,s(a-x)\,dx = U \,.\, V$$

c. q. f. d.

Dans les intégrales considérées jusqu'ici nous avons supposé, pour plus de simplicité, que les fonctions à intégrer, $u(x)$ et $v(x)$, étaient continues pour $x \geqq 0$. En vue du développement qui va suivre, nous ferons remarquer que le théorème IV — aussi bien que tous les théorèmes précédents — a lieu également pour deux intégrales $\int_0^\infty u(x)\,dx$ et $\int_0^\infty v(x)\,dx$ dans lesquelles $u(x)$ et $v(x)$, sont continues pour $x > 0$ mais infinies au point $x = 0$ de manière toutefois que $\int_0^a |u(x)|\,dx = \lim_{\varepsilon=0}\int_\varepsilon^a |u(x)|\,dx$ et $\int_0^a |v(x)|\,dx = \lim_{\varepsilon=0}\int_\varepsilon^a |v(x)|\,dx$ existent (a étant un nombre positif et fini quelconque)[2]).

Voici quelques exemples des applications à faire, dans le domaine de l'analyse, des théorèmes précédemment établis.

Premier exemple: A l'aide du théorème IV on peut trouver d'une manière bien simple la valeur de l'intégrale semi-convergente $\int_0^\infty \frac{e^{ix}}{\sqrt{x}}\,dx$. Posons, pour obtenir le carré de l'intégrale en question, $u(x) = v(x) = \frac{e^{ix}}{\sqrt{x}}$; nous aurons, $w(x)$

[1]) (1) et (2) se démontrent par différentiation. On aura, respectivement, les identités $w(x) = u(x)\,.\,t(0) + \int_0^x u(y)\,v(x-y)\,dy = w(x)$ et

$$W(a) = t(a)\,s(0) + \int_0^a t(x)\,u(a-x)\,dx = \int_0^a u(x)\,t(a-x)\,dx = W(a).$$

[2]) En examinant plus en détail les démonstrations ci-dessus données des théorèmes I, II, III, IV, on voit aisément que ces démonstrations restent valables dans les cas où $u(x)$ et $v(x)$ ont, dans le point 0, des singularités comme celle dont nous venons de parler.

$$= \int_0^x u(y)\, v(x-y)\, dy = \int_0^x \frac{e^{iy}}{\sqrt{y}} \cdot \frac{e^{i(x-y)}}{\sqrt{x-y}}\, dy = e^{ix} \cdot \int_0^x \frac{1}{\sqrt{y}} \cdot \frac{1}{\sqrt{x-y}}\, dy$$

$$= e^{ix} \cdot \int_0^1 \frac{1}{\sqrt{t}} \cdot \frac{1}{\sqrt{1-t}}\, dt = e^{ix} \cdot \int_0^{\frac{\pi}{2}} \frac{d \sin^2 \vartheta}{\sin\vartheta \cdot \cos\vartheta} = e^{ix} \cdot \int_0^{\frac{\pi}{2}} 2\, d\vartheta$$

$= \pi \cdot e^{ix}$. On voit que $\int_0^\infty w(x)\, dx = \int_0^\infty \pi e^{ix}\, dx$ n'est pas convergente. Cherchons maintenant la valeur de sommabilité que doit posséder notre intégrale d'après le théorème IV, nous obtenons: $W(x) = \int_0^x w(y)\, dy = \pi \cdot \int_0^x e^{iy}\, dy = \frac{\pi}{i}(e^{ix}-1)$,

$$\int_0^a W(x)\, dx = \frac{\pi}{i}\left[\frac{1}{i}(e^{ia}-1) - a\right], \quad \lim_{a=\infty} \frac{1}{a}\int_0^a W(x)\, dx = \frac{\pi}{i}[-1] = \pi i.$$

Il s'ensuit que $\left(\int_0^\infty \frac{e^{ix}}{\sqrt{x}}\, dx\right)^2 = \pi i = \pi \cdot e^{\frac{\pi i}{2}}$ d'où, puisqu'on voit immédiatement que $\int_0^\infty \frac{\sin x}{\sqrt{x}}\, dx$, le facteur de la composante imaginaire de $\int_0^\infty \frac{e^{ix}}{\sqrt{x}}\, dx$, est de signe positif:

$$\int_0^\infty \frac{e^{ix}}{\sqrt{x}}\, dx = \sqrt{\pi} \cdot e^{\frac{\pi i}{4}} = \sqrt{\frac{\pi}{2}}\,(1+i).$$

Comme nous l'avons dit dans l'Introduction (p. 214) la méthode dont nous nous sommes servi ici pour trouver la valeur d'une intégrale, méthode qui consiste à élever l'intégrale au carré et à en extraire ensuite la racine carrée, avait déjà été employée par Laplace cherchant la valeur de l'intégrale absolument convergente $\int_0^\infty e^{-x^2} dx$; mais les théorèmes que nous venons d'établir ont sensiblement étendu le champ d'application de cette méthode et celui de toute méthode analogue. A l'aide de ces théorèmes nous pouvons multiplier les intégrales que nous voudrons, n'étant plus astreints à nous en tenir aux intégrales absolument convergentes.

Considérons maintenant l'exemple plus général de la multiplication des intégrales $\int_0^\infty e^{ix} \cdot x^{\alpha-1}\, dx$ et $\int_0^\infty e^{ix} \cdot x^{\beta-1}\, dx$, ces intégrales

étant, comme l'on sait, semi-convergentes pour $0 < \frac{R(\alpha)}{R(\beta)} < 1$, $R(\gamma)$ représentant la composante réelle de γ.

Nous aurons:

$$w(x) = \int_0^x u(y)\, v(x-y)\, dy = \int_0^x e^{iy}\, y^{\alpha-1} \,.\, e^{i(x-y)}(x-y)^{\beta-1}\, dy$$
$$= e^{ix} \,.\, x^{\alpha+\beta-1} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} dt = e^{ix} \,.\, x^{\alpha+\beta-1} B(\alpha, \beta),$$

où B designe la fonction béta ordinaire. Il faut distinguer deux cas, savoir:

I) $R(\alpha) + R(\beta) = R(\alpha+\beta) < 1$.

$\int_0^\infty w(x)\, dx = B(\alpha, \beta) \,.\, \int_0^\infty e^{ix} \,.\, x^{\alpha+\beta-1}\, dx$ est une intégrale convergente. Donc:

$$\int_0^\infty e^{ix}\, x^{\alpha-1}\, dx \,.\, \int_0^\infty e^{ix} \,.\, x^{\beta-1}\, dx = B(\alpha, \beta) \int_0^\infty e^{ix} \,.\, x^{\alpha+\beta-1}\, dx.$$

II) $R(\alpha) + R(\beta) \geqq 1$.

$\int_0^\infty w(x)\, dx = B(\alpha, \beta) \int_0^\infty e^{ix} \,.\, x^{\alpha+\beta-1}\, dx$ n'est pas convergente. Toutefois, d'après le théorème IV, l'intégrale est sommable avec la valeur $\int_0^\infty e^{ix} \,.\, x^{\alpha-1}\, dx \,.\, \int_0^\infty e^{ix} \,.\, x^{\beta-1}\, dx$.

Posant, en particulier, dans le cas II, $\alpha + \beta = 1$, ce qui nous donne pour l'intégrale non convergente $\int_0^\infty e^{ix} \,.\, x^{\alpha+\beta-1}\, dx = \int_0^\infty e^{ix}\, dx$ une valeur de sommabilité très facile à calculer (la quantité $B(\alpha, \beta)$, qui est indépendante de x, peut être négligée dans les calculs, étant partout mise en facteur), il viendra:

$$W(x) = \int_0^x e^{ix}\, dx = \frac{1}{i}(e^{ix}-1),$$

$$\lim_{a=\infty} \frac{1}{a} \int_0^a W(x)\, dx = \lim_{a=\infty} \frac{1}{a\,.\,i} \left[\frac{1}{i}(e^{ia}-1) - a\right] = -\frac{1}{i} = i$$

et par suite:

$$\int_0^\infty e^{ix} \,.\, x^{\alpha-1}\, dx \,.\, \int_0^\infty e^{ix} \,.\, x^{\beta-1}\, dx = i B(\alpha, \beta).$$

Pour $\alpha = \beta = \frac{1}{2}$ notre intégrale se réduit à celle que nous avons traitée plus haut (p. 222).

12

L'intégrale $\varphi(\gamma) = \int_0^\infty e^{ix}\, x^{\gamma-1}\, dx$ étant dans son région de convergence une fonction analytique de γ, nous allons montrer que l'équation fonctionnelle $\varphi(a) \cdot \varphi(\beta) = B(a,\beta)\,\varphi(a+\beta)$, trouvée pour $0 < \dfrac{R(a)}{R(\beta)} < 1$ et $R(a) + R(\beta) < 1$, peut être utilisée pour la détermination de $\varphi(\gamma)$. Posant $\varphi(\gamma) = \Gamma(\gamma)\psi(\gamma)$ la susdite équation se transforme en celle-ci: $\psi(a) \cdot \psi(\beta) = \psi(a+\beta)$, qui entraîne comme conséquence immédiate (la solution $\psi(\gamma) = 0$ n'entrant pas en ligne de compte, puisque nous avons trouvé plus haut $\varphi\left(\frac{1}{2}\right) = \sqrt{\frac{\pi}{2}}\,(1+i)$) $\psi(\gamma) = e^{K \cdot \gamma}$; donc $\varphi(\gamma) = \Gamma(\gamma) \cdot e^{K\gamma}$. La valeur de $\varphi\left(\frac{1}{2}\right)$ que nous avons trouvée plus haut, à savoir $\varphi\left(\frac{1}{2}\right) = \sqrt{\pi} \cdot e^{\frac{\pi i}{4}}$, nous servira pour la détermination de la constante K; il viendra: $\sqrt{\pi} \cdot e^{\frac{\pi i}{4}} = \Gamma\left(\frac{1}{2}\right) e^{K \cdot \frac{1}{2}} = \sqrt{\pi} \cdot e^{\frac{1}{2}K}$; $e^{\frac{\pi i}{4}} = e^{\frac{1}{2}K}$; $K = \frac{\pi i}{2} + 4n\pi i$ (n entier); $\varphi(\gamma) = \Gamma(\gamma) \cdot e^{\gamma\left[\frac{\pi i}{2} + 4n\pi i\right]}$ [1]). Pour obtenir une détermination univoque de K, c'est-à-dire pour trouver la valeur n à employer il convient de faire remarquer que $\int_0^\infty \sin x \cdot x^{\gamma-1}\, dx$, qui représente pour γ réel entre 0 et 1 le facteur de la composante imaginaire de $\int_0^\infty e^{ix}\, x^{\gamma-1}\, dx$, se montre immédiatement positif pour toutes ces valeurs de γ. Or $\Gamma(\gamma)$ étant également

[1]) Dans l'intégrale recherchée $\varphi(\gamma) = \int_0^\infty e^{ix} \cdot x^{\gamma-1}\, dx$ nous supposons naturellement $x^{\gamma-1}$ défini par $e^{(\gamma-1)\operatorname{Log} x}$ où $\operatorname{Log} x$ désigne le logarithme réel du x positif compris entre 0 et ∞. Mais si nous déterminons $x^{\gamma-1}$ par $e^{(\gamma-1)(\operatorname{Log} x + 2p\pi i)}$ et que nous fassions dans cette hypothèse $\chi_p(\gamma) = \int_0^\infty e^{ix} \cdot x^{\gamma-1}\, dx$, nous aurons $\chi_p(\gamma) = \varphi(\gamma) \cdot e^{2\pi p i \cdot \gamma}$. Le seul point où nous ayons dû tenir compte jusqu'ici de ce fait que nous entendons par $x^{\gamma-1}$ $e^{(\gamma-1)\operatorname{Log} x}$ a éte le moment où, pour obtenir la détermination de $\varphi\left(\frac{1}{2}\right)$, nous avons employé la valeur positive de $\sqrt{x} = e^{\frac{1}{2}(\operatorname{Log} x + 2p\pi i)}$. Or comme $e^{\frac{1}{2} \cdot 2(2n)\pi i} = 1$, et que par conséquent $\chi_{2n}\left(\frac{1}{2}\right) = \varphi\left(\frac{1}{2}\right)$ (tandis que $\chi_{2n+1}\left(\frac{1}{2}\right) = -\varphi\left(\frac{1}{2}\right)$), il est clair que la connaissance de la valeur $\varphi\left(\frac{1}{2}\right)$ ne peut nous fournir la fonction $\varphi(\gamma)$ qu'à un facteur $e^{2\pi i \cdot 2n\gamma} = e^{4\pi i n\gamma}$ près.

positif pour γ réel entre 0 et 1 (de sorte que le signe de la composante imaginaire de $\Gamma(\gamma)e^{\gamma\left[\frac{\pi i}{2}+4n\pi i\right]}$ est déterminé par $\sin\left[\gamma\left(\frac{\pi}{2}+4n\pi\right)\right]$) il faut choisir n de manière que $\sin\left[\gamma\left(\frac{\pi}{2}+4n\pi\right)\right]$ soit positif pour toute valeur de γ réel entre 0 et 1, ce qui implique que n soit égal à zéro. Donc

$$\int_0^\infty e^{ix}\,.\,x^{\gamma-1}\,dx = \Gamma(\gamma)\,.\,e^{\frac{\pi i\gamma}{2}} \text{ [1]}.$$

Sans entreprendre ici une étude approfondie des questions qui se rattachent à ce sujet, nous allons encore montrer qu'il y a des cas où la notion de sommabilité peut nous être utile pour prolonger une fonction analytique, définie par une intégrale définie, jusqu'à la faire dépasser les limites du domaine où l'intégrale est convergente. Dans les cas de ce genre, on n'a pas eu, jusqu'ici, de moyen pour trouver, par la seule considération de l'intégrale divergente, la valeur de la fonction en dehors de la région de convergence de l'intégrale; on a dû recourir à d'autres représentations de la fonction en question. Nous allons voir qu'à l'aide du théorème IV on peut reconnaître immédiatement que l'intégrale ci-dessus considérée $\int_0^\infty e^{ix}\,.\,x^{\gamma-1}\,dx$, dont le domaine de convergence est compris entre des droites perpendiculaires à l'axe réel passant par 0 et 1 et qui est dans ce domaine égale a $\Gamma(\gamma)\,.\,e^{\frac{\pi i\gamma}{2}}$, que cette intégrale, disons-nous, est sommable dans la région que délimitent les droites menées par 1 et 2 perpendiculairement à l'axe réel ayant le prolongement naturel de la fonction pour valeur de sommabilité.

On a en effet

I) pour $0 < R(\gamma) < 1$ $\left(\int_0^\infty e^{ix}\,.\,x^{\frac{\gamma}{2}-1}\,dx\right)^2 : B\left(\frac{\gamma}{2},\frac{\gamma}{2}\right)$

$$= \int_0^\infty e^{ix}\,.\,x^{\gamma-1}\,dx = \Gamma(\gamma)\,.\,e^{\frac{\pi i\gamma}{2}}$$

[1]) La valeur de l'intégrale $\int_0^\infty e^{ix}x^{\gamma-1}\,dx$ est déjà connue. Elle peut se trouver à l'aide du théorème fondamental de Cauchy sur l'intégration complexe.

II) pour $1 \leqq R(\gamma) < 2$ $\left(\int_0^\infty e^{ix} \cdot x^{\frac{\gamma}{2}-1} dx\right)^2 : B\left(\frac{\gamma}{2}, \frac{\gamma}{2}\right)$

$=$ la valeur de sommabilité de $\int_0^\infty e^{ix} \cdot x^{\gamma-1} dx$.

La fonction du premier membre $\left[\left(\int_0^\infty e^{ix} \cdot x^{\frac{\gamma}{2}-1}\right)^2 : B\left(\frac{\gamma}{2}, \frac{\gamma}{2}\right)\right]$ étant analytique dans tout le domaine où $0 < R(\gamma) < 2$, il s'ensuit immédiatement des équations I et II que $\int_0^\infty e^{ix} \cdot x^{\gamma-1} dx$ a, pour $1 \leqq R(\gamma) < 2$ son prolongement naturel pour valeur de sommabilité. Et comme $\Gamma(\gamma) \cdot e^{\frac{\pi i \gamma}{2}}$ est également analytique dans tout le domaine où $0 < R(\gamma) < 2$, cette valeur de sommabilité est nécessairement $\Gamma(\gamma) \cdot e^{\frac{\pi i \gamma}{2}}$.

Revenons de ces exemples à la considération de deux intégrales semi-convergentes quelconques $\int_0^\infty u(x)\,dx$ et $\int_0^\infty v(x)\,dx$. Conformément à ce que nous avons trouvé pour la multiplication de deux intégrales dont l'une est absolument convergente, l'autre semi-convergente, nous pouvons ici, en considérant deux intégrales semi-convergentes, généraliser le théorème (théorème IV) qui a lieu pour la multiplication où la courbe limite coïncide avec la droite $x + y = z$ (Const.) jusqu'à lui faire comprendre les cas où les courbes limites du champ d'intégration de l'intégrale double $\iint u(x)\,v(y)\,d\omega$, sont d'un type général déterminé. Nous sommes à même de démontrer à ce sujet le théorème suivant:

Théorème V. Désignons par $\varphi(z)$ et par $\psi(z)$ deux fonctions d'une variable réelle, z, qui soient, pour $z \geqq 0$, continues et constamment croissantes (mettons pour plus de simplicité, $\varphi(0) = \psi(0) = 0$) et qui admettent, pour $z > 0$, des fonctions dérivées continues, différentes de zéro et, par conséquent, positives: $\varphi'(z)$ et $\psi'(z)$; — définissons en outre une fonction $W(r)$ la valeur qu'on obtient par l'intégration de $f(x, y) = u(x) \cdot v(y)$ dans le champ délimité par les axes et par la courbe $\varphi(x) + \psi(y) = r$; la fonction W aura,

si elle tend pour $r = \infty$ vers une valeur limite finie et déterminée, la valeur limite $U \,.\, V$, et plus généralement: on aura dans tous les cas:

$$\lim \frac{1}{a}\int_0^a W(r)\,dr = U \,.\, V.\ ^{1})$$

Posant, en particulier, $\varphi(z) = \psi(z) = z$ le théorème sera évidemment identique au théorème IV.

Démonstration du théorème V:

Transformons les deux intégrales considérées $\int_0^\infty u(x)\,dx$ et $\int_0^\infty v(y)\,dy$ — en posant $x = \varphi^{(-1)}(\alpha)$ (la fonction inverse de φ; $\alpha = \varphi(x)$) et $y = \psi^{(-1)}(\beta)$ — en $\int_0^\infty U(\alpha)\,d\alpha$ et $\int_0^\infty V(\beta)\,d\beta$, ce qui nous donne

$$U(\alpha) = u(\varphi^{(-1)}(\alpha))\frac{d\varphi^{(-1)}(\alpha)}{d\alpha} \quad \text{et} \quad V(\beta) = V(\psi^{(-1)}(\beta))\frac{d\psi^{(-1)}(\beta)}{d\beta};$$

le résultat que nous obtiendrons en intégrant la fonction $f(x, y) = u(x) \,.\, v(y)$ dans le champ du plan $X\text{-}Y$ que limite la courbe $\varphi(x) + \psi(y) = r$, sera le même que donne, dans le plan $A \,.\, B$, l'intégration de la fonction $F(\alpha, \beta) = U(\alpha) \,.\, V(\beta)$ dans le champ limité par la droite $\alpha + \beta = r$. L'identité des résultats est due à ce fait que dans la représentation qui aura lieu le rectangle infiniment petit $d\alpha \,.\, d\beta$ du plan $A \,.\, B$ répondra à un rectangle infiniment petit $dx \,.\, dy$ du plan $X\text{-}Y$ de grandeur $\left(\frac{d\varphi^{(-1)}(\alpha)}{d\alpha} \,.\, \frac{d\psi^{(-1)}(\beta)}{d\beta}\right) d\alpha \,.\, d\beta$. Il s'ensuit que $\iint f(x, y)\,dx \,.\, dy$ étendue au champ M du plan $X\text{-}Y$ est égale à

$$\iint f(\varphi^{(-1)}(\alpha), \psi^{(-1)}(\beta))\frac{d\varphi^{(-1)}(\alpha)}{d\alpha} \,.\, \frac{d\psi^{(-1)}(\beta)}{d\beta}\,d\alpha \,.\, d\beta = \iint F(\alpha,\beta)\,d\alpha \,.\, d\beta$$

étendue au champ N du plan $A \,.\, B$, qui correspond à M. Et comme nous avons démontré plus haut (théorème IV) qu'il est permis de multiplier une intégrale par une autre intégrale (dans l'espèce: $\int_0^\infty U(\alpha)\,d\alpha \,.\, \int_0^\infty V(\beta)\,d\beta$) en prenant pour courbe

[1]) Aux cas où $f(x)$ est une fonction continue et où $\lim\limits_{a = \infty} \frac{1}{a} \,.\, \int_0^a f(x)\,dx$ est égale à F, nous dirons dans ce qui suit, pour plus de simplicité, que $f(x)$ a, pour $x = \infty$, la valeur limite généralisée F. Le théorème V peut alors s'exprimer ainsi: $W(r)$ a pour valeur limite généralisée le produit $U \,.\, V$.

16

limite la droite $\alpha + \beta = r$, nous en pouvons conclure immédiatement à la validité du théorème V portant sur la multiplication qui prend pour courbe limite $\varphi(x) + \psi(y) = r$.[1])

Nous allons faire application du théorème V en nous en servant pour trouver la valeur de l'intégrale $\int_0^\infty e^{ix^2}dx$. Posant $u(x) = v(x) = e^{ix^2}$ — (cherchant le carré de $\int_0^\infty e^{ix^2}dx$) — et mettant $\varphi(z) = \psi(z) = z^2$ — (multipliant en prenant des cercles $x^2 + y^2 = r$ pour courbes limites) — on obtient, en employant des coordonnées polaires (ρ, θ):

$$W(r) = \int_0^{\frac{\pi}{2}} d\theta \,.\int_0^{\sqrt{r}} \rho d\rho \,.\, e^{i(\rho^2\cos^2\theta + \rho^2\sin^2\theta)} = \int_0^{\frac{\pi}{2}} d\theta \,.\int_0^{\sqrt{r}} \rho d\rho \,.\, e^{i\rho^2}$$
$$= \frac{\pi}{2}\cdot\frac{1}{2i}(e^{ir}-1).$$

On voit que $W(r)$ n'a pas de valeur limite ordinaire pour $r = \infty$. La valeur limite généralisée de $W(r)$ est, on le reconnaît immédiatement, $\frac{\pi}{2}\cdot\frac{1}{2i}(-1) = \frac{\pi i}{4}$, e^{ir} ayant sa valeur limite généralisée égale à zéro. Nous avons donc $\left(\int_0^\infty e^{ix^2}dx\right)^2 = \frac{\pi i}{4}$ d'où, le signe de la racine carrée extraite étant facile à trouver, $\int_0^\infty e^{ix^2}dx = \frac{\pi\sqrt{2}}{4}(1+i)$.

D'après le théorème général sur la multiplication de deux intégrales absolument convergentes, théorème qui peut s'exprimer par l'équation (3) (p. 214), aussi bien que d'après les théorèmes relatifs à la multiplication de deux intégrales dont

[1]) La seule condition que nous ayons imposée aux fonctions continues φ et ψ c'était d'avoir pour les valeurs d'argument supérieures à zéro une fonction dérivée différente de zéro. Si nous ne les avons pas assujetties à la même restriction pour $z = 0$, c'est qu'elle n'est pas nécessaire pour notre démonstration. On le voit aisément en se rappelant la remarque que nous avons faite plus haut en ayant en vue ce point justement. Nous disions (p. 222) que le théorème IV avait lieu même si les deux intégrales considérées avaient le point 0 pour point singulier. (La valeur numérique de la fonction était supposée intégrable pour $x = +0$). Il est donc permis d'employer par exemple $\varphi(z) = \psi(z) = z^2$ (multiplier en prenant des cercles $x^2 + y^2 = r$ pour courbes limites) quoique $\varphi'(0) = \psi'(0) = 0$.

l'une est absolument convergente et l'autre semi-convergente, théorèmes énoncés au chapitre I, l'intégrale double $\iint u(x)\,v(y)\,d\omega$ étendue aux divers champs considérés par les théorèmes en question, a immédiatement $U.V$ pour valeur limite quand le champ croît indéfiniment.

Par contre, les théorèmes IV et V qui traitent le cas de deux intégrales semi-convergentes, n'impliquent pas que $\iint u(x)\,v(y)\,d\omega$ ait la valeur limite $U.V$ quand le champ d'intégration croît à l'infini des diverses manières indiqueés par ces théorèmes, mais seulement que la valeur de l'intégrale double „oscille autour de la valeur $U.V$" de sorte qu'on peut obtenir cette valeur par suite de certaines opérations d'ajustage. Il en résulte qu'en opérant avec les théorèmes IV et V nous ne sommes pas libres de considérer $\iint u(x) v(y) d\omega$ — la fonction W — comme fonction de n'importe quelle variable[1]. Dans le cas traité par le théorème V et dans celui du théorème IV qui n'en est qu'un cas particulier, la variable, indépendante, de la fonction W est la valeur de $\varphi(x)+\psi(y)$, constante pour chaque courbe. Or cette quantité n'est pas dans un rapport simple avec l'image géométrique et on pourrait se demander par exemple si le théorème V n'aurait pas lieu tel quel si on regardait la fonction W — l'intégrale double $\iint u(x)\,v(y)\,d\omega$ étendue au champ limité par la courbe $\varphi(x)+\psi(y)$ = Const. — comme fonction de l'abscisse, que découpe la courbe. Un problème qui s'impose à quiconque voudra approfondir ce sujet, est celui des modifications déterminées dans une intégrale sommable par les changements de la variable indépendante. Les questions qui s'y rattachent ont été traitées par Cesàro dans un mémoire intitulé: Contribution à la théorie

[1]) Remarquons à titre d'exemple que, considérée comme fonction de x, $e^x \cos(e^x)$ a la valeur limite généralisée 0. $\left(\lim \frac{1}{a}\int_0^a e^x \,.\, \cos e^x \,.\, dx = 0\right)$ tandis que considérée comme fonction de $y = e^x$ cette même quantité n'a pas de limite généralisée, $\frac{1}{a}\int_0^a y \,.\, \cos y \,.\, dy$ se comportant pour les a infiniment grands comme $\sin a$.

18

des limites" [1]). Cependant les résultats obtenus par M. Cesàro sont susceptibles d'amplifications assez considérables mais qui n'entreraient pas dans les cadres de la présente étude [2]); je me permettrai seulement de donner ici un théorème que je dois à une application particulière de mes propres résultats au problème traité ci-dessus:

Théorème VI. Soient a, b, m, n des nombres positifs quelconques. L'intégrale double $W(s) = \iint u(x)\,v(y)\,d\omega$ étendue au champ limité par la courbe $ax^m + by^n = r$ aura — si nous prenons pour variable indépendante l'abscisse ou l'ordonnée s découpée par la courbe — la valeur limite généralisée $U.V$, quand r et, par suite, s tendent vers ∞.

Autrement dit: $\lim\limits_{s=\infty} \frac{1}{z}\int_0^z W(s)\,ds = U.V.$

Posant, en particulier, $a = b = 1$, $m = n = 2$, ce théorème énonce que si l'on multiplie en prenant des cercles $x^2 + y^2 = \text{Const.}$, pour courbes limites on peut considérer le rayon du cercle comme variable indépendante.

REMARQUES FINALES.

Abstraction faite des théorèmes V et VI, les propositions ci-dessus données pour la multiplication de deux intégrales $\int_0^\infty u(x)\,dx$ et $\int_0^\infty v(x)\,dx$ forment un ensemble très analogue à celui des théorèmes connus qui ont lieu pour la multiplication de deux séries infinies $\sum\limits_0^\infty u_n$ et $\sum\limits_0^\infty v_n$, théorèmes que nous devons à MERTENS, STIELTJES, ABEL et CESÀRO [3]). En intro-

[1]) Darboux, Bulletin, 1889.

[2]) J'espère pouvoir bientôt publier mes recherches sur cette question.

[3]) Cette analogie devient surtout manifeste si, en multipliant les deux séries $\sum u_n$ et $\sum v_n$ on regarde $u_p . v_q$ comme une fonction du couple de nombres (p, q) et qu'on se figure les couples de nombres successifs représentés par des points sur le quart de plan positif d'un système de coordonnées rectangulaires.

duisant une nouvelle notion — on pourrait la désigner sous le nom de séries à indices arbitraires — où les séries infinies ordinaires rentreraient comme un cas particulier et dont les intégrales définies représenteraient un cas limite, il devient possible de réunir sous un point de vue commun la multiplication des séries et celle des intégrales. En effet on peut démontrer que les cas considérés par ces théorèmes sont tels qu'on peut établir pour toute la classe de séries à indices arbitraires un nouveau théorème comprenant comme un cas particulier le théorème des séries et comme cas limite le théorème des intégrales. — En opérant avec les séries plus générales à indices arbitraires au lieu des séries infinies ordinaires on obtient de pouvoir se servir d'un procédé parfaitement analogue à la transformation des intégrales définies, on pourra donc, au cas où les deux séries considérées sont semi-convergentes, amplifier la théorie de la multiplication des séries par des théorèmes qui correspondent du tout au tout aux deux théorèmes que nous avons désignés dans ce qui précède comme les théorèmes V et VI.

Nous touchons ici à un sujet qui demanderait des recherches ultérieures; j'espère pour ma part que j'aurai bientôt l'occasion d'y revenir.

Om en Udvidelse af en kendt Konvergenssætning.

Af Harald Bohr.

Ved Hjælp af den Abel'ske delvise Summationsformel har P. Du-Bois-Reymond*) bevist følgende Konvergenssætning:

Rækken $\sum\limits_0^\infty a_n \cdot b_n$ er konvergent, naar $\sum\limits_0^\infty a_n$ og $\sum\limits_1^\infty |b_n - b_{n-1}|$ er konvergente.

Det skal her vises, hvorledes denne Sætning kan udvides til følgende almindeligere:

Er $\sum\limits_0^\infty a_n$ konvergent og er

$$\begin{matrix} b_{0,0} & & & \\ b_{1,0} & b_{1,1} & & \\ \vdots & & & \\ b_{n,0} & b_{n,1} & \ldots & b_{n,n} \\ \vdots & \vdots & & \vdots \end{matrix}$$

en Samling komplekse Tal, der opfylder følgende 2 Betingelser:

1) $\lim\limits_{m \text{ konst. } n=\infty} b_{n,m} = b_m$ 2) $\sum\limits_{r=1}^{r=n} |b_{n,r} - b_{n,r-1}| < B$ (for alle n)

da har $a_0 b_{n,0} + a_1 b_{n,1} + \cdots + a_n \cdot b_{n,n}$ for $n = \infty$ en Grænseværdi lig $\sum\limits_0^\infty a_n \cdot b_n$.

Vi vil begynde med at vise, at $\sum\limits_0^\infty a_n \cdot b_n$ under de givne Forudsætninger er konvergent.

*) Antrittsprogramm, Freiburg 1871.

Lad os for Kortheds Skyld betegne

$|a_{m+1} \cdot b_{n,m+1} + a_{m+2} \cdot b_{n,m+2} + \cdots + a_{m+p} \cdot b_{n,m+p}|$ med $f(n, m, p)$.

Vi har da, idet $s_r = a_0 + a_1 + \cdots + a_r$, og idet vi benytter den Abel'ske Omskrivning:

$$f(n, m, p) = |a_{m+1} \cdot b_{n,m+1} + \cdots + a_{m+p} \cdot b_{n,m+p}| =$$
$$|(b_{n,m+1} - b_{n,m+2})(s_{m+1} - s_m) + \cdots +$$
$$(b_{n,m+p-1} - b_{n,m+p})(s_{m+p-1} - s_m) + b_{n,m+p}(s_{m+p} - s_m)| \leq$$
$$M_m(|b_{n,m+1} - b_{n,m+2}| + \cdots + |b_{n,m+p-1} - b_{n,m+p}| + |b_{n,m+p}|),$$

hvor M_m betegner Maksimum af

$$|s_{m+1} - s_m|, \ldots |s_{m+p} - s_m| \ldots \lim_{m=\infty} M_m = 0.$$

Af Ligningen

$$b_{n,m+p} = b_{n,0} + (b_{n,1} - b_{n,0}) + \cdots + (b_{n,m+p} - b_{n,m+p-1})$$

faas

$$|b_{n,m+p}| < B + |b_{n,0}|,$$

og følgelig

$f(n, m, p) \leq M_m(2B + |b_{n,0}|)$, hvor $|b_{n,0}|$ er endelig ($<$ Konst.) for alle n, da $\lim\limits_{n=\infty} b_{n,0} = b_0$.

Der findes derfor sikkert, naar ε er et vilkaarligt opgivet nok saa lille positivt Tal, et helt Tal m', saaledes at $f(n, m, p) < \varepsilon$, naar $m \geq m'$. For ethvert saadant m og et vilkaarligt p, faar man derfor ogsaa

$$\lim_{n=\infty} f(n, m, p) = |a_{m+1} \cdot b_{m+1} + \cdots + a_{m+p} \cdot b_{m+p}| \leq \varepsilon;$$

men dette udsiger netop, at Rækken $\sum\limits_0^\infty a_n \cdot b_n$ er konvergent. Lad os kalde dens Sum G.

Vi kan nu sikkert, naar ε atter betegner et vilkaarligt positivt Tal, finde et Tal m_1, saaledes at baade

$$f(n, m_1, p) < \frac{\varepsilon}{3} \text{ og } |G - (a_0 b_0 + a_1 b_1 + \cdots + a_{m_1} \cdot b_{m_1})| < \frac{\varepsilon}{3}.$$

Vi kan endvidere derefter, idet m_1 jo er et bestemt fastlagt Tal, finde et Tal N, saaledes at for $n \geq N$,

$$g(n) = |(a_0 b_0 + \cdots + a_{m_1} \cdot b_{m_1}) - (a_0 \cdot b_{n,0} + \cdots + a_{m_1} \cdot b_{n,m_1})| < \frac{\varepsilon}{3}.$$

For ethvert $n \geqq N$ haves nu

$$|G-(a_0 b_{n,0}+a_1 b_{n,1}+\cdots+a_n \cdot b_{n,n})| \leqq$$
$$|G-(a_0 b_0+a_1 b_1+\cdots+a_{m_1} \cdot b_{m_1})|+g(n)+$$
$$f(n, m_1, n-m_1)<3 \cdot \frac{\varepsilon}{3}=\varepsilon.$$

Altsaa

$$\lim_{n=\infty}(a_0 b_{n,0}+a_1 b_{n,1}+\cdots+a_n \cdot b_{n,n})=\sum_0^\infty a_n \cdot b_n$$

q. e. d.

Sættes $b_{n,m}=b_m$ for alle n, faas Du Bois-Reymonds Sætning.

Den ovenfor opstillede Grænseovergangssætning kan i flere Tilfælde anvendes med Fordel. Her skal vises, hvorledes man ved Hjælp af den umiddelbart kan bevise de 2 Hovedsætninger for Multiplikation af 2 uendelige Rækker, der henholdsvis skyldes Mertens og Cesàro.

Den Merten'ske Sætning hedder:

Er $U=\sum_0^\infty u_n$ konvergent og er $V=\sum_0^\infty v_n$ absolut konvergent $\left(\sum_0^\infty |v_n|=V_1\right)$ samt sættes

$$w_u=u_0 \cdot v_u+u_1 \cdot v_{u-1}+\cdots+u_u \cdot v_0,$$

da er $\sum_0^\infty w_n$ konvergent og lig $U \cdot V$.

Bevis:

$$W_n=w_0+w_1+\cdots+w_n=u_0 \cdot t_n+u_1 \cdot t_{n-1}+\cdots+u_n \cdot t_0,$$

idet $t_r=v_0+v_1+\cdots+v_r$.

Da nu

1) $\sum_0^\infty u_n$ konvergent; 2) $\lim_{r \text{ konst. } n=\infty} t_{n-r}=V$;

3) $|t_n-t_{n-1}|+\cdots+|t_1-t_0|=\sum_1^n |v_r| \leqq V_1$

faas:

$$\lim W_n=u_0 \cdot V+u_1 V+\cdots+u_n \cdot V+\cdots=U \cdot V$$

q. e. d.

Den Cesàro'ske Sætning hedder, idet der benyttes samme Betegnelser som ovenfor:

Er $\sum\limits_0^\infty u_n$ og $\sum\limits_0^\infty v_n$ konvergente, da er $\sum\limits_0^\infty w_n$ summabel med Værdien $U \cdot V$ d. v. s. da er

$$\lim \frac{1}{n}(W_0 + W_1 + \cdots + W_n) = U \cdot V.$$

Bevis: $W_n = u_0 t_n + u_1 t_{n-1} + \cdots u_n t_0$

$$\frac{1}{n}[W_0 + \cdots + W_n] = u_0 \frac{t_0 + t_1 + \cdots + t_n}{n} + u_1 \frac{t_0 + t_1 + \cdots + t_{n-1}}{n} + \cdots + u_n \frac{t_0}{n}.$$

Da nu

1) Σu_n konvergent; 2) $\lim\limits_{r = \text{konst.},\, n = \infty} \frac{t_0 + t_1 + \cdots + t_{n-r}}{n} = V$;

3) $\frac{|t_n| + |t_{n-1}| + \cdots + |t_1|}{n} < \text{Konst}$,

da Størrelsen har Grænseværdien $|V|$ for $n = \infty$ faas

$$\lim \frac{1}{n}(W_0 + W_1 + \cdots + W_n) = u_0 \cdot V + u_1 V + \cdots + u_n \cdot V + \cdots = U \cdot V$$

q. e. d.

Ved Undersøgelser af Produktrækken ordnet paa anden Maade end ved de 2 ovenfor omhandlede Sætninger, vil den her givne udvidede Konvergenssætning ligeledes kunne anvendes med Fordel.

Nogle Bemærkninger om formel Regning.

Af Harald Bohr.

Da Docent A. F. Andersen i sin Tid forsvarede for den filosofiske Doktorgrad sin udmærkede Afhandling „Studier over Cesàro's Summabilitetsmetode" — der, som den senere Litteratur om summable Rækker viser, har vakt megen Opmærksomhed og vundet stor Anerkendelse ude i Verden — diskuterede Docent Andersen og jeg (som Opponent ex auditorio) bl. a. nogle Spørgsmaal om Fordelene ved formel Regning med uendelige Rækkeudviklinger. De følgende Betragtninger tager deres Udgangspunkt i denne Diskussion. Jeg begynder med nogle almindelige orienterende Bemærkninger, gaar derefter over til at belyse de Spørgsmaal, Talen er om, ved et specielt typisk Eksempel — Udledelsen af den Møbius'ske Omvendingsformel — for sluttelig udførligt at behandle en af de interessante Sætninger i Docent Andersens Disputats omhandlende Regning med Differenser af brudden Orden.

1° Naar man vil studere en eller anden talteoretisk Funktion $a(n)$, altsaa en vis Talfølge $a(1)$, $a(2)$, $a(3), \cdots$, er det som bekendt i talrige Tilfælde bekvemt at anbringe Elementerne i Talfølgen som Koefficienter i en eller anden uendelig Rækkeudvikling, f. Eks. at danne Potensrækken

$$a(1)x+a(2)x^2+\cdots+a(n)x^n+\cdots$$

eller den Dirichletske Række

$$a(1)+\frac{a(2)}{2^x}+\cdots+\frac{a(n)}{n^x}+\cdots$$

eller lignende; hvilken Rækketype, man med størst Fordel kan benytte, afhænger naturligvis af Forholdene i det enkelte Tilfælde.

Ved mange Undersøgelser er den videre Fremgangsmaade da den, at man studerer den ved Rækken fremstillede Funktion $f(x)$, altsaa betragter x som en (reel eller kompleks) Variabel, og saa derefter ud fra de fundne Egenskaber ved denne Funktion $f(x)$ søger at slutte tilbage til Koefficienternes Egenskaber, f. Eks. deres asymptotiske Forhold. Et fundamentalt Eksempel paa Anvendelsen af denne Metode er i nyere Tid givet ved Hardy-Littlewood's berømte Undersøgelser indenfor den additive Talteori. Men ogsaa i saadanne Tilfælde, hvor den betragtede Række ikke konvergerer — eller paa simpel Maade kan summeres — for nogen Værdi af x, og hvor man altsaa ikke kan betragte x som en Variabel (et foranderligt Tal), men blot som en Parameter (et Bogstav, man regner med efter de sædvanlige elementære Regneregler), kan det være af stor Fordel og medføre betydelig Overskuelighed ved Udledelsen af Formler at betragte Rækkeudviklinger som de nævnte, selvom man jo ikke her til den enkelte Række kan knytte nogen Funktion $f(x)$ som Rækkens „Sum" og derfor er henvist til „formel" Regning med Rækkerne selv. Det er Spørgsmaal vedrørende saadanne formelle Regninger, jeg i det følgende skal omtale. Jeg begynder med at betragte et specielt typisk Eksempel, der tydeligt belyser Sagen.

2° Ved flere talteoretiske Undersøgelser træffer man paa den saakaldte Møbius'ske talteoretiske Funktion $\mu(n)$ defineret ved

$$\mu(n) = \begin{cases} 1 \text{ for } n = 1. \\ (-1)^m, \text{ hvis } n \text{ er et Produkt af } m \text{ forskellige Primtal.} \\ 0, \text{ hvis } n \text{ indeholder mindst et Primtal i højere end } 1^{\text{ste}} \\ \quad \text{Potens.} \end{cases}$$

Betydningen af denne Funktion hænger paa det nøjeste sammen med den saakaldte Møbius'ske Omvendingsformel, som udsiger, at hvis $a(n)$ er en eller anden talteoretisk Funktion og $b(n)$ den nye talteoretiske Funktion, der dannes udfra $a(n)$ ved Formlen

$$b(n) = \sum_{d/n} a(d), \tag{1}$$

hvor d/n under Summationstegnet angiver, at Summen skal udstrækkes over alle positive hele Divisorer d i n, da kan man

omvendt udfra $b(n)$ komme tilbage til $a(n)$ ved Formlen

$$(2) \qquad a(n) = \sum_{d|n}{}' \mu(d)\, b\left(\frac{n}{d}\right).$$

Denne Formel (eller, om man vil, disse uendelig mange Formler svarende til $n = 1, 2, \cdots$) kan man naturligvis uden større Vanskelighed verificere ved en direkte elementær Regning, men man faar en langt klarere Forstaaelse af den egentlige „Grund" til, at der gælder en saadan Formel, og sparer sig enhver kedelig Regning ved paa den ovenfor omtalte Maade at anbringe de talteoretiske Funktioner, Talen er om, som Koefficienter i en uendelig Rækkeudvikling hvortil her naturligt anvendes en Dirichletsk Række, da der her er Tale om Summer udstrakt over et Tals Divisorer, og saadanne Summer jo af sig selv optræder ved Multiplikation netop af Dirichletske Rækker.

Vi danner altsaa, udfra den givne talteoretiske Funktion $a(n)$, den Dirichletske Række $\sum' \frac{a(n)}{n^x}$ og kommer da umiddelbart ved formel Regning til den nye Dirichletske Række $\sum' \frac{b(n)}{n^x}$, hvis Koefficienter $b(n)$ er bestemt ved Formlen (1), ved at udføre Multiplikationen

$$(3) \qquad \sum{}' \frac{a(n)}{n^x} \cdot \sum{}' \frac{1}{n^x} = \sum{}' \frac{b(n)}{n^x}.$$

Vil vi nu omvendt komme fra Funktionen $b(n)$ tilbage til Funktionen $a(n)$, maa vi se at faa Rækken $\sum' \frac{b(n)}{n^x}$ udtrykt ved Rækken $\sum' \frac{a(n)}{n^x}$, altsaa se at faa Faktoren $\sum' \frac{1}{n^x}$ skaffet over paa den anden Side af Lighedstegnet i (3). Dette gøres ved først at omskrive Rækken $\sum' \frac{1}{n^x}$ til et uendeligt Produkt paa den bekendte Maade

$$\sum{}' \frac{1}{n^x} = \prod_p \left(1 + \frac{1}{p^x} + \frac{1}{p^{2x}} + \cdots\right) = \prod_p \frac{1}{1 - \frac{1}{p^x}},$$

hvor p gennemløber Primtallene $2, 3, 5, \cdots$. Idet det uendelige Produkt

$$\prod_p \left(1 - \frac{1}{p^x}\right)$$

ved formel Udregning jo netop giver den Dirichletske Række $\sum \frac{\mu(n)}{n^x}$ (denne Formel $\Pi\left(1-\frac{1}{p^x}\right) = \sum \frac{\mu(n)}{n^x}$ er forøvrigt den simpleste Maade at udtrykke selve Definitionen af den Møbius'ske Funktion paa), finder vi, naar vi i (3) dividerer paa begge Sider af Lighedstegnet med $\sum \frac{1}{n^x}$, Formlen

$$\sum \frac{a(n)}{n^x} = \sum \frac{b(n)}{n^x} \cdot \sum \frac{\mu(n)}{n^x}. \tag{4}$$

Denne Formel giver os, naar vi formelt udfører Multiplikationen paa højre Side af Lighedstegnet og derefter sætter tilsvarende Koefficienter paa begge Sider af Lighedstegnet lig med hinanden, netop den Møbius'ske Omvendingsformel (2).

Men, spørger vi os nu, med hvilken Ret kan vi betragte den ovenstaaende Udledning ved Hjælp af formelle Regninger med uendelige Rækker og Produkter som et virkeligt Bevis for den Møbius'ske Formel. Eller er der her slet ikke Tale om et Bevis, men blot om en heuristisk Betragtning, som vel kan være overmaade nyttig til at komme paa Sporet efter Formlen, men som bagefter maa komplementeres med et virkelig strengt Bevis som f. Eks. det direkte elementære Bevis, jeg ovenfor hentydede til. Naturligvis, hvis vi i vore Regninger kunde opfatte x som en Variabel (og ikke blot som en Parameter), og de Rækker, vi regnede med, for tilstrækkelig store Værdier af x var absolut konvergente, kunde vi let give den ovenstaaende formelle Udledelse en reel Gyldighed. Hvad de benyttede „Hjælperækker" $\zeta(x) = \sum \frac{1}{n^x}$ og $\frac{1}{\zeta(x)} = \sum \frac{\mu(n)}{n^x}$ angaar, er der intetsomhelst i Vejen; disse Rækker er absolut konvergente for $x > 1$. Men nu Rækken $\sum \frac{a(n)}{n^x}$? Hvis denne Række ogsaa konvergerede absolut for tilstrækkelig store Værdier af x (lad os sige for $x > x_0 > 1$) kunde vi umiddelbart slutte saaledes: For $x > x_0$ gælder (ikke blot formelt) Ligningen

$$\sum \frac{a(n)}{n^x} = \sum \frac{a(n)}{n^x} \cdot \zeta(x) \cdot \frac{1}{\zeta(x)} = \left(\sum \frac{a(n)}{n^x} \cdot \sum \frac{1}{n^x}\right) \cdot \sum \frac{\mu(n)}{n^x},$$

altsaa idet vi multiplicerer de to (absolut konvergente) Rækker indenfor Parentesen paa højre Side af Lighedstegnet

$$\sum \frac{a(n)}{n^x} = \sum \frac{b(n)}{n^x} \cdot \sum \frac{\mu(n)}{n^x},$$

og heraf følger da, ved fornyet Multiplikation af de to (ligeledes absolut konvergente) Rækker paa højre Side og paafølgende Koefficientsammenligning, den Møbius'ske Formel (2), idet vi benytter Entydighedssætningen for Dirichletske Rækker, der udsiger, at hvis to for tilstrækkelig store Værdier af x konvergente Dirichletske Rækker fremstiller den samme Funktion, da er de formelt identiske.

Men dette Bevis kan jo ikke opretholdes, hvis den givne Funktion $a(n)$ er saaledes beskaffen (f. Eks. $a_n = n!$), at Rækken $\sum \frac{a(n)}{n^x}$ ikke konvergerer for nogen, selv nok saa stor, Værdi af x. Og paa den anden Side ved vi jo, f. Eks. udfra den elementære Begrundelse af den Møbius'ske Omvendingsformel, at Gyldigheden af denne Formel ikke i mindste Maade afhænger af $a(n)$'s Størrelsesorden, men at den bestaar for enhver talteoretisk Funktion $a(n)$. Den ovenstaaende Maade at omforme den formelle Udledning til et virkeligt Bevis, kan derfor ikke betragtes som fyldestgørende eller naturlig; man maa forlange et Bevis, som i den Henseende svarer til Formlens Natur, at det slet ikke kommer ind paa Spørgsmaalet om $a(n)$'s Størrelsesorden, d. v. s. er ganske uafhængig af, om Rækken $\sum \frac{a(n)}{n^x}$ konvergerer eller divergerer. Og en Fortolkning af vor formelle Udledelse, der tilfredsstiller dette Krav, ligger da ogsaa lige for. Vi behøver blot at bemærke, at naar en Dirichletsk Række $\sum \frac{\gamma_n}{n^x}$ dannes ved formel Multiplikation af to andre Dirichletske Rækker $\sum \frac{\alpha_n}{n^x}$ og $\sum \frac{\beta_n}{n^x}$ bestemmes Koefficienterne γ_n i Produktrækken ved en Formel

$$\gamma_n = \sum_{d|n} \alpha_d \, \beta_{\frac{n}{d}},$$

hvor der for et fast n kun indgaar et endeligt Antal af Koefficienterne α_ν og β_ν. Det er da nemlig klart, at vi, naar vi skal bevise Formlen (2) for et fast $n = N$ simpelthen kan skære $\sum \frac{a(n)}{n^x}$ af ved $n = N$, d. v. s. kan nøjes med, i Stedet for hele

Rækken $\sum_{1}^{\infty}\frac{a(n)}{n^x}$, kun at betragte den endelige Række $\sum_{1}^{N}\frac{a(n)}{n^x}$. Og lad os tilføje, at hvis vi vilde gøre Udledelsen ganske elementær — og derved befri den for ethvert Skin af at have at gøre med uendelige Processer — kunde (og burde) vi ogsaa undlade at benytte os af den „tilfældige“ Konvergens af Rækkerne $\sum\frac{1}{n^x}$ og $\sum\frac{\mu(n)}{n^x}$ ved ogsaa, omend naturligvis kun ved den endelige Fortolkning af Resultaterne, at tænke os disse Rækker, og de tilsvarende Produkter, skaaret af paa passende Maade efter et endeligt Antal Led.

3° Jeg skal nu sluttelig gaa over til at omtale — og dette er som ovenfor sagt Udgangspunktet og det egentlige Emne for denne lille Note — hvorledes man ved at benytte formel Regning med Rækkeudviklinger i den ovennævnte Forstand kan modificere et (typisk) af de af Docent Andersen givne Beviser for Regneregler for Differenser af vilkaarlig Orden, saaledes at man i visse Henseender opnaar en større Overskuelighed og i hvert Fald sparer enhver detailleret Benyttelse af den nøjagtige Størrelsesorden af de i den omhandlede Differensregning indgaaende Binomialkoefficienter.

Vi begynder med nogle Betegnelser. Lad $\varepsilon_0, \varepsilon_1, \varepsilon_2, \cdots$ være en vilkaarlig Talfølge og $\Delta^1\varepsilon_\nu$ $(\nu = 0, 1, 2, \cdots)$ betegne Differensen $\varepsilon_\nu - \varepsilon_{\nu+1}$. Vi bemærker, at disse Differenser kan karakteriseres som Koefficienter til $\frac{1}{x^\nu}$ $(\nu = 0, 1, 2 \cdots)$ i Rækkeudviklingen

$$(1-x)\left(\varepsilon_0+\frac{\varepsilon_1}{x}+\frac{\varepsilon_2}{x^2}+\cdots\right) = -\varepsilon_0 x+\left(\Delta^1\varepsilon_0+\frac{\Delta^1\varepsilon_1}{x}+\frac{\Delta^1\varepsilon_2}{x^2}+\cdots\right).$$

Idet vi paa sædvanlig Maade betegner de itererede Differenser med $\Delta^2\varepsilon_\nu = \Delta^1(\Delta^1\varepsilon_\nu)$, $\Delta^3\varepsilon_\nu = \Delta^1(\Delta^2\varepsilon_\nu), \cdots$ finder vi umiddelbart, at Differensen $\Delta^r\varepsilon_\nu (\nu = 0, 1, 2, \cdots)$ for et vilkaarligt positivt helt r kan karakteriseres som Koefficient til $\frac{1}{x^\nu}$ i Rækkeudviklingen

$$(1-x)^r\left(\varepsilon_0+\frac{\varepsilon_1}{x}+\frac{\varepsilon_2}{x^2}+\cdots\right) = P(x)+\Delta^r\varepsilon_0+\frac{\Delta^r\varepsilon_1}{x}+\frac{\Delta^r\varepsilon_2}{x^2}+\cdots,$$

hvor $P(x)$ er en (iøvrigt endelig) Sum af Led med positive Eksponenter, hvis Koefficienter vi ikke interesserer os for. Her er foreløbig kun Tale om ren formel Regning, idet de enkelte

Koefficienter $\Delta^r \varepsilon_\nu$ jo dannes ved Kombination af kun et endeligt Antal ε^{er} og et endeligt Antal Binomialkoefficienter. Det ligger nu nær at definere „Differensen" $\Delta^r \varepsilon_\nu$ for et fuldkommen vilkaarligt reelt Tal r ved den tilsvarende Formel

$$\text{(I)} \quad (1-x)^r \left(\varepsilon_0 + \frac{\varepsilon_1}{x} + \frac{\varepsilon_2}{x^2} + \cdots\right) = \cdots - \Delta^r \varepsilon_0 + \frac{\Delta^r \varepsilon_1}{x} + \frac{\Delta^r \varepsilon_2}{x^2} + \cdots,$$

hvor Prikkerne før det konstante Led paa højre Side af Lighedstegnet angiver Leddene med positive Eksponenter, eller rettere tilkendegiver, at vi ikke interesserer os for disse Led og derfor slet ikke tænker os dem dannede. Idet vi til Afkortning betegner den (uendelige) Binomialrække for $(1-x)^r$ med $r_0 + r_1 x + r_2 x^2 + \cdots$, er den ovenstaaende Definition altsaa ensbetydende med, at vi har sat

$$\text{(II)} \qquad \Delta^r \varepsilon_\nu = \sum_{p=\nu}^{\infty} \varepsilon_p r_{p-\nu}$$

Dette er imidlertid en uendelig Række, og ved denne Definition maa der derfor i Følge Sagens Natur indføres Begrebet Grænseovergang. Vi vedtager, at vi da og kun da vil tillægge Symbolet $\Delta^r \varepsilon_\nu$ en Mening, naar Rækken paa højre Side af Lighedstegnet i (II) er konvergent i den angivne Orden af Leddene. Definitionen af $\Delta^r \varepsilon_\nu$ ved Ligningen (II) (med Forlangendet om Konvergens) er nøjagtig den af Andersen benyttede. Vi har her taget vort Udgangspunkt i Ligning (I) indeholdende Parameteren x for paa naturlig Maade at faa samtlige Differenser $\Delta^r \varepsilon_\nu$ ($\nu = 0, 1, 2, \cdots$) frem paa én Gang, nemlig som Koefficienter i en Rækkeudvikling dannet paa simpel Maade udfra en Rækkeudvikling med $\varepsilon^{\text{erne}}$ selv som Koefficienter. Den i Ligning (I) omhandlede Multiplikation er formel i den Forstand, at det ganske vist forlanges, at de ved Dannelsen af Koefficienterne paa højre Side optrædende Summer skal være konvergente, men ikke forlanges, at de to Potensrækker med Koefficienter ε_ν og $\Delta^r \varepsilon_\nu$ skal konvergere for noget x.

Eks. Vi nævner, efter Dr. Andersen, at Differenserne $\Delta^r \varepsilon_\nu$ ($\nu = 0, 1, 2, \cdots$) sikkert eksisterer i det Tilfælde, hvor $\varepsilon^{\text{erne}}$ er begrænsede, $|\varepsilon_\nu| < k$, og $r > 0$. Thi i dette Tilfælde er jo Rækken Σr_ν (Binomialrækken for $(1-x)^r$ i Punktet $x = 1$) konvergent, og da alle Leddene r_ν har samme Fortegn paa nær et

endeligt Antal, er Rækken (II) derfor ogsaa konvergent, endda absolut konvergent.

Dr. Andersen indfører endvidere, hvad han kalder de ved n afbrudte Differenser af r^{te} Orden $\Delta_n^r \varepsilon_\nu$ $(\nu = 0, 1, 2, \cdots n)$, hvor r ligesom før er et vilkaarligt reelt Tal, medens n er et positivt helt Tal; disse defineres som endelige Summer, nemlig som Afsnit i Rækken (II)

$$\text{(III)} \qquad \Delta_n^r \varepsilon_\nu = \sum_{p=\nu}^{n} \varepsilon_p \, r_{p-\nu}.$$

For vort Formaal er det bekvemt at bemærke, i Tilknytning til den ovenstaaende Definition (I) af selve Differenserne $\Delta^r \varepsilon_\nu$, at disse afbrudte Differenser kan defineres ved Ligningen

$$(1-x)^r \left(\varepsilon_0 + \cdots + \frac{\varepsilon_n}{x^n}\right) = \cdots + \left(\Delta_n^r \varepsilon_0 + \frac{\Delta_n^r \varepsilon_1}{x} + \cdots + \frac{\Delta_n^r \varepsilon_n}{x^n}\right).$$

De afbrudte Differenser eksisterer altsaa (som endelige Summer) ligegyldigt, hvilke Tal $\varepsilon^{\text{erne}}$ end er. Vi tilføjer af Hensyn til det følgende, at hvis vi kun interesserer os for Differensen $\Delta_n^r \varepsilon_0$, kan vi naturligvis ligesaavel skære Binomialrækken af efter Leddet med x^n i Stedet for at skære „ε-Rækken" af efter Leddet med $\frac{1}{x^n}$, d. v. s. vi kan ligesaavel karakterisere $\Delta_n^r \varepsilon_0$ som det konstante Led i Rækkeudviklingen

$$(r_0 + r_1 x + \cdots + r_n x_n)\left(\varepsilon_0 + \frac{\varepsilon_1}{x} + \cdots\right) = \cdots + \Delta_n^r \varepsilon_0 + \cdots.$$

Det er klart udfra Definitionerne (II) og (III), at $\Delta^r \varepsilon_\nu$ da og kun da eksisterer, naar den ved n afbrudte Differens $\Delta_n^r \varepsilon_\nu$ for $n \to \infty$ nærmer sig til en bestemt Grænseværdi, og at denne Grænseværdi da netop bestemmer os $\Delta^r \varepsilon_\nu$.

Den omhandlede Sætning af Dr. Andersen udsiger nu:

Hvis $|\varepsilon_\nu| < k$ $(\nu = 0, 1, 2, \cdots)$, vil der gælde Ligningen

$$\text{(A)} \qquad \Delta^s \Delta^r \varepsilon_\nu = \Delta^{r+s} \varepsilon_\nu,$$

dersom $r > 0$, $s > -1$ og $r+s > 0$, d. v. s. de angivne Differenser eksisterer og er ligestore.

[Det interessante ved denne Sætning ligger ikke mindst deri, at den, som vist af Dr. Andersen, i en vis nærmere præciseret Forstand er den „bedst mulige" i sin Slags].

Medens Dr. Andersen i sit direkte og i og for sig meget instruktive Bevis for denne Sætning arbejder med selve Definitionen (II) for den enkelte Differens $\Delta^r \varepsilon_\nu$ (og med den tilsvarende Definition (III) for den afbrudte Differens $\Delta^r_n \varepsilon_\nu$)*), skal jeg nedenfor i mit modificerede Bevis benytte Definitionen af Differenserne ved Hjælp af Rækkeudviklingen (I) og udnytte Fordelene ved at have en Parameter x indgaaende i Regningerne; derved opnaaes, at man ikke behøver at benytte andet om de optrædende Binomialkoefficienter end netop det, at de er Udviklingskoefficienter i de uendelige Rækkeudviklinger for Potenser af $1-x$.

Bevis. Til Afkortning sættes (formelt)

$$\varepsilon_0 + \frac{\varepsilon_1}{x} + \frac{\varepsilon_2}{x^2} + \cdots = E(x)$$

samt

$$(1-x)^r = r_0 + r_1 x + r_2 x^2 + \cdots = R(x)$$
$$(1-x)^s = s_0 + s_1 x + s_2 x^2 + \cdots = S(x)$$
$$(1-x)^t = t_0 + t_1 x + t_2 x^2 + \cdots = T(x) \qquad (t = r+s);$$

endvidere betegner vi med $S_n(x)$ den ved x^n afbrudte Rækkeudvikling for $(1-x)^s$

$$S_n(x) = s_0 + s_1 x + \cdots + s_n x^n .$$

Det eneste angaaende Binomialkoefficienterne r_ν, s_ν, t_ν, vi kommer til at benytte, er de simple Kendsgerninger: 1) Koefficienterne i hver af de tre angivne Rækker har alle samme Fortegn fra et vist Trin af, lad os sige for $\nu > c$. 2) Rækkerne for $(1-x)^r$ og $(1-x)^t$ er begge konvergente, og altsaa absolut konvergente, i selve Punktet 1 med Summen 0, idet r og $t = r+s$ i Følge Antagelse begge er >0. 3) Rækken for $(1-x)^s$ behøver vel ikke at være konvergent i Punktet 1 (den er divergent for $s<0$), men det gælder om den, at dens Koefficienter s_ν aftager mod 0 for $\nu \to \infty$, fordi $s > -1$.

Ved Beviset for Ligningen (A) kan vi naturligvis nøjes med at betragte Tilfældet $\nu = 0$, d. v. s. nøjes med at bevise Ligningen

$$\Delta^s \Delta^r \varepsilon_0 = \Delta^t \varepsilon_0 ;$$

men ved Dannelsen af $\Delta^s(\Delta^r \varepsilon_0)$ kommer vi dog undervejs til at betragte alle de uendelig mange Differenser af r^{te} Orden $\Delta^r \varepsilon_\nu$

*) I et nylig udkommet Arbejde „Comparison theorems in the theory of Cesàro summability" (Proc. Lond. Math. Soc. (2) vol. 27 pag. 39–71, se særlig pag. 58–60) har Dr. Andersen givet sit oprindelige Bevis for en lignende Sætning en simplere Form, men han betjener sig ogsaa dér af en direkte Bevismetode.

($\nu = 0, 1, 2, \cdots$). Idet $r > 0$ og $t > 0$ samt $|\varepsilon_\nu| < k$ ved vi (se Eksemplet ovenfor), at Differenserne $\Delta^r \varepsilon_\nu$ og $\Delta^t \varepsilon_\nu$ alle eksisterer, og at disse Differenser er Koefficienter i Rækkeudviklingerne

$$E(x)\,R(x) = \cdots + \Delta^r \varepsilon_0 + \frac{\Delta^r \varepsilon_1}{x} + \frac{\Delta^r \varepsilon_2}{x^2} + \cdots$$

og

$$E(x)\,T(x) = \cdots + \Delta^t \varepsilon_0 + \frac{\Delta^t \varepsilon_1}{x} + \frac{\Delta^t \varepsilon_2}{x^2} + \cdots.$$

Det gælder om at vise, at $\Delta^s \Delta^r \varepsilon_0$ eksisterer og er lig med $\Delta^t \varepsilon_0$, altsaa at den ved n afbrudte Differens $\Delta^s_n \Delta^r \varepsilon_0 \to \Delta^t \varepsilon_0$ for $n \to \infty$. Ved vor formelle Regning er imidlertid $\Delta^s_n \Delta^r \varepsilon_0$ det konstante Led i Rækkeudviklingen

$$\left(\Delta^r \varepsilon_0 + \frac{\Delta^r \varepsilon_1}{x} + \frac{\Delta^r \varepsilon_2}{x^2} + \cdots\right) S_n(x)$$

eller, hvad der i Følge det ovenstaaende er det samme, det konstante Led i Rækkeudviklingen for $E(x)\,R(x) \cdot S_n(x)$. Idet endvidere $\Delta^t \varepsilon_0$ er det konstante Led i Udviklingen for $E(x)\,T(x)$, bliver Differensen $\Delta^s_n \Delta^r \varepsilon_0 - \Delta^t \varepsilon_0$ det konstante Led i Rækkeudviklingen for

$$E(x)\,R(x)\,S_n(x) - E(x)\,T(x) = E(x)\,(R(x)\,S_n(x) - T(x));$$

her har vi ved Udregningen af de forskellige konstante Led kun udført absolut konvergente Regninger, fordi $\sum |r_\nu|$ og $\sum |t_\nu|$ er konvergente (og $|\varepsilon_\nu| < k$), medens $S_n(x)$ jo kun indeholder et endeligt Antal Led. Lad os betegne Rækkeudviklingen for $R(x) \cdot S_n(x)$ med

$$R(x)\,S_n(x) = \tau_0 + \tau_1 x + \tau_2 x^2 + \cdots.$$

Idet $t = r + s$ og $S_n(x)$ er Binomialudviklingen for $(1-x)^s$ afbrudt efter Leddet med x^n, er det klart, at $\tau_\nu = t_\nu$ for $0 \leqq \nu \leqq n$. Vi finder altsaa

$$\text{(IV)} \qquad R(x)\,S_n(x) - T(x) = \sum_{\nu = n+1}^{\infty} (\tau_\nu - t_\nu) x^\nu.$$

Idet endvidere Rækkeudviklingerne for $(1-x)^r$ og $(1-x)^t$ begge er konvergente ogsaa i Punktet $x = 1$ med Summerne $R(1) = T(1) = 0$, er det klart, at denne sidste Rækkeudvikling (IV) ligeledes er (absolut) konvergent i Punktet $x = 1$ med Summen 0, altsaa at

$$\sum_{\nu=n+1}^{\infty}(\tau_\nu - t_\nu) = 0.$$

Nu ved vi imidlertid, at $\sum\limits_{\nu=n+1}^{\infty} t_\nu$ (Restleddet i en konvergent Række) $\to 0$ for $n \to \infty$; følgelig vil ogsaa $\sum\limits_{\nu=n+1}^{\infty} \tau_\nu \to 0$ for $n \to \infty$ [Vi tilføjer for at undgaa Misforstaaelse, at $\tau_\nu (\nu > n)$ i Modsætning til t_ν ikke blot afhænger af ν, men ogsaa af n, fordi den ene Faktor $S_n(x)$ i Produktet $R(x)\,S_n(x)$ jo er en ved n afbrudt Binomialrække]. Den Størrelse $\Delta_n^s \Delta^r \varepsilon_0 - \Delta^t \varepsilon_0$, vi skal undersøge, er som ovenfor angivet lig med det konstante Led i Rækkeudviklingen for $E(x)\,(R(x)\,S_n(x) - T(x))$, altsaa i Følge (IV) lig med

$$\sum_{\nu=0}^{\infty} \varepsilon_\nu (\tau_\nu - t_\nu) = \sum_{\nu=n+1}^{\infty} \varepsilon_\nu (\tau_\nu - t_\nu).$$

Idet $|\varepsilon_\nu| < k$ og $\sum\limits_{\nu=n+1}^{\infty} |t_\nu| \to 0$ er det klart, at $\sum\limits_{\nu=n+1}^{\infty} \varepsilon_\nu t_\nu \to 0$. Hvad vi har at bevise, er altsaa blot, at $\sum\limits_{\nu=n+1}^{\infty} \varepsilon_\nu \tau_\nu \to 0$, og dette er øjensynlig bevist, naar vi har vist, at

(V) $$\sum_{\nu=n+1}^{\infty} |\tau_\nu| \to 0 \quad \text{for } n \to \infty.$$

Grænseligningen (V) er imidlertid let at bevise udfra det ovenfor fundne Resultat

$$\sum_{\nu=n+1}^{\infty} \tau_\nu \to 0 \quad \text{for } n \to \infty,$$

idet vi let kan vise, at Størrelsen

(VI) $$\sum_{\nu=n+1}^{\infty} |\tau_\nu| - \left| \sum_{\nu=n+1}^{\infty} \tau_\nu \right|$$

gaar mod 0 for $n \to \infty$, ved at benytte, at alle Leddene i enhver af de to uendelige Rækker Σr_ν og Σs_ν har konstant Fortegn for $\nu > c$. Den (absolut konvergente) Proces, der fører til $\sum\limits_{\nu=n+1}^{\infty} \tau_\nu$ bestaar jo nemlig deri, at vi (i en eller anden Orden) danner Summen af alle Produkter $r_\alpha s_\beta$ for hvilke

$$\alpha+\beta>n \quad \text{og} \quad \beta \leqq n.$$

Disse Produkter har imidlertid „næsten alle“ samme Fortegn, idet r_α jo har konstant Fortegn for $\alpha>c$, ligesom s_β har konstant Fortegn for $\beta>c$. Det samlede Bidrag fra de Produkter $r_\alpha s_\beta$ der har „ureglementeret“ Fortegn — og dermed ogsaa Størrelsen (VI), eller rettere Halvdelen af denne Størrelse — er derfor numerisk mindre end

$$\sum_{\beta=1}^{c} | s_\beta | \cdot \sum_{\alpha=n-c}^{\infty} | r_\alpha | + \sum_{\alpha=1}^{c} | r_\alpha | \cdot \sum_{\beta=n-c}^{n} | s_\beta | .$$

Men denne sidste Størrelse gaar øjensynlig mod 0 for $n \to \infty$, idet $\sum_{\beta=1}^{c} | s_\beta |$ og $\sum_{\alpha=1}^{c} | r_\alpha |$ er Konstanter (uafhængige af n), medens $\sum_{\alpha=n-c}^{\infty} | r_\alpha | \to 0$, fordi det er Restleddet i en konvergent Række, og $\sum_{\beta=n-c}^{n} | s_\beta | \to 0$, fordi det er en Sum af et endeligt Antal (c) Led, hvoraf hver enkelt gaar mod 0, idet jo $s_\nu \to 0$ for $\nu \to \infty$.

THE ARITHMETIC AND GEOMETRIC MEANS

HARALD BOHR*.

Let n be a positive integer and $a_1, \ldots, a_n$ positive numbers; then the famous theorem of the arithmetic and geometric means states that

$$\frac{a_1+\ldots+a_n}{n} \geqslant \sqrt[n]{(a_1 \ldots a_n)},$$

i.e.

$$(a_1+\ldots+a_n)^n \geqslant n^n a_1 \ldots a_n.$$

Of this theorem a number of different proofs are known [*cf.* Hardy, Littlewood, and Pólya, *Inequalities* (Cambridge, 1934)]. In this note I shall give a new one. Even if this proof is more curious than simple, and moreover does not show that the sign of equality holds only in the case $a_1 = \ldots = a_n$, it may perhaps be worth while to indicate one more way of arriving at the classical inequality in question.

It is trivial that

$$(a_1+ \ldots +a_n)^n \geqslant n!\, a_1 \ldots a_n,$$

since the product on the right side is just one of the terms of the polynomial development of the left side. We have to prove that the factor $n!$ may be replaced by n^n. This can be done by help of the following artifice.

Let $q > 1$ be another integer. From the polynomial development we have immediately

$$(a_1+\ldots+a_n)^{qn} \geqslant \frac{(qn)!}{(q!)^n} a_1{}^q \ldots a_n{}^q.$$

But†

$$\frac{(qn)!}{(q!)^n} > \frac{n^{qn}}{(qn)^n},$$

and hence

$$(a_1+\ldots+a_n)^{qn} > \frac{n^{qn}}{(qn)^n} a_1{}^q \ldots a_n{}^q = \frac{(n^n a_1 \ldots a_n)^q}{(qn)^n},$$

i.e.

$$(a_1+\ldots+a_n)^n > \frac{n^n a_1 \ldots a_n}{(qn)^{n/q}}.$$

Now let $q \to \infty$. Then $(qn)^{n/q} \to 1$, and we get the desired result.

Maglevaenget 9,
Charlottenlund, Denmark.

* Received 12 February, 1935; read 14 February, 1935.

† In fact the numerator $(qn)!$ is evidently greater than $\{(q-1)!\, n^{q-1}\}^n$, since each of the n products $k(k+n)\ldots\{k+(q-1)n\}$ $[k=1, \ldots, n]$ is greater than

$$1.n.2n \ldots (q-1)n = (q-1)!\, n^{q-1}.$$

Om almindelige Konvergenskriterier for Rækker med positive Led.

Af Harald Bohr.

Allerede i Professor Nørlunds helt unge Aar, da han som Student var knyttet til og boede paa Observatoriet, var hans videnskabelige Interesser sjældent omfattende, og ved udstrakte og dybtgaaende Studier lagde han Grunden til sit senere fremragende og mangesidige Livsværk. Blandt den unge Student Nørlunds Undersøgelser indenfor den rene Matematik indtog hans Studier indenfor Rækkelæren en fremskudt Plads, og paa dette Omraade kom vi i nær videnskabelig Berøring med hinanden, idet den af Nørlund udviklede Transformationsteori for Fakultetsrækker viste sig at staa i nøje Sammenhæng med en af mig udviklet Summabilitetsteori for de med Fakultetsrækkerne beslægtede Dirichletske Rækker. Ved for nylig at gennemse nogle gamle Papirer stødte jeg paa et ret omfattende, ikke offentliggjort, Manuskript fra disse unge Aar, som jeg havde benævnet „Om Rækker med vilkaarlige Indices“, og som netop havde sit Udgangspunkt i de førnævnte Studier over Dirichletske Rækker. I det følgende skal jeg give et lille Uddrag af et enkelt af Manuskriptets Kapitler, omhandlende saakaldte almindelige Konvergens- og Divergenskriterier for Rækker med positive Led, idet den paagældende Undersøgelse formentlig endnu kan frembyde nogen Interesse og maaske tillige kan være egnet til at genkalde Stemninger fra de svundne Dage. I min Fremstilling skal jeg holde mig nær opad det gamle Manuskript og bl. a. ogsaa benytte Betegnelserne derfra, selvom disse maaske nok nu kan virke lidt selvlavede og antikverede.

Det var den nøje Sammenhæng mellem Læren om uendelige Rækker og uendelige Integraler, der førte mig til (først og fremmest med Henblik paa Studiet af de almindelige Dirichletske Rækker af Formen $\Sigma a_n e^{x_n s}$, hvor $x_1 < x_2 \cdots < x_n \to \infty$) at opstille Begrebet uendelig Række med vilkaarlige Indices, der som specielt Tilfælde omfatter de sædvanlige uendelige Rækker $\sum\limits_1^\infty u_n$ og som Grænsetilfælde de uendelige Integraler $\int_1^\infty u(x)\,dx$.

Svarende til de til „Indexfølgen“ 0, 1, 2, 3, $\cdots$, og en dertil knyttet Funktion u_ν, hørende Begreber

Dedicated to N. E. Nørlund.

$$\sum_{\nu=1}^{n} u_\nu, \quad u_\nu^{(1)} = u_\nu - u_{\nu-1} \quad \text{og} \quad \prod_{\nu=1}^{n} u_\nu, \quad {}^{(1)}u_\nu = \frac{u_\nu}{u_{\nu-1}},$$

forbundet ved Relationerne

$$\sum_{\nu=1}^{n} u_\nu^{(1)} = u_n - u_0 \quad \text{og} \quad \prod_{\nu=1}^{n} {}^{(1)}u_\nu = \frac{u_n}{u_0},$$

indføres svarende til en vilkaarlig monotont voksende Indexfølge $x_0 < x_1 < x_2 \cdots$, og en dertil knyttet Funktion u_{x_ν}, følgende Begreber:

„Summen af Funktionen u fra x_0 til x_n m. H. t. $x_0, x_1, \cdots, x_n$“, nemlig

$$S_{(x_0, x_1, \cdots, x_n)}^{u} = u_{x_1}(x_1 - x_0) + u_{x_2}(x_2 - x_1) + \cdots + u_{x_n}(x_n - x_{n-1}),$$

og „Differenskvotienten af u i Punktet x_ν“, nemlig

$$u_{x_\nu}^{(1)} = \frac{u_{x_\nu} - u_{x_{\nu-1}}}{x_\nu - x_{\nu-1}},$$

forbundet ved Relationen

$$S_{(x_0, x_1, \cdots, x_n)}^{u^{(1)}} = u_{x_n} - u_{x_0},$$

samt, idet u antages at være en positiv Funktion,

„Produktet af Funktionen u fra x_0 til x_n m. H. t. $x_0, x_1, \cdots, x_n$“, nemlig

$$P_{(x_0, x_1, \cdots, x_n)}^{u} = u_{x_1}{}^{(x_1 - x_0)} \cdots u_{x_n}{}^{(x_n - x_{n-1})},$$

og „Kvotientroden af u i Punktet x_ν“, nemlig

$${}^{(1)}u_{x_\nu} = \sqrt[x_\nu - x_{\nu-1}]{\frac{u_{x_\nu}}{u_{x_{\nu-1}}}},$$

forbundet ved Relationen

$$P_{(x_0, x_1, \cdots, x_n)}^{{}^{(1)}u} = \frac{u_{x_n}}{u_{x_0}}.$$

Herefter indføres naturligt (hvad enten den monotone Følge x_n gaar mod en endelig Grænseværdi eller mod ∞) Begrebet *uendelig Række med vilkaarlige Indices* $x_0, x_1, \cdots$

$$S_{(x_0, x_1, \cdots)}^{u} = \lim_{n \to \infty} S_{(x_0, x_1, \cdots, x_n)}^{u}$$

(forudsat at Grænseværdien eksisterer), hvor det afgørende er, at Summanden $u_{x_n}(x_n - x_{n-1})$ ikke opfattes som det n^{te} Led i en sædvanlig uendelig Række, men at u_{x_n} opfattes som en Funktion af x_n (og ikke af n), og Faktoren $x_n - x_{n-1}$ som en Summationsfaktor (Intervallængde). Paa tilsvarende Maade defineres, for en positiv Funktion u, det uendelige Produkt med vilkaarlige Indices

$$P^{\,u}_{(x_0,\, x_1, \cdots)} = \lim_{n \to \infty} P^{\,u}_{(x_0,\, x_1, \cdots,\, x_n)}.$$

Jeg skal nu ved et typisk Eksempel vise den Nytte, man kan have af disse Begreber indenfor den sædvanlige Rækkelære, og vælger hertil den Brug, man kan gøre af dem til Udledelse af saakaldte almindelige Konvergens- og Divergenskriterier.

Som „almindelige" Konvergens- og Divergenskriterier for Rækker $\sum\limits_{1}^{\infty} v_n$ med positive Led — i Modsætning til „specielle" — betegner man Kriterier, hvori der indgaar „vilkaarlige Funktioner" paa en saadan Maade, at man ved Specialisering af disse kan gøre Kriterierne saa omfattende, som man vil, i den Forstand, at man kan gøre dem principielt anvendelige til Bedømmelse af Konvergensen eller Divergensen af en hvilkensomhelst forelagt Række med positive Led. Saadanne almindelige Kriterier var allerede tidligt angivet af forskellige Matematikere, som Kummer, Dini og du Bois Reymond, men det var Pringsheims Undersøgelser paa dette Omraade[1]), der særlig havde vakt Opmærksomhed og Interesse, idet han mente, at det var lykkedes ham — i Modsætning til tidligere Forskere — at udvikle en Teori, der i systematisk Henseende maatte betragtes som fuldendt. I det Kapitel af det gamle Manuskript, som jeg her skal give et Uddrag af, viser jeg, hvorledes man, naar man lægger Rækkerne med vilkaarlige Indices til Grund, kan komme til de omhandlede almindelige Kriterier paa en, som det forekommer mig, særlig naturlig Maade, og tillige paaviser jeg, hvorledes Pringsheims Udledelsesmaade lider af væsentlige Mangler, ikke mindst netop i systematisk Henseende.

Den Ide, der ligger til Grund for den i Manuskriptet foreslaaede Metode til Udledelse af almindelige Kriterier for Rækker med positive Led, er følgende: Medens man, naar Talen er om

[1]) A. Pringsheim, Allgemeine Theorie der Divergenz und Convergenz von Reihen mit positiven Gliedern, Mathematische Annalen Bd. 35, 1890.

et uendeligt Integral $\int^{\infty} u(x)\,dx$ $(u(x) > 0)$, ikke har nogen egentlig Anledning til at skelne mellem „specielle" og „almindelige" Konvergens- og Divergenskriterier, idet man ved blot at anvende en monoton Transformation $x = \varphi(t)$ paa Integralet $\int^{\infty} u(x)\,dx$ kan gøre Konvergensen (eller Divergensen) af det transformerede Integral $\int^{\infty} u(\varphi(t))\,\varphi'(t)\,dt$ vilkaarlig stærk eller svag, stiller Sagen sig anderledes for en sædvanlig uendelig Række Σu_n, der i sin Stivhed jo ikke tillader nogen saadan monoton Transformation. Men betragter man Rækker med vilkaarlige Indices, er Muligheden for en monoton Transformation til Ændring af Konvergensens (eller Divergensens) Styrke naturligvis umiddelbart til Stede, ganske som ved Integralerne, idet man jo kan transformere en forelagt Række $S_{(x_0, x_1, \cdots)}^{u}$ hørende til Indexfølgen $x_0, x_1, x_2, \cdots$ over i en Række $S_{(y_0, y_1, \cdots)}^{v}$ hørende til en vilkaarlig anden Indexfølge $y_0, y_1, y_2, \cdots$, hvor Funktionen v_{y_n} da er bestemt ved

$$v_{y_n} = u_{x_n} \frac{x_n - x_{n-1}}{y_n - y_{n-1}}.$$

Man kan derfor for Rækker med vilkaarlige Indices, ganske som ved Integralerne, umiddelbart faa et „specielt" Kriterium omdannet til et „almindeligt" Kriterium ved blot at anvende det specielle Kriterium, ikke paa selve den forelagte Række, men paa den Række, der fremkommer af denne ved at underkaste den en vilkaarlig monoton Transformation.

Inden jeg gaar over til at belyse denne Fremgangsmaade ved nogle enkelte typiske Eksempler, forudskikkes nogle Ord om Konvergens- og Divergenskriterier for sædvanlige uendelige Rækker med positive Led. Man skelner sædvanligvis mellem Kriterier af 1. Art og Kriterier af 2. Art. Begge Typer af Kriterier bygger paa en simpel Sammenligning mellem Leddene i den til Bedømmelse forelagte Række Σu_n med de tilsvarende Led i en fast valgt konvergent Række Σc_n, eller en fast valgt divergent Række Σd_n. Kriteriernes Form er simpelthen:

1. Art.	$u_n \leqq c_n$ (Kvg.)	$u_n \geqq d_n$ (Dvg.),
2. Art.	$\dfrac{u_n}{u_{n-1}} \leqq \dfrac{c_n}{c_{n-1}}$ (Kvg.)	$\dfrac{u_n}{u_{n-1}} \geqq \dfrac{d_n}{d_{n-1}}$ (Dvg.).

Benyttes som konvergent Række Σc_n og som divergent Række Σd_n f. Eks. en Kvotientrække Σa^n henholdsvis med Kvotienten $a < 1$ og Kvotienten $a \geqq 1$, finder vi saaledes:

1. Art. $\sqrt[n]{u_n} \begin{cases} \leqq a < 1 & \text{(Kvg.)} \\ \geqq 1 & \text{(Dvg.)}, \end{cases}$

2. Art. $\dfrac{u_n}{u_{n-1}} \begin{cases} \leqq a < 1 & \text{(Kvg.)} \\ \geqq 1 & \text{(Dvg.)}. \end{cases}$

Ganske svarende til Kriterierne for en sædvanlig uendelig Række Σu_n (altsaa en Række hørende til Indexfølgen $0, 1, 2, \cdots$) har man for en Række $S^{\,u}_{(x_0, x_1, \cdots)}$ med positive Led, som hører til en vilkaarlig Indexfølge $x_0 < x_1 < x_2 \cdots$, følgende Kriterier, hvor c_{x_n} og d_{x_n} betegner positive Tal, saaledes at $S^{\,c}_{(x_0, x_1, \cdots)}$ og $S^{\,d}_{(x_0, x_1, \cdots)}$ er henholdsvis konvergent og divergent,

1. Art. $u_{x_n} \leqq c_{x_n}$ (Kvg.) $\quad u_{x_n} \geqq d_{x_n}$ (Dvg.),

2. Art. ${}^{(1)}u_{x_n} \leqq {}^{(1)}c_{x_n}$ (Kvg.) $\quad {}^{(1)}u_{x_n} \geqq {}^{(1)}d_{x_n}$ (Dvg.),

af hvilke Kriterierne af 1. Art er indlysende, og f. Eks. Konvergenskriteriet af 2. Art umiddelbart følger af, at Uligheden ${}^{(1)}u_{x_n} \leqq {}^{(1)}c_{x_n}$ jo medfører Uligheden

$$\frac{u_{x_n}}{u_{x_0}} = P^{\;{}^{(1)}u}_{(x_0, \ldots, x_n)} \leqq P^{\;{}^{(1)}c}_{(x_0, \ldots, x_n)} = \frac{c_{x_n}}{c_{x_0}}.$$

Vi skal nu først opstille nogle „specielle“ Kriterier for en Række $S^{\,u}_{(x_0, x_1, \cdots)}$ hørende til den (vilkaarlig valgte, men faste) Indexfølge $(0 <) x_0 < x_1 \cdots$, derved at vi til vore Sammenligningsrækker $S^{\,c}_{(x_0, x_1, \cdots)}$ og $S^{\,d}_{(x_0, x_1, \cdots)}$ vælger nogle særlig simple Typer paa henholdsvis konvergente og divergente Rækker. Som det simplest mulige betragter vi (svarende til Kvotientrækken Σa^n for Indexfølgen $0, 1, 2, \cdots$) Rækketypen $S^{\,a^x}_{(x_0, x_1, \cdots)}$, der er konvergent for $a < 1$ (hvad der er indlysende, hvis $\lim x_n$ er $< \infty$, og en umiddelbar Følge af Konvergensen af $\int^{\infty} a^x\,dx$, dersom $\lim x_n = \infty$), medens den, dog kun saafremt $x_n \to \infty$, er divergent for $a \geqq 1$ (idet jo $a^x \geqq 1$ for $x > 0$). Herved faas da de *specielle Kriterier* for en Række $S^{\,u}_{(x_0, x_1, \cdots)}$ med positive Led

1. Art. $u_{x_n} \leqq a^{x_n} (a < 1)$ (Kvg.) $u_{x_n} \geqq a^{x_n} (a \geqq 1,\ x_n \to \infty)$ (Dvg.)

eller

$$\sqrt[x_n]{u_{x_n}} \begin{cases} \leqq a < 1 \text{ (Kvg.)} \\ \geqq 1 \ (x_n \to \infty) \text{ (Dvg.)}, \end{cases}$$

2. Art. ${}^{(1)}u_{x_n} \leqq {}^{(1)}a^{x_n} = a < 1$ (Kvg.)

$${}^{(1)}u_{x_n} \geqq {}^{(1)}a^{x_n} = a \geqq 1 \ (x_n \to \infty) \text{ (Dvg.)}$$

d. v. s.

$$\sqrt[x_n - x_{n-1}]{\frac{u_{x_n}}{u_{x_{n-1}}}} \begin{cases} \leqq a < 1 \text{ (Kvg.)} \\ \geqq 1 \ (x_n \to \infty) \text{ (Dvg.)}. \end{cases}$$

Vi kommer nu til det afgørende Punkt, nemlig Overførelsen af de fundne specielle Kriterier til almindelige Kriterier ved blot, før Kriteriernes Anvendelse, at transformere den forelagte Række $S_{(x_0, x_1, \cdots)}^{u}$ fra Indexfølgen $x_0, x_1, \cdots$ over til en anden Indexfølge $y_0, y_1, \cdots$. Vi vil dog kun — andet har vi i den foreliggende Sammenhæng ikke Brug for — udføre dette i det Tilfælde, hvor den forelagte Række er en sædvanlig uendelig Række Σv_n og ikke en vilkaarlig Række $S_{(x_0, x_1, \cdots)}^{u}$; vi antager med andre Ord, at den givne Indexfølge specielt er Indexfølgen $0, 1, 2, \cdots$, og vil da, for Simpelheds Skyld, betegne den nye Indexfølge, hvortil vi transformerer, med $x_0, x_1, \cdots$ og ikke med $y_0, y_1, \cdots$. Er Σv_n den vilkaarligt forelagte sædvanlige uendelige Række med positive Led, hvis Konvergens resp. Divergens vi skal undersøge, transformerer vi den altsaa om til en Række $S_{(x_0, x_1, \cdots)}^{u}$, hvorved

$$u_{x_n} = \frac{v_n}{x_n - x_{n-1}}.$$

Anvender vi nu paa den transformerede Række $S_{(x_0, x_1, \cdots)}^{u}$ de ovenstaaende simple specielle Kriterier, faar vi da straks de følgende *almindelige Kriterier* for Rækken Σv_n

1. Art.
$$\sqrt[x_n]{\frac{v_n}{x_n - x_{n-1}}} \begin{cases} \leqq a < 1 \text{ (Kvg.)} \\ \geqq 1 \ (x_n \to \infty) \text{ (Dvg.)}, \end{cases}$$

2. Art.
$$\sqrt[x_n - x_{n-1}]{\frac{v_n}{v_{n-1}} \cdot \frac{x_{n-1} - x_{n-2}}{x_n - x_{n-1}}} \begin{cases} \leqq a < 1 \text{ (Kvg.)} \\ \geqq 1 \ (x_n \to \infty) \text{ (Dvg.)}. \end{cases}$$

Grunden til, at disse Kriterier er „almindelige" er — for at fremhæve, hvad der allerede er antydet ovenfor — at den i Kriterierne indgaaende monotone Følge $x_0, x_1, \cdots$ ikke er forud bestemt, men kan vælges vilkaarligt, efter at Rækken Σv_n er forelagt.

De ovenfor udledte almindelige Kriterier er (i noget anden Form) velkendte, og findes f. Eks. ogsaa hos Pringsheim. Han benævner det første det „almindeligste Kriterium af 1. Art"; det andet derimod fremtræder ikke hos ham — som ovenfor — som det tilsvarende Kriterium af 2. Art. Dette gør derimod et helt andet almindeligt Kriterium, det saaakaldte Kummerske Kriterium, der udsiger, at Σv_n er konvergent, dersom

$$\frac{v_{n-1}}{v_n}\varphi_{n-1} - \varphi_n > K > 0,$$

hvor φ_n er en vilkaarlig positiv Funktion af n.

Den Maade, hvorpaa Pringsheim udleder dette, meget smukke, Kriterium er imidlertid, forekommer det mig, kun lidet systematisk, ja nærmest „tilfældig". Pringsheim opstiller som Type paa et Konvergenskriterium af 2. Art for en Række Σv_n Uligheden

$$\underline{\lim}\left(\frac{1}{c_{n-1}}\frac{v_{n-1}}{v_n} - \frac{1}{c_n}\right) > 0, \text{ d. v. s. } \frac{1}{c_{n-1}}\frac{v_{n-1}}{v_n} - \frac{1}{c_n} > K > 0.$$

Ved forskellige Omformninger af ikke helt simpel Karakter beviser han dernæst, at dette Konvergenskriterium forbliver gyldigt, naar man deri erstatter Tallene c_n taget fra en vilkaarlig konvergent Række med Leddene d_n i en vilkaarlig divergent Række; det vises med andre Ord, at ogsaa Uligheden

$$\frac{1}{d_{n-1}}\frac{v_{n-1}}{v_n} - \frac{1}{d_n} > K > 0$$

er et Konvergenskriterium. Og da for en hvilkensomhelst Følge af positive Tal φ_n Rækken $\sum \frac{1}{\varphi_n}$ jo enten maa være konvergent eller divergent, er Kummers Kriterium dermed udledt. Pringsheim tilføjer: „Først gennem denne Udledelse synes Kriteriets sande Grundlag og dets Stilling indenfor hele Konvergensteorien at være fuldstændig klarlagt". Dette forekommer mig dog ikke

med Rimelighed at kunne hævdes. Selvom man, som det tilsyneladende fremgaar af Pringsheims Fremstilling, i den almindelige Type paa et Konvergenskriterium af 2. Art kunde erstatte c_n (Leddet i en konvergent Række) med d_n (Leddet i en divergent Række), vilde Tilladeligheden af denne Ombytning — da et Konvergenskriterium jo i Følge sin Natur beror paa en Sammenligning af Leddene i den forelagte Række med Leddene i en ***konvergent*** Række – næppe kunde siges at give en Forklaring af Kriteriets sande Grundlag. Men hertil kommer yderligere, at hele Pringsheims Udledelse er baseret paa, at han, umotiveret, har givet sit Udgangskriterium Formen $\frac{1}{c_{n-1}}\frac{v_{n-1}}{v_n}-\frac{1}{c_n}>K>0$, i Stedet for den naturlige Form

$$\frac{v_n}{v_{n-1}}\leqq\frac{c_n}{c_{n-1}}\quad \text{d. v. s.}\quad \frac{1}{c_{n-1}}\frac{v_{n-1}}{v_n}-\frac{1}{c_n}\geqq 0,$$

og havde han benyttet denne „korrekte“ Form, havde det ikke været muligt at gaa frem som det gøres i hans Udledelse, idet c_n da naturligvis ikke kunde være erstattet med d_n.

Jeg skal nu til Slut vise, hvad Kummers Kriterium, udfra det Synspunkt, jeg ovenfor har lagt til Grund, „egentlig“ udsiger.

Lad $S_{(x_0,x_1,\cdots)}^{u}$ være en vilkaarlig (konvergent eller divergent) Række, hvori den positive Funktion u_{x_n}, der summeres, antages stadig aftagende, d. v. s. $u_{x_n}^{(1)}$ negativ. Rækken $S_{(x_0,x_1,\cdots)}^{-u^{(1)}}$ med positive Led vil da i alle Tilfælde være konvergent, idet jo

$$S_{(x_0,\cdots,x_n)}^{-u^{(1)}}=u_{x_0}-u_{x_n}<u_{x_0}\quad\text{for alle } n.$$

For en Række $S_{(x_0,x_1,\cdots)}^{u}$, hvor den positive Funktion u_{x_n} er aftagende, faas derfor umiddelbart følgende *specielle Konvergenskriterium*

$$u_{x_n}<k(-u_{x_n}^{(1)})\quad\text{eller}\quad\frac{-u_{x_n}^{(1)}}{u_{x_n}}>K>0;$$

herved er det forøvrigt unødvendigt at anføre Kravet om, at u_{x_n} skal være aftagende, d. v. s. $u_{x_n}^{(1)}<0$, idet dette jo af sig selv er opfyldt, naar Uligheden $u_{x_n}<k(-u_{x_n}^{(1)})$ er tilfredsstillet, da u_{x_n} er positiv.

Vi skal vise, at dette overmaade simple, specielle, Kriterium for en Række med vilkaarlige Indices er det, hvorpaa det almindelige Kummerske Konvergenskriterium bygger. Betegner nemlig Σv_n en vilkaarlig forelagt Række med positive Led, og transformeres den om til $S_{(x_0, x_1, \cdots)}^{u}$, hvor $x_0, x_1, \cdots$ er en vilkaarlig monotont voksende Talfølge, faas ved Indsættelse af $u_{x_n} = \frac{v_n}{x_n - x_{n-1}}$ i ovenstaaende specielle Kriterium det almindelige Konvergenskriterium

$$\frac{-u_{x_n}^{(1)}}{u_{x_n}} = \frac{\frac{v_n}{x_n - x_{n-1}} - \frac{v_{n-1}}{x_{n-1} - x_{n-2}}}{x_n - x_{n-1}} : \frac{v_n}{x_n - x_{n-1}} > K > 0$$

d. v. s.

$$\frac{v_{n-1}}{v_n}\left(\frac{1}{x_{n-1} - x_{n-2}}\right) - \frac{1}{x_n - x_{n-1}} > K > 0.$$

Dette er imidlertid netop Kummers Kriterium; sættes $\frac{1}{x_\nu - x_{\nu-1}} = \varphi_\nu$, hvorved φ_ν bliver en vilkaarlig positiv Funktion, faar vi den ovenfor angivne Form.

I denne Udledelse fremtræder det Kummerske Kriterium saaledes overhovedet ikke som et normalt Sammenligningskriterium af samme simple Natur som de tidligere udledte, idet den Række $S_{(x_0, x_1, \cdots)}^{-u^{(1)}}$, udfra hvis Konvergens Kriteriet udledtes, ***afhænger af Funktionen*** u, d. v. s. afhænger af den Række $S_{(x_0, x_1, \cdots)}^{u}$, hvis Konvergens skal afgøres. Herved forstaas tillige, hvorledes Kummers Kriterium kan komme til at indeholde 2 Led af Rækken Σv_n (nemlig baade v_{n-1} og v_n), skønt det jo egentlig nærmest har Karakter af et Kriterium af 1. Art, idet man jo direkte sammenlignede selve de enkelte Led u_{x_n} og $-u_{x_n}^{(1)}$ i de to Rækker $S_{(x_0, x_1, \cdots)}^{u}$ og $S_{(x_0, x_1, \cdots)}^{-u^{(1)}}$ med hinanden.

ENCYCLOPÆDIA ARTICLE

II C 8. DIE NEUERE ENTWICKLUNG DER ANALYTISCHEN ZAHLENTHEORIE.

Von

H. BOHR IN KOPENHAGEN (DÄNEMARK) UND **H. CRAMÉR** IN STOCKHOLM (SCHWEDEN)

Dieser Artikel, welcher den 1900 abgeschlossenen *Bachmann*schen Artikel (I C 3) weiterführen soll, besteht aus zwei Teilen, von denen der erste, der von *Bohr* ausgearbeitet ist, insofern einen vorbereitenden Charakter trägt, als er sich ausschließlich mit den für die Behandlung der zahlentheoretischen Probleme nötigen funktionen- und reihentheoretischen Hilfsmitteln beschäftigt, während der zweite, welcher von *Cramér* herrührt, die betreffenden Probleme selbst behandelt.

Es wurde von den Verfassern zweckmäßig gefunden, dem Artikel, obwohl er sich nur in geringem Grade mit der älteren, in dem *Bachmann*schen Artikel behandelten Literatur befaßt, jedoch eine in sich abgerundete Form zu geben, so daß er gewissermaßen als ein selbständiges Ganzes hervortritt.*)

Inhaltsübersicht.

Erster Teil.

I. Allgemeine Theorie der Dirichletschen Reihen.

*) Bei der Ausarbeitung ist uns die von dem Meister des Gebietes, *J. Hadamard*, in der französischen Ausgabe der Encyklopädie gegebene Bearbeitung und Weiterführung des *Bachmann*schen Artikels von großer Bedeutung gewesen. Dasselbe gilt von dem klassischen Werk von *E. Landau*, Handbuch der Lehre von der Verteilung der Primzahlen, Bd. 1—2, Leipzig und Berlin 1909, welches wir im folgenden einfach mit „Handbuch" zitieren werden.

Erster Teil.

In diesem Teil, der, wie in den einleitenden Worten gesagt, einen rein analytischen Charakter trägt, d. h. von den zahlentheoretischen Anwendungen prinzipiell absieht, wird die Theorie der *Dirichletschen Reihen* besprochen, welche sich — obwohl ihre wesentliche Bedeutung in ihrer Stellung als besonders geeignetes Hilfsmittel zur funktionentheoretischen Behandlung von zahlentheoretischen Aufgaben zu ersehen ist, und sie immer noch ihre meisten Problemstellungen der analytischen Zahlentheorie verdankt — doch im Laufe der letzten Jahrzehnte zu einem selbständigen Abschnitt der allgemeinen Reihenlehre entwickelt hat. Das Referat ist in zwei Kapitel eingeteilt, von denen das erste die Theorie der allgemeinen *Dirichlet*schen Reihen behandelt, während das zweite der für das Studium der Primzahlen fundamentalen speziellen *Dirichlet*schen Reihe, welche die *Riemann*sche Zetafunktion darstellt, gewidmet ist. Bei der Abfassung ist mehr Gewicht auf eine bequeme Übersicht der wichtigeren Resultate als auf strenge Vollständigkeit gelegt.

I. Allgemeine Theorie der Dirichletschen Reihen.[1])

1. Definition einer Dirichletschen Reihe. Unter *einer allgemeinen Dirichletschen Reihe* wird eine unendliche Reihe der Form

$$f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s} \tag{1}$$

verstanden; hierbei bedeutet $s = \sigma + it$ eine komplexe, unabhängige Variable, die Koeffizienten a_n sind beliebige komplexe Zahlen, während die Exponentenfolge $\{\lambda_n\}$ eine reelle monoton wachsende Zahlenfolge mit $\lambda_n \to \infty$ bezeichnet.[2]) Für die folgende Darstellung wird es bequem sein, die (unwesentliche) Annahme $\lambda_1 \geqq 0$ zu machen. Für $\lambda_n = n$ ist (1) eine Potenzreihe in der Variablen e^{-s}. In dem beson-

1) Betreffs vieler Einzelheiten in der Theorie sei der Leser auf *E. Landau,* Handbuch, und *G. H. Hardy-M. Riesz,* The general theory of Dirichlet's series, Cambridge tracts, Nr. 18 (1915), verwiesen.

2) *W. Schnee,* Über irreguläre Potenzreihen und Dirichletsche Reihen, Dissertation, Berlin 1908, und *K. Väisälä,* Verallgemeinerung des Begriffes der Dirichletschen Reihen, Acta Universitatis Dorpatensis (1921), betrachten auch Reihen mit *komplexen* Exponenten λ_n und untersuchen, unter welchen Bedingungen solche Reihen sich „ähnlich" benehmen wie Reihen mit reellen Exponenten.

ders wichtigen Spezialfall $\lambda_n = \log n$ erhalten wir die *gewöhnlichen Dirichletschen Reihen*

$$(2) \qquad \sum_{n=1}^{\infty} a_n e^{-s\log n} = \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

Die spezielle Reihe (2), bei welcher $a_n = 1$ ist für alle n, also die Reihe

$$(3) \qquad \sum \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots,$$

definiert die *Riemannsche Zetafunktion*, deren Theorie in einem besonderen Kapitel behandelt wird. Als ein anderes wichtiges Beispiel einer gewöhnlichen *Dirichlet*schen Reihe (2) sei eine solche erwähnt[3]), bei der die Koeffizienten a_n sich periodisch wiederholen (etwa mit der Periode k), und die Summe der Koeffizienten erstreckt über eine Periode gleich 0 ist, wo also

$$(4) \qquad a_n = a_m \quad \text{für} \quad m \equiv n \pmod k, \quad \sum_{n=1}^{k} a_n = 0.$$

Zu diesem Typus gehört z. B. die Zetareihe mit abwechselndem Vorzeichen

$$(5) \qquad \sum \frac{(-1)^{n+1}}{n^s} = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \cdots,$$

welche durch formale Multiplikation der Zetareihe (3) mit dem Faktor $1 - 2^{1-s}$ entsteht. Andere wichtige Typen gewöhnlicher *Dirichlet*scher Reihen werden in Nr. 7 besprochen.

2. Die drei Konvergenzabszissen. Eine *Dirichlet*sche Reihe (1), die in einem Punkte $s_0 = \sigma_0 + it_0$ *absolut konvergiert*, wird offenbar in jedem Punkte $s = \sigma + it$ mit $\sigma \geqq \sigma_0$ absolut konvergieren; denn es ist ja, $s - s_0 = s'$ gesetzt,

$$(6) \qquad \sum a_n e^{-\lambda_n s} = \sum a_n e^{-\lambda_n s_0} \cdot e^{-\lambda_n s'}$$

und $|e^{-\lambda_n s'}| \leqq 1$ für $\Re(s') \geqq 0$. Jede Reihe (1) besitzt daher eine *absolute Konvergenzabszisse* σ_A derart, daß (1) für $\sigma > \sigma_A$ absolut konvergiert, für $\sigma < \sigma_A$ dagegen nicht; hierbei sind, den Werten $+\infty$ und $-\infty$ von σ_A entsprechend, diejenigen Fälle mit inbegriffen, wo die Reihe nirgends bzw. überall absolut konvergiert.

Tiefer liegt der Satz von *Jensen*[4]), daß, wenn die Reihe (1) im Punkte $s_0 = \sigma_0 + it_0$ *konvergiert*, sie dann auch in der ganzen Halbebene $\sigma > \sigma_0$ konvergiert. Diesen Hauptsatz der Theorie beweist *Jensen*

3) *G. Lejeune Dirichlet*, Recherches sur diverses applications de l'Analyse infinitésimale à la Théorie des Nombres, Crelles J. 19 (1839), p. 324—369 = Werke, Bd. 1, p. 411 u. f.

4) *J. L. W. V. Jensen*, Om Rækkers Konvergens, Tidsskr. for Math. (5) 2 (1884), p. 63—72.

von (6) aus, indem er mit Hilfe partieller (*Abel*scher) Summation nachweist, daß bei festem s' mit $\Re(s') > 0$ die Zahlenfolge $\{e^{-\lambda_n s'}\}$ eine „konvergenzerhaltende" ist in dem Sinne, daß aus der Konvergenz einer Reihe $\sum b_n$ die Konvergenz der „multiplizierten" Reihe $\sum b_n e^{-\lambda_n s'}$ folgt. Es gibt also auch eine *Konvergenzabszisse* σ_B ($\leqq \sigma_A$) derart, daß (1) für $\sigma > \sigma_B$ konvergiert, für $\sigma < \sigma_B$ divergiert.

Cahen[5]), der zuerst die *Dirichlet*schen Reihen einer systematischen Untersuchung unterworfen hat, zeigt, daß (1) in jedem Gebiete $\sigma > \sigma_B + \varepsilon$, $|s| < K$ *gleichmäßig konvergiert* und somit in der Konvergenzhalbebene $\sigma > \sigma_B$ eine *reguläre analytische Funktion* $f(s)$ darstellt. Im allgemeinen konvergiert aber eine Reihe (1) *nicht* gleichmäßig in der *ganzen* Halbebene $\sigma > \sigma_B + \varepsilon$, und *Bohr*[6]) hat daher die *gleichmäßige Konvergenzabszisse* σ_G eingeführt, welche definiert wird als die untere Grenze aller Abszissen σ_0, für die (1) in der ganzen Halbebene $\sigma > \sigma_0$ gleichmäßig konvergiert. Hierbei ist offenbar $-\infty \leqq \sigma_B \leqq \sigma_G \leqq \sigma_A \leqq +\infty$, und es können die drei Konvergenzabszissen alle Werte tatsächlich haben, welche mit diesen Ungleichungen verträglich sind.[7])

Die drei Konvergenzabszissen einer Reihe (1) können leicht aus den Koeffizienten und Exponenten der Reihe bestimmt werden. Für die Abszisse σ_B gilt nach *Cahen*[8]) der Satz: Falls $\sigma_B > 0$ ist[9]), wird

5) *E. Cahen*, Sur la fonction $\zeta(s)$ de Riemann et sur des fonctions analogues, Ann. Éc. Norm. (3) 11 (1894), p. 75—164.

6) *H. Bohr*, a) Sur la convergence des séries de Dirichlet, Paris C. R. 151 (1910), p. 375—377; b) Über die gleichmäßige Konvergenz Dirichletscher Reihen, Crelles J. 143 (1913), p. 204—211; c) Nogle Bemærkninger om de Dirichletske Rækkers ligelige Konvergens, Mat. Tidsskr. B 1921, p. 51—55.

7) *L. Neder*, Über die Lage der Konvergenzabszissen einer Dirichletschen Reihe zur Beschränktheitsabszisse ihrer Summe, Arkiv för Mat., Astr. och Fys. 16 (1922), No. 20.

8) *E. Cahen*, a. a. O. 5). Ein Teil des Satzes findet sich schon bei *J. L. W. V. Jensen*, Sur une généralisation d'une théorème de Cauchy, Paris C. R. 106 (1888), p. 833—836.

9) Die Bedingung $\sigma_B > 0$ bedeutet keine wesentliche Einschränkung der Allgemeinheit, weil ja die Konvergenzabszisse σ_B, falls sie $> -\infty$ ist, immer durch die einfache Transformation $s = s' - c$ um eine Konstante c vergrößert werden kann. Ausdrücke für σ_B, die im Falle $\sigma_B < 0$ oder sogar für jede Lage von σ_B gelten, sind gegeben von *S. Pincherle*, Alcune spigolature nel campo delle funzioni determinanti, Atti d. IV Congr. intern. d. Mat. 2 (Rom 1908), p. 44—48; *K. Knopp*, Über die Abszisse der Grenzgeraden einer Dirichletschen Reihe, Sitzungsber. Berl. Math. Ges. 10 (1910), p. 1—7; *W. Schnee*, Über die Koeffizientendarstellungsformel in der Theorie der Dirichletschen Reihen, Gött. Nachr. 1910, p. 1—42; *T. Kojima*, a) On the convergence-abscissa of general Dirichlet's series, Tôhoku J. 6 (1914), p. 134—139; b) Note on the convergence-abscissa of

sie durch den Ausdruck

$$\sigma_B = \limsup_{n\to\infty} \frac{\log|S_n|}{\lambda_n} \qquad \left(S_n = \sum_1^n a_m\right) \tag{7}$$

gegeben; d. h. σ_B ist die untere Grenze aller positiven Zahlen α, für welche die „summatorische" Funktion S_n gleich $O(e^{\lambda_n \alpha})$ ist.[10])

Aus (7) ergibt sich sofort, daß im Falle $\sigma_A > 0$

$$\sigma_A = \limsup_{n\to\infty} \frac{\log R_n}{\lambda_n}. \qquad \left(R_n = \sum_1^n |a_m|\right)$$

Für die gleichmäßige Konvergenzabszisse σ_G gilt schließlich, falls $\sigma_G > 0$ ist, die entsprechende Formel[11]):

$$\sigma_G = \limsup_{n\to\infty} \frac{\log T_n}{\lambda_n},$$

wo T_n, bei festem n, die obere Grenze von $\left|\sum_1^n a_m e^{-\lambda_m i t}\right|$ für $-\infty < t < \infty$ bezeichnet.

Für Reihen (1), bei denen die Exponentenfolge $\{\lambda_n\}$ hinreichend schnell ins Unendliche wächst (z. B. für die Potenzreihen, wo $\lambda_n = n$ ist), gilt immer die Gleichung $\sigma_A = \sigma_B$ ($= \sigma_G$), d. h. sie besitzen keinen bedingten Konvergenzstreifen. Die genaue notwendige und hinreichende Bedingung, die eine Exponentenfolge erfüllen muß, damit jede zu ihr gehörige *Dirichlet*sche Reihe der Bedingung $\sigma_A = \sigma_B$ genügt, ist

$$\lim_{n\to\infty} \frac{\log n}{\lambda_n} = 0. \tag{8}$$

Dirichlet's series, Tôhoku J. 9 (1916), p. 28—37; *M. Fujiwara*, a) On the convergence-abscissa of general Dirichlet's series, Tôhoku J. 6 (1914), p. 140—142; b) Über Konvergenzabszisse der Dirichletschen Reihe, Tôhoku J. 17 (1920), p. 344—350; *E. Lindh* (bei *Mittag-Leffler*), Sur un nouveau théorème dans la théorie des séries de Dirichlet, Paris C. R. 160 (1915), p. 271—273; *B. Malmrot*, Sur une formule de M. Fujiwara, Arkiv för Mat., Astr. och Fys. 14 (1919), No. 4, p 1—10.

10) Soll die Reihe (1) noch in Punkten *auf* der Konvergenzgeraden $\sigma = \sigma_B$ (> 0) konvergieren, ist es nach *Jensen*, a. a. O. 8), notwendig (aber nicht hinreichend, vgl. Nr. 5), daß die summatorische Funktion S_n der Bedingung $S_n = o(e^{\lambda_n \sigma_B})$ genügt.

11) Für *gewöhnliche Dirichlet*sche Reihen ($\lambda_n = \log n$) bei *H. Bohr*, Darstellung der gleichmäßigen Konvergenzabszisse einer Dirichletschen Reihe $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ als Funktion der Koeffizienten der Reihe, Arch. Math. Phys. (3) 21 (1913), p. 326—330, für *beliebige Dirichlet*sche Reihen bei *M. Kuniyeda*, Uniform convergence-abscissa of general Dirichlet's series, Tôhoku J. 9 (1916), p. 7—27. In der letzten Arbeit sind auch Formeln für σ_G angegeben, die für jede Lage von σ_G gelten. (Vgl. Note 9).)

Allgemein gilt der Satz[12]), daß *die maximale Breite M des bedingten Konvergenzstreifens* $\sigma_B \leqq \sigma \leqq \sigma_A$ für alle zu einer gegebenen Exponentenfolge gehörigen Reihen (1) durch den Ausdruck

$$M = \limsup_{n \to \infty} \frac{\log n}{\lambda_n}$$

gegeben wird. Für die gewöhnlichen *Dirichlet*schen Reihen (2) ist somit die maximale Breite $M = 1$. Diese Breite 1 wird z. B. bei jeder Reihe (2), die den Bedingungen (4) genügt, erreicht; in der Tat ist hier $\sigma_A = 1$, $\sigma_B = 0$.

3. Der Eindeutigkeitssatz. Aus der einfachen Bemerkung, daß die Funktion $e^{-\lambda s} = e^{-\lambda(\sigma + it)}$ $(\lambda > 0)$ für $\sigma \to \infty$ um so schneller gegen 0 abnimmt, je größer der Exponent λ ist, ergibt sich leicht: falls eine *Dirichlet*sche Reihe (1) mit $\sigma_B < \infty$ die Bedingung $\sigma_A < \infty$ oder nur die Bedingung $\sigma_G < \infty$[6c]) erfüllt, dann *überwiegen für $\sigma \to \infty$ die Anfangsglieder der Reihe den Rest,* d. h. es gilt, bei jedem festen N, für $\sigma \to \infty$ *gleichmäßig in t* die Limesgleichung

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n s} = \sum_{n=1}^{N} a_n e^{-\lambda_n s} + o(e^{-\lambda_N \sigma}); \tag{9}$$

hieraus folgt sofort, daß, wenn nicht sämtliche Koeffizienten a_n gleich 0 sind, die Summe $f(s)$ bei hinreichend großem K in der ganzen Halbebene $\sigma > K$ von 0 verschieden sein wird. Für *Reihen* (1) *mit* $\sigma_G < \infty$ gilt daher der folgende *Eindeutigkeitssatz*: Sind zwei *Dirichlet*sche Reihen $f(s) = \sum a_n e^{-\lambda_n s}$ und $g(s) = \sum b_n e^{-\mu_n s}$ gleichgroß in allen Punkten einer Zahlenfolge $\{s_n = \sigma_n + it_n\}$ mit $\sigma_n \to \infty$, dann sind die beiden Reihen identisch; denn in der *Dirichlet*schen Reihe $\sum c_n e^{-\nu_n s}$, welche durch Subtraktion von $f(s)$ und $g(s)$ entsteht, müssen ja alle Koeffizienten c_n gleich 0 sein.

Für eine *beliebige Dirichlet*sche Reihe (1) mit $\sigma_B < \infty$ gilt die Limesgleichung (9) für $\sigma \to \infty$ im allgemeinen *nicht* gleichmäßig in t, wenn t das *ganze* Intervall $-\infty < t < \infty$ durchläuft. Dagegen gilt (9), wie von *Perron*[13]) bewiesen, gleichmäßig in t, wenn t durch eine Bedingung der Form $|t| < e^{k\sigma}$ beschränkt wird, wo k eine beliebige Konstante bedeutet. In diesem *allgemeinen Fall* finden wir daher den folgenden *Eindeutigkeitssatz*: Wenn zwei *Dirichlet*sche Reihen mit

---

12) *E. Cahen,* a. a. O. 5). Vgl. auch *Hardy-Riesz,* a. a. O. 1), p. 9.

13) *O. Perron,* Zur Theorie der Dirichletschen Reihen, Crelles J. 134 (1908), p. 95—143. Daß die Limesgleichung (9) für ein *festes t* gilt, steht schon bei *Dirichlet,* Vorlesungen über Zahlentheorie, herausgegeben von *Dedekind,* Braunschweig 1863, p. 410—414. Vgl. auch eine (in Math. Ztschr. bald erscheinende) Arbeit von *L. Neder,* Über Gebiete gleichmäßiger Konvergenz Dirichletscher Reihen.

$\sigma_B < \infty$ in den Punkten einer Zahlenfolge $\{s_n\}$ mit $\sigma_n \to \infty$ und $|t_n| < e^{k\sigma}$ gleichgroß sind, so sind die beiden Reihen identisch. Hier kann die Forderung $|t_n| < e^{k\sigma_n}$ nicht weggelassen werden, denn es existieren tatsächlich Reihen (1), deren Koeffizienten nicht alle 0 sind, die jedoch eine Folge von Nullstellen $\{s_n\}$ mit $\sigma_n \to \infty$ besitzen.[14])

4. Die Koeffizientendarstellungsformel. Aus dem Eindeutigkeitssatze in Nr. 3 folgt sofort: wenn eine in einer gewissen Halbebene $\sigma > \sigma_0$ reguläre analytische Funktion $f(s)$ durch eine konvergente *Dirichlet*sche Reihe darstellbar ist, dann müssen die Exponenten λ_n und die Koeffizienten a_n dieser Reihe aus der Funktion $f(s)$ eindeutig bestimmt werden können. Die tatsächliche Bestimmung dieser beiden Zahlenfolgen $\{\lambda_n\}$ und $\{a_n\}$ wird durch den unten folgenden Satz gegeben, dessen *formale* Herleitung[15]) sich aus der bekannten, für jedes positive c gültigen Formel

$$\frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty}\frac{e^{\alpha s}}{s}\,ds = \begin{cases} 1 & \text{für } \alpha > 0 \\ 0 & \text{für } \alpha < 0 \end{cases}$$

ergibt, während seine strenge Begründung zuerst von *Hadamard* und *Perron*[16]) gegeben wurde. Dieser Satz lautet: *Es sei* (1) *eine beliebige Dirichletsche Reihe mit der Konvergenzabszisse $\sigma_B < \infty$ und c eine positive Zahl $> \sigma_B$. Dann gilt für jedes x im Intervalle $\lambda_N < x < \lambda_{N+1}$ die Formel*

$$\sum_1^N a_n = \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} f(s)\frac{e^{xs}}{s}\,ds. \tag{10}$$

Es ist also das auf der rechten Seite stehende Integral $J(x)$ streckenweise konstant (für $0 < x < \infty$) und die Exponenten λ_n sind die Unstetigkeitsstellen von $J(x)$, während die Koeffizienten a_n sich

14) *H. Bohr*, Beweis der Existenz Dirichletscher Reihen, die Nullstellen mit beliebig großer Abszisse besitzen, Palermo Rend. 31 (1911), p. 235—243.

15) Vgl. *L. Kronecker*, Notiz über Potenzreihen, Monatsber. Akad. Berlin (1878), p. 53—58, und *E. Cahen*, a. a. O. 5). Ein Spezialfall kommt schon bei *B. Riemann*, Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse, Monatsber. Akad. Berlin 1859, p. 671—680 = Werke, p. 145—153, vor.

16) *J. Hadamard*, Sur les séries de Dirichlet, Palermo Rend. 25 (1908), p. 326—330, beweist den Satz unter der Annahme, daß die Reihe eine *absolute* Konvergenzhalbebene besitzt (also $\sigma_A < \infty$) und *O. Perron*, a. a. O. 13) für den allgemeinen Fall. Vgl. auch *E. Phragmén*, Über die Berechnung der einzelnen Glieder der Riemannschen Primzahlformel, Oefvers. af Kgl. Vetensk. Förh. 48 (Stockholm 1891), p. 721—744 und *H. v. Mangoldt*, Auszug aus einer Arbeit unter dem Titel: Zu Riemanns Abhandlung „Über die Anzahl der Primzahlen unter einer gegebenen Größe", Sitzungsber. Akad. Berlin 1894, p. 883—896.

als die Sprünge in den Punkten λ_n ergeben.[17]) In einer Unstetigkeitsstelle λ_n selbst ist das Integral $J(x)$ wohl nicht direkt konvergent, es hat aber einen Hauptwert, definiert durch $\lim\limits_{T\to\infty}\frac{1}{2\pi i}\int\limits_{c-iT}^{c+iT}$, und dieser Hauptwert ist gleich dem Mittelwert $\frac{1}{2}(J(\lambda_n+0)+J(\lambda_n-0))$.

Das Integral in (10) konvergiert im allgemeinen nur bedingt. Bei verschiedenen Untersuchungen ist es deshalb bequem, statt (10) die Formel

$$\frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} f(s)\frac{e^{xs}}{s^2}\,ds = \sum_1^N a_n(x-\lambda_n) \tag{11}$$

zu benutzen, wo das Integral (wenigstens im Falle $\sigma_G < \infty$, vgl. Nr. 6) absolut konvergiert. Die Formeln (10) und (11) sind übrigens Spezialfälle der allgemeinen Formel[18])

$$\frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} f(s)\frac{e^{xs}}{s^\alpha}\,ds = \frac{1}{\Gamma(\alpha)}\sum_1^N a_n(x-\lambda_n)^{\alpha-1}, \tag{12}$$

wo $\alpha \geqq 1$ ist.

5. Beziehung zwischen der Reihe auf der Konvergenzgeraden und der Funktion bei Annäherung an die Konvergenzgerade. In den Punkten der Konvergenzgeraden $\sigma = \sigma_B$ einer *Dirichlet*schen Reihe (1) kann das Verhalten der Reihe sehr verschiedenartig sein. Wie im Spezialfall einer Potenzreihe ($\lambda_n = n$) bestehen aber auch bei den allgemeinen *Dirichlet*schen Reihen wichtige Zusammenhänge zwischen dem Verhalten der *Reihe* in einem Punkte der Konvergenzgeraden und dem Verhalten der dargestellten *Funktion* $f(s)$, wenn die Variable s sich diesem Punkte nähert. Da dies Problem im Spezialfall $\lambda_n = n$ im Artikel II C 4 ausführlich besprochen ist, sollen hier nur einige Hauptresultate erwähnt werden. Zuerst nennen wir den Satz (Analogon zum *Abel-Stolz*schen Satze über Potenzreihen): wenn die Reihe (1) in einem Punkte s_0 der Konvergenzgeraden $\sigma = \sigma_B$ *konvergiert* mit der Summe A, dann existiert der *Grenzwert* $\lim f(s)$ und ist $= A$, wenn s sich von rechts längs einer horizontalen Geraden oder sogar

17) Eine andere, von *Hadamard* herrührende Methode, um die Koeffizienten a_n einer *Dirichlet*schen Reihe aus der durch die Reihe dargestellten Funktion zu bestimmen, wird in Nr. 9 besprochen; diese letzte Methode — und nicht die oben angegebene — ist übrigens als die unmittelbare Verallgemeinerung der *Cauchy*schen Methode zur Bestimmung der Koeffizienten einer Potenzreihe anzusehen.

18) *J. Hadamard,* Sur la distribution des zéros de la fonction $\zeta(s)$ et ses conséquences arithmétiques, Bull. Soc. math. France 24 (1896), p. 199—220. Wegen der strengen Begründung im Falle $\sigma_A = \infty$ vgl. *O. Perron,* a. a. O. 13).

in einem der Halbebene $\sigma > \sigma_B$ ganz angehörenden Winkelraum dem Punkte s_0 nähert.[19]) Dieser Satz läßt sich natürlich *nicht* ohne weiteres *umkehren*, d. h. aus der Existenz des Grenzwertes folgt nicht die Konvergenz der Reihe im Punkte s_0. Bedingungen, unter welchen die Umkehrung erlaubt ist, wurden von *Landau*, *Schnee*, *Littlewood* und *Hardy-Littlewood* gegeben.[20]) Hier sei nur der tiefliegende Satz von *Littlewood* erwähnt, wonach die Bedingung

$$a_n e^{-\lambda_n s_0} = O\left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n}\right) \qquad (\text{für } n \to \infty)$$

für die besprochene Umkehrung genügt.

Von etwas anderer Art — weil *Regularität* im Punkte s_0 statt Grenzwert für $s \to s_0$ vorausgesetzt wird — ist ein für verschiedene Anwendungen sehr wichtiger Satz von *M. Riesz*[21]), der als die Verallgemeinerung eines *Fatou*schen Satzes über Potenzreihen ($\lambda_n = n$) anzusehen ist, und der besagt, daß, falls eine *Dirichlet*sche Reihe (1) mit $\sigma_B > 0$ die Bedingung

$$(13) \qquad S_n = a_1 + \cdots a_n = o(e^{\lambda_n \sigma_B})$$

erfüllt, sie in jedem Punkte der Konvergenzgeraden $\sigma = \sigma_B$, in welchem die Funktion $f(s)$ *regulär* ist, *konvergiert*, und zwar gleichmäßig in jedem Regularitätsintervall. Die Bedeutung dieses Satzes zeigt sich

19) Für Annäherung längs einer horizontalen Geraden siehe *Dirichlet-Dedekind*, a. a. O. 13), p. 410—414; für Annäherung im Winkelraum *E. Cahen*, a. a. O. 5).

20) *E. Landau*, a) Über die Konvergenz einiger Klassen von unendlichen Reihen am Rande des Konvergenzgebietes, Monatsh Math. Phys. 18 (1907), p. 8—28; b) Über einen Satz des Herrn Littlewood, Palermo Rend. 35 (1913), p 265—276; *W. Schnee*, Über Dirichletsche Reihen, Palermo Rend. 27 (1909), p. 87—116; *J. Littlewood*, The converse of Abel's theorem on power series, Proc. London math. Soc. (2) 9 (1910), p. 434—448; *G. H. Hardy* u. *J. Littlewood*, Tauberian theorems concerning power series and Dirichlet's series whose coefficients are positive, Proc. London math. Soc. (2) 13 (1913), p 174—191.

21) *M. Riesz*, a) Sur les séries de Dirichlet et les séries entières, Paris C. R. 149 (1909), p 309—312; b) Ein Konvergenzsatz für Dirichletsche Reihen, Acta Math. 40 (1916), p. 349—361. Ein Beweis des Spezialfalls $\lambda_n = \log n$ wurde schon früher (nach einer Mitteilung von *Riesz*) von *E. Landau*, Über die Bedeutung einiger neuer Grenzwertsätze der Herren Hardy und Axer, Prac. Mat. Fiz. 21 (1910), p. 97—177, veröffentlicht. Vgl. auch *D. Kojima*, On the double Dirichlet series, Reports Tôhoku University 9 (1920), p. 351—400.

Riesz hat bedeutende Verallgemeinerungen seines Satzes in Aussicht gestellt. Vgl. eine demnächst in den Acta Univ. hung. Francesco-Jos. erscheinende Arbeit. Eine besonders wichtige dieser Verallgemeinerungen — welche den Fall Summabilität statt Konvergenz behandelt, vgl. Nr. 13 — ist in der zahlentheoretischen Arbeit von *H. Cramér*, Über das Teilerproblem von Piltz, Ark. f. Mat., Astr. och Fys. 16 (1922), No. 21, nach einer Mitteilung von *Riesz* veröffentlicht. Vgl. auch *A. Walfisz*, Über die summatorischen Funktionen einiger Dirichletscher Reihen, Diss. Göttingen 1922, p. 1—56.

schon darin, daß die Bedingung (13), wie früher[10]) erwähnt, *notwendig* ist, damit die Gerade $\sigma = \sigma_B$ überhaupt eine Konvergenzstelle der Reihe enthalte.

An die erstgenannten Sätze schließt sich eine Reihe von weiteren Sätzen an, wo an Stelle der Konvergenz der Reihe im Punkte s_0 und der Existenz des Grenzwertes der Funktion bei Annäherung an diesen Punkt, bestimmte Art von (eigentlicher) *Divergenz* der Reihe im Punkte s_0 und entsprechende bestimmte Art von *Unendlichwerden* der Funktion bei Annäherung an den Punkt tritt. Solche Sätze, die durch Vergleich mit speziellen einfachen Typen *Dirichlet*scher Reihen abgeleitet werden, verdankt man besonders *Knopp*[22]) und *Schnee*[23]). Als ein einfaches Beispiel für eine gewöhnliche *Dirichlet*sche Reihe (2) sei der folgende Satz genannt (wo es sich um den Punkt $s_0 = 0$ handelt). Aus

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \cdots a_n}{\log^\alpha n} = A \qquad (\alpha > 0)$$

folgt

$$\lim_{s\to 0} s^\alpha f(s) = A\,\Gamma(\alpha + 1),$$

wo s durch positive Werte gegen 0 strebt. Mit der viel schwierigeren Frage nach der *Umkehrung* solcher Sätze haben sich *Hardy* und *Littlewood*[24]) beschäftigt. So haben sie z. B. die Umkehrung des eben erwähnten Satzes in dem Falle bewiesen, wo die Koeffizienten a_n sämtlich *positiv* sind. Der weitestgehende von *Hardy* und *Littlewood* bewiesene Satz, welcher den allgemeinen Typus *Dirichlet*scher Reihen (1) betrifft (wo jedoch $\lambda_n : \lambda_{n+1} \to 1$ vorausgesetzt wird) besagt[24b]), daß, wenn eine Reihe (1) mit der Konvergenzabszisse $\sigma_B = 0$ die Limesgleichung

$$\lim_{s\to 0} s^\alpha f(s) = A \qquad (\alpha \geqq 0)$$

erfüllt, und ihre Koeffizienten a_n reell sind und der „einseitigen" Bedingung $a_n > -K\lambda_n^{\alpha-1}(\lambda_n - \lambda_{n-1})$ genügen[25]), die Gleichung gilt:

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \cdots a_n}{\lambda_n^\alpha} = \frac{A}{\Gamma(\alpha+1)}.$$

22) *K. Knopp,* a) Grenzwerte von Reihen bei der Annäherung an die Konvergenzgrenze, Diss. Berlin 1907; b) Divergenzcharaktere gewisser Dirichletscher Reihen, Acta Math. 34 (1911), p. 165—204; c) Grenzwerte von Dirichletschen Reihen bei der Annäherung an die Konvergenzgrenze, Crelles J. 138 (1910), p. 109—132.

23) *W. Schnee*, a) a. a. O. 2); b) a. a. O. 20). In der letzten Arbeit gibt *Schnee* einige interessante spezielle Typen *Dirichlet*scher Reihen an, die als „Vergleichsreihen" besonders geeignet sind.

24) Vgl. insbesondere *G. H. Hardy* u. *J. Littlewood*, a) a. a. O. 20); b) Some theorems concerning Dirichlet's series, Mess. of math. 43 (1914), p. 134—147.

25) Hieraus folgt sofort als Corollar, daß der Satz, im Falle komplexer Koeffizienten, gültig ist, falls die oben angegebene „einseitige" Bedingung durch

Mit den obigen Fragestellungen eng verwandt ist das Problem nach der Beziehung des Verhaltens der Funktion bei Annäherung an einen Punkt s_0 auf der Konvergenzgeraden $\sigma = \sigma_B$ und der Art der Divergenz der Reihe in einem Punkte s_1, welcher *links* von dieser Geraden in derselben Höhe wie s_0 gelegen ist; der einfachen Formulierung halber seien beide Punkte auf der reellen Achse angenommen, und zwar $s_1 = 0$ (also $s_0 = \sigma_B > 0$), so daß die Partialsummen im Punkte s_1 die Werte der summatorischen Funktion $S_n = a_1 + \cdots a_n$ ergeben. Hier ist vor allem ein Satz von *Dirichlet*[26]) über gewöhnliche *Dirichlet*sche Reihen (mit $\sigma_B = 1$) zu erwähnen, der besagt, daß aus

$$\frac{S_n}{n} \to A \qquad \text{(für } n \to \infty\text{)}$$

folgt
$$f(s)(s-1) \to A. \qquad \text{(für zu 1 abnehm. } s\text{)}$$

Auch dieser Satz läßt sich nicht ohne weiteres umkehren[27]), und zwar nicht einmal, wenn den Koeffizienten der Reihe Bedingungen der Art auferlegt werden (z. B. daß sie alle positiv sein sollen), welche beim vorhergehenden Problem für die Gültigkeit des Umkehrsatzes genügten; es läßt sich im allgemeinen nur behaupten[28]), daß aus $f(s)(s-1) \to A$ folgt

$$\limsup_{n\to\infty} \frac{S_n}{n} \geqq A \quad \text{und} \quad \liminf_{n\to\infty} \frac{S_n}{n} \leqq A.$$

Bei den obigen Sätzen, wo aus dem Verhalten der Funktion auf das Verhalten der Reihe geschlossen wurde, bezog sich die Annahme über die Funktion stets auf ihr Verhalten in der Nähe eines *einzigen* Punktes auf der Konvergenzgeraden. Von *Landau*[29]) rührt der folgende

die „allseitige" Bedingung $a_n = O(\lambda_n^{\alpha-1}(\lambda_n - \lambda_{n-1}))$ ersetzt wird. Im speziellen Falle $\alpha = 0$ reduziert sich dieser letzte Satz auf den oben erwähnten *Littlewood*schen Satz (über Konvergenz).

26) *G. Lejeune Dirichlet,* Sur un théorème relatif aux séries, J. de math. (2) 1 (1856), p. 80—81 = Werke, Bd. 2, p. 195—200. Verallgemeinerungen solcher Sätze finden sich z. B. bei *A. Pringsheim,* Zur Theorie der Dirichletschen Reihen, Math. Ann. 37 (1890), p. 38—60; *A. Berger*, Recherches sur les valeurs moyennes dans la théorie des nombres, Nova Acta Upsala (3) 14 (1891), Nr. 2; *J. Franel,* Sur la théorie des séries, Math. Ann. 52 (1899), p 529—549.

27) Wäre dies der Fall, so „würde das ganze Gebäude der Primzahltheorie mit großer Geschwindigkeit errichtet werden können" (*Landau*, Handbuch, Bd. 1, p. 114).

28) *O. Hölder,* Grenzwerte von Reihen an der Convergenzgrenze, Math. Ann. 20 (1882), p. 535—549. Vgl. auch *E. Landau,* Über die zu einem algebraischen Zahlkörper gehörige Zetafunktion und die Ausdehnung der Tschebyschefschen Primzahlentheorie auf das Problem der Vertheilung der Primideale, Crelles J. 125 (1903), p. 64—188.

29) *E. Landau,* Beiträge zur analytischen Zahlentheorie, Palermo Rend. 26 (1908), p. 169—302. Eine Verschärfung seines Satzes gab *Landau* a. a. O. 21).

tiefliegende Satz her, in welchem Voraussetzungen über die Funktion bei Annäherung an *alle* Punkte der Konvergenzgeraden gemacht werden und daraus ein sehr genaues Resultat über das Verhalten der Reihe (nämlich Umkehrung des obigen *Dirichlet*schen Satzes) hergeleitet wird: Es sei eine gewöhnliche *Dirichlet*sche Reihe (2) mit positiven Koeffizienten (und $\sigma_B = 1$) in allen Punkten der Konvergenzgeraden $\sigma = 1$ regulär mit Ausnahme des Punktes $s = 1$, wo sie einen Pol erster Ordnung mit dem Residuum A besitzt; ferner sei für $\sigma \geqq 1$ (und $|t| \to \infty$) die Relation $f(s) = O(|t|^k)$ bei passender Wahl einer Konstanten k erfüllt. Dann ist

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \cdots a_n}{n} = A.$$

Landau[30]) hat später diesen Satz auf beliebige *Dirichlet*sche Reihen (1) übertragen. Eine Verallgemeinerung dieses *Landau*schen Satzes und andere ähnliche Sätze haben auf anderem Wege *Hardy* und *Littlewood*[31]) gefunden.

6. Das Konvergenzproblem. In Nr. 2 wurde besprochen, wie die drei Konvergenzabszissen σ_A, σ_B, σ_G von den Koeffizienten und Exponenten der Reihe aus bestimmt werden können. Wir wenden uns nun zu einem viel schwierigeren Problem, dem sogenannten *Konvergenzproblem* der *Dirichlet*schen Reihen, nämlich zur Frage, ob und in welcher Weise die Lage dieser Abszissen (und vor allem der Konvergenzabszisse σ_B) mit einfachen analytischen Eigenschaften der durch die Reihe dargestellten Funktion $f(s)$ zusammenhängt. Im speziellen Fall $\lambda_n = n$ (Potenzreihe in e^{-s}) ist diese Frage ja einfach dahin zu beantworten, daß die Reihe genau so weit konvergiert, wie die Funktion $f(s)$ *regulär* bleibt; in der Tat, es liegt ja hier immer ein singulärer Punkt auf der Konvergenzgeraden $\sigma = \sigma_B$ $(= \sigma_A = \sigma_G)$. Es gilt aber nicht nur in dem ganz speziellen Fall $\lambda_n = n$, sondern für alle solche *Dirichlet*sche Reihen (1), deren Exponentenfolge die Bedingung

$$\lim_{n\to\infty} \frac{\log n}{\lambda_n} = 0 \tag{14}$$

erfüllt (wo also, nach Nr. 2, $\sigma_B = \sigma_A$ ist), daß das Konvergenzproblem in einfachster Weise zu lösen ist; die Funktion $f(s)$ braucht wohl hier *nicht* auf (oder in unendlicher Nähe links von) der Konvergenzgeraden

30) *E. Landau,* Handbuch, p. 874.

31) *G. H. Hardy* u. *J. Littlewood,* a) New proofs of the prime-number theorem and similar theorems, Quart. J. 46 (1915), p. 215—219; b) Contributions to the theory of the Riemann Zetafunction and the theory of the distributions of primes, Acta Math. 41 (1918), p. 119—196.

$\sigma = \sigma_B$ Singularitäten zu besitzen, es gilt aber der fast ebenso einfache Satz, daß die Reihe genau so weit konvergiert, wie die Funktion $f(s)$ *regulär und beschränkt* bleibt, d. h. es ist $\sigma_B\ (= \sigma_A) = \sigma_b$, wo σ_b (wie überall im folgenden) *die untere Grenze aller Zahlen* σ_0 *bezeichnet, für welche* $f(s)$ *in der Halbebene* $\sigma > \sigma_0$ *regulär ist und einer Ungleichung* $|f(s)| < K = K(\sigma_0)$ *genügt.*[32]) Für Reihen (1), deren Exponentenfolge „sehr" schnell ins Unendliche wächst, gilt übrigens, daß die Funktion $f(s)$ überhaupt nicht über die Konvergenzgerade hinaus fortgesetzt werden kann; es läßt sich nämlich, wie zuerst *Wennberg*[33]) und später allgemeiner *Carlson* und *Landau*[34]) und *Szász*[35]) gezeigt haben, der *Hadamard-Fabry*sche Lückensatz für Potenzreihen auf beliebige *Dirichlet*sche Reihen übertragen. Der Satz lautet hier, daß für jede zu einer Exponentenfolge mit $\lambda_n : n \to \infty$ und $\liminf (\lambda_{n+1} - \lambda_n) > 0$ gehörige Reihe (1) die Konvergenzgerade $\sigma = \sigma_B\ (= \sigma_A)$ eine *wesentlich singuläre Linie* ist.*)

In anderer Richtung — weil Voraussetzungen über die *Koeffizienten* und nicht über die Exponenten gemacht werden — liegt ein

32) Dieser Satz wurde zuerst von *H. Bohr*, a. a. O 6 b) bewiesen. Einen äußerst einfachen Beweis gab *E. Landau*, Über die gleichmäßige Konvergenz Dirichletscher Reihen, Math. Ztschr. 11 (1921), p. 317—318. Der Satz umfaßt offenbar den für die Potenzreihen ($\lambda_n = n$) gültigen Satz als Spezialfall, denn im Falle $\lambda_n = n$ ist ja $f(s)$ *periodisch* mit der Periode $2\pi i$, und $f(s)$ wird daher von selbst in jeder Halbebene $\sigma \geq \sigma_0$ beschränkt sein, wenn sie dort regulär ist.

Zur Definition der Abszisse σ_b vgl. auch die Arbeit von *H. Bohr*, Ein Satz über Dirichletsche Reihen, Münch Sitzungsber. 1913, p. 557—562, worin bewiesen wird, daß, falls die durch eine beliebige *Dirichlet*sche Reihe (mit $\sigma_A < \infty$) definierte Funktion $f(s)$ nur in irgendeiner *Viertelebene* $\sigma > \sigma_0$, $t > t_0$ regulär und beschränkt ist, sie von selbst in der ganzen *Halbebene* $\sigma > \sigma_0$ regulär und beschränkt bleiben wird.

33) *S. Wennberg*, Zur Theorie der Dirichletschen Reihen, Diss. Upsala 1920

34) *F. Carlson* u. *E. Landau*, Neuer Beweis und Verallgemeinerungen des Fabryschen Lückensatzes, Gött. Nachr. 1921, p. 184—188. Vgl. hierzu auch *L. Neder*, Über einen Lückensatz für Dirichletsche Reihen, Math. Ann. 85 (1922), p. 111—114.

35) *O. Szász*, Über Singularitäten von Potenzreihen und Dirichletschen Reihen am Rande des Konvergenzbereiches, Math. Ann. 85 (1922), p. 99—110

*) In einer soeben erschienenen interessanten Abhandlung von *A. Ostrowski*, Über vollständige Gebiete gleichmäßiger Konvergenz von Folgen analytischer Funktionen, Hamburger Seminar 1 (1922), p. 327—350, die sich allgemein mit den Abschnittsfolgen einer *Dirichlet*schen Reihe beschäftigt, wird u. a. auch ein *Lückensatz* bewiesen, wo die Exponentenfolge $\{\lambda_n\}$ nur „ab und zu" große Lücken aufweist; es wird gezeigt, daß die den Lücken entsprechende Abschnittsfolge so weit konvergiert, wie es von vornherein überhaupt gehofft werden konnte, d. h. so weit, wie die Funktion sich regulär verhält. Vgl hierzu auch *H. Bohr*, a. a. O.[44])

wichtiger Satz von *Landau*[36]), der ebenfalls die Verallgemeinerung eines bekannten (*Vivanti*schen) Satzes über Potenzreihen darstellt und der besagt, daß, wenn alle Koeffizienten a_n *positiv* sind, der Punkt σ_B, worin die Konvergenzgerade durch die reelle Achse geschnitten wird, immer ein *singulärer* Punkt der Funktion ist.

Für solche *Dirichlet*sche Reihen (1), für welche die Exponentenfolge $\{\lambda_n\}$ die Bedingung (14) *nicht* erfüllt, z. B. für die *gewöhnlichen Dirichlet*schen Reihen (2), stellt sich das Konvergenzproblem (wenn keine besonderen Bedingungen über die Koeffizienten gemacht werden) viel schwieriger, und es scheint hier überhaupt zweifelhaft, ob es möglich ist, die Lage der Konvergenzgeraden $\sigma = \sigma_B$ durch „einfache" analytische Eigenschaften der dargestellten Funktion genau zu charakterisieren.[37]) Bevor wir über die vorliegenden Resultate berichten können, müssen einige charakteristische Eigenschaften erörtert werden, die einer jeden von einer *Dirichlet*schen Reihe (1) dargestellten Funktion zukommen, und die das Verhalten dieser Funktion $f(s)$ für ins Unendliche wachsende Werte der Ordinate t betreffen. Zuerst nennen wir den Satz, daß jede solche Funktion $f(s)$ in der Halbebene $\sigma > \sigma_B + \varepsilon$ die Limesgleichung

$$(15) \qquad f(s) = f(\sigma + it) = o(|t|) \qquad (\text{für } |t| \to \infty)$$

erfüllt, sogar gleichmäßig in σ.[38]) Es bezeichne nunmehr hier (und überall im folgenden) σ_e ($\leqq \sigma_B$) *die untere Grenze aller Abszissen* σ_0,

---

36) *E. Landau*, Über einen Satz von Tschebyschef, Math. Ann. 61 (1905), p. 527—550. Verallgemeinerungen des *Landau*schen Satzes sind gegeben von *M. Fekete*, a) Sur les séries de Dirichlet, Paris C. R. 150 (1910), p. 1033—1036; b) Sur une théorème de M. Landau, Paris C. R. 151 (1910), p. 497—500.

Für die von *Landau* betrachteten Reihen mit $a_n > 0$ ist offenbar $\sigma_A = \sigma_B$; es sei beiläufig bemerkt, daß das bloße Bestehen dieser Gleichung $\sigma_A = \sigma_B$ *nicht* genügt um zu schließen, daß die Konvergenzgerade einen singulären Punkt enthält. *H. Bohr*, Über die Summabilität Dirichletscher Reihen, Gött. Nachr. 1909, p. 247—262.

37) So kennt man z. B. keinen allgemeinen Satz über gewöhnliche *Dirichlet*sche Reihen (2), der uns aus einfachen analytischen Eigenschaften der durch die Zetareihe mit abwechselndem Vorzeichen (5) definierten ganzen transzendenten Funktion $\zeta(s)(1 - 2^{1-s})$ darüber Aufschluß gibt, daß diese Reihe eben die Konvergenzabszisse $\sigma_B = 0$ besitzt. Anders verhält es sich, wie aus den späteren Ausführungen hervorgehen wird, mit der gleichmäßigen Konvergenzabszisse $\sigma_G = 1$ und der absoluten Konvergenzabszisse $\sigma_A = 1$ dieser Reihe.

38) *E. Landau*, Handbuch, Bd. 2, p. 824. Der Satz findet sich schon, wie von *Landau* angegeben, implizite bei *O. Perron*, a. a. O. 13). Wie von *H. Bohr*, Bidrag til de Dirichlet'ske Rækkers Theori, Habilitationsschrift, Kopenhagen 1910, p. 32, bewiesen, läßt sich die Gleichung $f(s) = o(|t|)$ durch keine Gleichung der Form $f(s) = o(|t|^{\alpha})$ mit $\alpha < 1$ ersetzen.

für welche $f(s)$ in der Halbebene $\sigma > \sigma_0$ regulär und von endlicher Größenordnung in bezug auf t ist, d. h. gleich $O(|t|^k)$ bei passender Wahl von $k = k(\sigma_0)$. Für jedes feste $\sigma > \sigma_e$ definieren wir alsdann die „Größenordnung" $\mu = \mu(\sigma)$ von $f(s)$ auf der vertikalen Geraden mit der Abszisse σ als die untere Grenze aller Zahlen α, für die $f(\sigma + it) = O(|t|^\alpha)$ ist. Die somit für $\sigma > \sigma_e$ definierte *Funktion* $\mu(\sigma)$ ist nach (15) gewiß $\leqq 1$ für $\sigma > \sigma_B$, und sie ist ferner, wie leicht zu sehen[39]), immer $\geqq 0$ für $\sigma > \sigma_B$. Die genaue Bestimmung der zu einer gegebenen *Dirichlet*schen Reihe gehörigen μ-Funktion ist im allgemeinen ein sehr schwieriges Problem. Doch läßt sich mit Hilfe der bekannten allgemeinen Sätze von *Phragmén* und *Lindelöf* (Artikel II C 4, Nr. **10**) über das Verhalten analytischer Funktionen in der Nähe einer wesentlich singulären Stelle (hier des Punktes $s = \infty$) leicht zeigen, *daß $\mu(\sigma)$ im ganzen Definitionsintervall $\sigma > \sigma_e$ eine stetige konvexe Funktion ist, die überall $\geqq 0$ ist, und die mit abnehmendem σ niemals abnimmt.* Wenn nicht nur $\sigma_B < \infty$, sondern auch $\sigma_G < \infty$ ist (was ja z. B. für jede gewöhnliche *Dirichlet*sche Reihe mit $\sigma_B < \infty$ der Fall ist), wird übrigens $\mu(\sigma)$ gleich 0 sein für alle hinreichend großen σ, nämlich mindestens für $\sigma > \sigma_G$.[40])

Kehren wir jetzt zum Konvergenzproblem zurück. *Landau*[41]) war der erste, der mit Erfolg die Frage angegriffen hat, inwiefern man aus der Kenntnis der Größenordnung der durch eine *Dirichlet*sche Reihe dargestellten Funktion (d. h. aus ihrer μ-Funktion) Schlüsse über die Lage der Konvergenzgeraden $\sigma = \sigma_B$ ziehen kann. Das Problem wurde später von *Schnee*[42]) in einer bedeutsamen Arbeit und von *Landau*[43]) selbst weiter verfolgt. Die Untersuchungen umfassen

39) *K. Ananda-Rau*, Note on a property of Dirichlet's series, London math. Soc. (2) 19 (1920), p. 114—116; *T. Jansson*, Über die Größenordnung Dirichletscher Reihen, Arkiv f. Mat., Astr. och Fys. 15 (1920), No. 6.

40) Die angeführten Resultate über die μ-Funktion finden sich im wesentlichen implizite bei *E. Lindelöf*, Quelques remarques sur la croissance de la fonction $\zeta(s)$, Bull. de Soc. math. (2) 32 (1908), p. 341—356. Vgl. auch *H. Bohr*, a. a. O. 38), p. 28—36; *G. H. Hardy-M. Riesz*, a. a. O. 1), p. 16—18, und die a. a. O. 39) erwähnten Abhandlungen.

Eine sich auf das Verhalten der oberen Grenze $L(\sigma)$ der Funktion $|f(s)|$ im Intervall $\sigma > \sigma_G$ beziehende Ergänzung des *Lindelöf*schen Satzes über die Konvexität der μ-Funktion ist von *G. Doetsch*, Über die obere Grenze des absoluten Betrages einer analytischen Funktion auf Geraden, Math. Ztschr. 8 (1920), p. 237—240, gegeben.

41) *E. Landau*, a. a. O. 29).

42) *W. Schnee*, Zum Konvergenzproblem der Dirichletschen Reihen, Math. Ann. 66 (1909), p. 337—349.

43) *E. Landau*, a) Über das Konvergenzproblem der Dirichletschen Reihen,

nicht den allgemeinsten Typus *Dirichlet*scher Reihen, sondern es wird der Exponentenfolge $\{\lambda_n\}$ die (für $\lambda_n = \log n$ erfüllte) Bedingung

$$\frac{1}{\lambda_{n+1} - \lambda_n} = O(e^{\lambda_n k}) \qquad (k > 0) \tag{16}$$

auferlegt, welche offenbar darauf hinausläuft, daß die Exponenten nirgends allzu dicht aufeinander folgen dürfen.[44]) Indem wir uns der Einfachheit halber auf die gewöhnlichen *Dirichlet*schen Reihen (2) beschränken, besagt das allgemeinste Resultat von *Landau* und *Schnee*: Es sei die Reihe (2) in einer gewissen Halbebene $\sigma > \sigma_0$ nicht nur absolut konvergent, sondern „so deutlich" absolut konvergent, daß $a_n n^{-\sigma_0}$ gleich $O(n^{-1+\varepsilon})$ bei jedem $\varepsilon > 0$ ist; es sei ferner die durch die Reihe dargestellte Funktion $f(s)$ für $\sigma > \sigma_0 - \alpha$ $(\alpha > 0)$ regulär und gleich $O(|t|^k)$. Dann konvergiert die Reihe jedenfalls für

$$\sigma > \sigma_0 - \frac{\alpha}{1+k}.$$

Hierin ist speziell das Resultat (von *Schnee*[43])) enthalten, daß, falls $f(s)$ für $\sigma > \sigma_1$ $(= \sigma_0 - \alpha)$ regulär und, bei jedem $\delta > 0$, gleich $O(|t|^\delta)$ ist, $\sigma_B \leqq \sigma_1$ ist, d. h. eine *Dirichlet*sche Reihe (2) ist mindestens so weit nach links konvergent, wie die zugehörige μ-Funktion gleich 0 ist. Die genannten Sätze geben, mit Hilfe der μ-Funktion, hinreichende Bedingungen für die Konvergenz der Reihe in einer gewissen Halbebene, aber keine Bedingungen, die zugleich notwendig und hinreichend sind. Solche Bedingungen gibt es aber überhaupt nicht, d. h. es ist nicht möglich, von der bloßen Kenntnis der μ-Funktion zu einer *genauen* Bestimmung der Konvergenzabszisse σ_B zu gelangen; in der Tat[45]), es existieren *Dirichlet*sche Reihen, sogar vom Typus (2), die dieselbe μ-Funktion, aber verschiedene Konvergenzabszissen σ_B besitzen.

Palermo Rend. 28 (1909), p. 113—151; b) Neuer Beweis eines Hauptsatzes aus der Theorie der Dirichletschen Reihen, Leipziger Ber. 69 (1917), p. 336—348.

44) Die Bedingung (16) ist übrigens nicht die von *Landau* und *Schnee* benutzte; sie wurde erst später von *H. Bohr,* Einige Bemerkungen über das Konvergenzproblem Dirichletscher Reihen, Palermo Rend. 37 (1914), p. 1—16, eingeführt, der zeigte, daß sie die für die betreffenden Untersuchungen „genau richtige" Bedingung ist, d. h. die für die Gültigkeit der *Landau-Schnee*schen Sätze notwendige und hinreichende.

Zur Orientierung sei bemerkt, daß eine Exponentenfolge $\{\lambda_n\}$, die der Bedingung (16) genügt, auch der Bedingung $\limsup \log n : \lambda_n < \infty$ genügt (aber nicht umgekehrt), so daß (nach Nr. 2) jede Reihe (1), die (16) erfüllt, gewiß ein *absolutes* Konvergenzgebiet besitzt, falls sie überhaupt ein Konvergenzgebiet besitzt.

45) *H. Bohr,* a. a. O. 38), p. 34.

Ganz anders verhält es sich mit dem Problem der Bestimmung der *gleichmäßigen* Konvergenzabszisse σ_G. Hier gilt nach *Bohr*[46]) der einfache Satz, daß jede *Dirichlet*sche Reihe (1), deren Exponentenfolge die Bedingung (16) erfüllt[47]), also speziell jede gewöhnliche *Dirichlet*sche Reihe (2), so weit nach links gleichmäßig konvergiert, wie von vornherein überhaupt gehofft werden konnte, d. h. es ist $\sigma_G = \sigma_b$, wo σ_b die oben definierte „Regularitäts- und Beschränktsheitsabszisse" bedeutet.

Es erübrigt die Frage nach dem Zusammenhang der Lage der *absoluten* Konvergenzgeraden $\sigma = \sigma_A$ mit den analytischen Eigenschaften der dargestellten Funktion zu erörtern. Diese Frage kann auch so gestellt werden, daß es sich um die Bestimmung der Breite des Streifens $\sigma_b \leqq \sigma \leqq \sigma_A$ handelt, in welchem die Funktion $f(s)$ über die absolute Konvergenzhalbebene hinaus regulär und beschränkt bleibt, und dann natürlich vor allem um den maximalen Wert dieser Breite bei gegebener Exponentenfolge $\{\lambda_n\}$. Diese letztere Frage, zu deren Behandlung Hilfsmittel ganz anderer Art herangezogen werden müssen als diejenigen, worauf die oben referierten Untersuchungen beruhen, wird am Ende der nächsten Nummer besprochen. Dabei werden wir uns wesentlich auf die gewöhnlichen *Dirichlet*schen Reihen (2) beschränken; bei diesen Reihen ist, nach dem obigen, $\sigma_b = \sigma_G$, und der besprochene Streifen $\sigma_b \leqq \sigma \leqq \sigma_A$ kann daher auch als derjenige Streifen charakterisiert werden, in welchem die Reihe gleichmäßig konvergiert ohne absolut zu konvergieren.

7. Anwendung der Theorie der diophantischen Approximationen. Die Rolle, welche die diophantischen Approximationen beim Studium der *Dirichlet*schen Reihen spielen, tritt am deutlichsten hervor bei der Aufgabe, die Menge der Werte zu bestimmen, welche eine gewöhnliche *Dirichlet*sche Reihe (2) annimmt, wenn die Variable s eine feste vertikale Gerade $\sigma = \sigma_0$ durchläuft. Hierbei umkreist offenbar jedes *einzelne* Glied, d. h. sein Bildpunkt in einer komplexen Ebene, einen festen *Kreis*; in der Tat, es ist, $a_n = \varrho_n e^{i\varphi_n}$ gesetzt,

$$\frac{a_n}{n^{\sigma_0+it}} = \frac{\varrho_n}{n^{\sigma_0}} \cdot e^{i\{\varphi_n - t\log n\}},$$

46) *H. Bohr*, a. a. O. 6a) und b).

47) Bei diesem Problem — im Gegensatz zu dem obigen — ist die Bedingung (16) übrigens *nicht* die „genau richtige", d. h. die für die Gültigkeit des Satzes notwendige und hinreichende. Eine wesentliche Erweiterung der Bedingung (16) ist von *E. Landau*, a. a. O. 32) gegeben. Vgl. hierzu auch *L. Neder*, a) a. a. O. 7); b) Zum Konvergenzproblem der Dirichletschen Reihen beschränkter Funktionen, Math. Ztschr. 14 (1922), p. 149—158.

wo der Modul $r_n = \varrho_n n^{-\sigma_0}$ nicht von t abhängt. Wie unmittelbar zu sehen, bewegen sich aber die Glieder nicht in der Weise „quasi unabhängig" voneinander jedes auf seinem Kreise, daß man bei passender Wahl der Variablen t erreichen kann, daß eine beliebig vorgegebene Anzahl N dieser Glieder beliebig nahe an N beliebig gegebene Punkte der entsprechenden N Kreisperipherien fallen; es ist ja dies z. B. für die drei Glieder $\frac{a_2}{2^s}, \frac{a_3}{3^s}, \frac{a_6}{6^s}$ gewiß nicht der Fall, denn aus der Gleichung $\frac{1}{2^s} \cdot \frac{1}{3^s} = \frac{1}{6^s}$ folgt sofort, daß, wenn die Bildpunkte der beiden ersten Glieder „sehr" nahe an zwei festen Punkten P_2 und P_3 auf ihren respektiven Kreisen liegen, der Bildpunkt des dritten Gliedes von selbst sehr nahe an einen festen, von P_2 und P_3 abhängigen, Punkt P_6 auf seiner Kreisperipherie fallen wird. Betrachten wir aber nicht die Größen $\frac{1}{n^s}$, wo n die sämtlichen Zahlen $1, 2, 3 \cdots$ durchläuft, sondern nur die Größen $\frac{1}{p_n^s}$, wo p_n die *Primzahlen* $2, 3, 5 \cdots$ durchläuft, so stellt die Sache sich ganz anders. Hier können wir nämlich, bei passender Wahl von t, erreichen, daß die Bildpunkte der N Größen $\frac{1}{2^s}, \frac{1}{3^s} \cdots \frac{1}{p_N^s}$ mit beliebig vorgegebener Genauigkeit in N beliebig gegebene Punkte ihrer N Kreisperipherien fallen; die Amplituden dieser Größen sind nämlich durch $-t \log 2$, $-t \log 3, \cdots -t \log p_N$ gegeben, und weil die Primzahllogarithmen — wegen der eindeutigen Zerlegbarkeit einer ganzen Zahl in Primfaktoren — im rationalen Körper *linear unabhängig* sind, können die genannten N Amplituden nach einem berühmten *Kronecker*schen Satz über diophantische Approximationen beliebig nahe (modulo 2π) an N beliebig gegebene Größen gebracht werden. Von dieser Bemerkung ausgehend hat *Bohr*[48]) die Bedeutung der diophantischen Approximationen für verschiedene Probleme in der Theorie der *Dirichlet*schen Reihen gezeigt; es sollen im folgenden die wesentlichsten Resultate dieser Untersuchung kurz angegeben werden.

Es bezeichne $p_{n_1}^{\nu_1}\, p_{n_2}^{\nu_2} \cdots p_{n_r}^{\nu_r}$ die Zerlegung der ganzen Zahl n in Primfaktoren, und es sei in der beliebig gegebenen gewöhnlichen *Dirichlet*schen Reihe

$$\sum \frac{a_n}{n^s} = \sum a_n \left(\frac{1}{p_{n_1}^s}\right)^{\nu_1} \left(\frac{1}{p_{n_2}^s}\right)^{\nu_2} \cdots \left(\frac{1}{p_{n_r}^s}\right)^{\nu_r}$$

48) Vgl. insb. *H. Bohr*, Über die Bedeutung der Potenzreihen unendlich vieler Variabeln in der Theorie der Dirichletschen Reihen $\sum \frac{a_n}{n^s}$, Gött. Nachr. 1913, p. 441—488.

$\frac{1}{p_1^s} = x_1, \frac{1}{p_2^s} = x_2, \cdots \frac{1}{p_m^s} = x_m \cdots$ gesetzt, wodurch die Reihe die Form annimmt:

$$P(x_1, x_2, \cdots x_m \cdots) = \sum_{n=1}^{\infty} a_n x_{n_1}^{\nu_1} x_{n_2}^{\nu_2} \cdots x_{n_r}^{\nu_r}$$

$$= c + \sum c_\alpha x_\alpha + \sum c_{\alpha,\beta} x_\alpha x_\beta + \sum c_{\alpha,\beta,\gamma} x_\alpha x_\beta x_\gamma + \cdots$$

wo $c = a_1$, $c_\alpha = a_{p_\alpha}$, $c_{\alpha,\beta} = a_{p_\alpha p_\beta}$, $\cdots$ ist. Hier sind vorläufig die Größen x_m alle Funktionen der einen Variablen s. Nun denken wir uns aber — weil ja oben gesehen wurde, daß die $x_m = p_m^{-s}$ sich in gewisser Beziehung „fast" so benehmen, als wären sie unabhängig voneinander — das Band zwischen den x_m ganz aufgelöst, d. h. wir fassen die x_m als *voneinander unabhängige* Variablen auf. Die obige Reihe $P(x_1, x_2, \cdots x_m \cdots)$ wird dann offenbar *eine Potenzreihe in den unendlich vielen Variabeln* $x_1, x_2, \cdots$, von der wir sagen werden, daß sie der gegebenen *Dirichlet*schen Reihe (2) entspricht. Betreffs der am Anfang des Paragraphen gestellten Frage nach *dem Verhalten der Reihe* (2) *auf einer vertikalen Geraden* $\sigma = \sigma_0$ ergibt sich dann der Satz: Es sei $\sigma_0 > \sigma_A$ (oder nur $\sigma_0 > \sigma_G$), und es bezeichne $U(\sigma_0)$ bzw. $W(\sigma_0)$ die Menge der Werte, welche die Reihe $f(s)$ auf bzw. in unendlicher Nähe[49]) der Geraden $\sigma = \sigma_0$ annimmt. Ferner bezeichne $M = M(\sigma_0)$ die Menge der Werte, welche die der *Dirichlet*schen Reihe entsprechende Potenzreihe $P(x_1, x_2 \cdots)$ annimmt, wenn die Variabeln $x_1, x_2 \cdots$ *unabhängig voneinander* die Kreise $|x_m| = p_m^{-\sigma_0}$ $(m = 1, 2 \cdots)$ durchlaufen. Dann gilt, 1. daß die Menge U in der Menge M *überall dicht* liegt, und 2. daß die Menge W mit der Menge M *identisch* ist. Die Wirkungsweise dieses Satzes wird durch seine später zu erwähnende Anwendung auf die Zetareihe deutlich hervorgehen.

Über die (in Nr. 6 erwähnte) Frage nach *der oberen Grenze T der Differenz $\sigma_A - \sigma_b$ für alle Dirichletschen Reihen* (2), findet man ferner mit Hilfe der Theorie der diophantischen Approximationen den Satz: Es ist

$$T = \frac{1}{S},$$

wo S die obere Grenze aller positiven Zahlen α mit der Eigenschaft bezeichnet, daß jede in einem Gebiete $|x_m| \leqq G_m$ $(m = 1, 2 \cdots)$ *beschränkte*[50]) Potenzreihe $P(x_1, x_2 \cdots)$ im Gebiet $|x_m| \leqq \varepsilon_m G_m (m = 1, 2 \ldots)$

---

49) Dies letzte so zu verstehen, daß eine Zahl w dann und nur dann zur Menge $W(\sigma_0)$ gehört, falls die Gleichung $f(s) = w$ in jedem Streifen $\sigma_0 - \varepsilon < \sigma < \sigma_0 + \varepsilon$ eine Lösung besitzt.

50) Eine Potenzreihe $P(x_1, x_2 \ldots)$ in unendlichvielen Variabeln heißt — nach *D. Hilbert*, Wesen und Ziele einer Analysis der unendlichvielen unabhängigen

absolut konvergiert, wenn nur $\Sigma \varepsilon_m^{\alpha}$ konvergiert (und $0 < \varepsilon_m < 1$). Es ist hierdurch die Bestimmung der „Maximalbreite" T auf die Bestimmung der (in der Theorie der Potenzreihen wesentlichen) Konstanten S zurückgeführt. Über diese Konstante S findet man sofort, daß sie $\geqq 2$ ist, woraus folgt, daß $T \leqq \frac{1}{2}$ ist.[51]) Die besonders wichtige Frage, ob nicht $T = 0$ ist (d. h. ob nicht immer $\sigma_A = \sigma_b$ ist), wurde von *Toeplitz*[52]) gelöst, der durch Untersuchungen über quadratische Formen mit unendlichvielen Variabeln zeigte, daß $S \leqq 4$, also $T \geqq \frac{1}{4}$ ist. Das Problem, S (und damit T) genau zu bestimmen, ist noch ungelöst.

Ein bemerkenswertes Resultat ergibt sich, wenn man den besprochenen Zusammenhang zwischen *Dirichlet*schen Reihen und Potenzreihen mit unendlichvielen Variabeln nicht auf die allgemeinen *Dirichlet*schen Reihen vom Typus (2), sondern auf zwei *spezielle Klassen* solcher Reihen anwendet, nämlich auf diejenigen Reihen (2), die formal eine Zerlegung in Addenden bzw. in Faktoren derart zulassen, daß dadurch *die einzelnen Primzahlen separiert werden*, d. h deren Koeffizienten entweder die Bedingung: $a_n = 0$ für alle n, die mindestens zwei verschiedene Primzahlen enthalten, oder die Bedingung: $a_m a_l = a_{ml}$ für teilerfremde m und l erfüllen. Für diese beiden Typen *Dirichlet*scher Reihen — die übrigens fast alle in der analytischen Zahlentheorie vorkommenden Reihen (2) umfassen — gilt immer die Gleichung $\sigma_A = \sigma_b$, d. h. eine jede *Dirichlet*sche Reihe einer dieser Typen ist (im Gegensatz zu einer beliebigen Reihe (2)) *genau so weit*

Variabeln, Palermo Rend. 27 (1909), p. 59—74 — *beschränkt* in einem Gebiete $|x_m| \leqq G_m (m = 1, 2 \cdots)$, wenn 1. bei jedem festen m der m^{te} „Abschnitt" $P_m(x_1, \cdots x_m)$ im Gebiete $|x_1| \leqq G_1, \cdots |x_m| \leqq G_m$ absolut konvergiert, und 2. eine absolute Konstante K derart existiert, daß bei jedem m und $|x_1| \leqq G_1, \cdots |x_m| \leqq G_m$ die Ungleichung $|P_m(x_1, \cdots x_m)| < K$ besteht.

51) *H. Bohr*, a. a. O. 48). Dies spezielle Resultat $T \leqq \frac{1}{2}$ ist, wie *G. H. Hardy*, The application of Abel's method of summation to Dirichlet's series, Quart. J. of math. 47 (1916), p. 176—192 gezeigt hat, kein tiefliegendes, d. h. es läßt sich auch ohne Zurückgreifen auf die Theorie der Potenzreihen mit unendlichvielen Variabeln leicht herleiten. Vgl. auch eine interessante Note von *F. Carlson*, Sur les séries de Dirichlet, Paris C. R. 172 (1921), p. 838—840, und die eben erschienene Arbeit von *K. Grandjot*, Über das absolute Konvergenzproblem der Dirichletschen Reihen, Diss. Göttingen 1922, in welcher ein dem *Schnee-Landau*schen Satze über das Konvergenzproblem (vgl. Nr. 6) entsprechender Satz über das absolute Konvergenzproblem abgeleitet wird.

52) *O. Toeplitz*, Über eine bei den Dirichletschen Reihen auftretende Aufgabe aus der Theorie der Potenzreihen von unendlichvielen Veränderlichen, Gött. Nachr. 1913, p 417—432.

absolut konvergent, wie die dargestellte Funktion regulär und beschränkt bleibt.[53])

Die oben erwähnten Untersuchungen können von den gewöhnlichen *Dirichlet*schen Reihen (2) auf den allgemeinen Typus (1) erweitert werden.[54]) Eine ähnliche Rolle, wie die von den Primzahllogarithmen gebildete Zahlenfolge für die spezielle Exponentenfolge $\{\lambda_n = \log n\}$, spielt im Falle einer *beliebigen Exponentenfolge* $\{\lambda_n\}$ eine sogenannte *Basis* dieser Folge $\{\lambda_n\}$, d. h. eine (aus endlich oder abzählbarvielen Zahlen bestehende) Folge von *linear unabhängigen* Zahlen $\beta_1, \beta_2, ..$ mit der Eigenschaft, daß jeder der Exponenten λ_n als lineare Funktion endlichvieler β mit rationalen Koeffizienten darstellbar ist. Ein besonders einfacher Fall liegt vor, wenn die Exponenten λ_n *selbst* linear unabhängig sind (also selbst eine Basis bilden). Hier gilt ganz allgemein der Satz, daß $\sigma_A = \sigma_b$ ist.[55]) Dies ist die Verallgemeinerung eines obigen Satzes über gewöhnliche *Dirichlet*sche Reihen (2), nach welchem die Gleichung $\sigma_A = \sigma_b$ immer gilt, wenn $a_n = 0$ ist für zusammengesetztes n.

8. Über die Darstellbarkeit einer Funktion durch eine Dirichletsche Reihe. Beim Konvergenzproblem in Nr. 6 (und Nr. 7) waren wir von einer Funktion $f(s)$ ausgegangen, von der *vorausgesetzt* wurde, daß sie in einer gewissen Halbebene durch eine *Dirichlet*sche Reihe dargestellt war, und es handelte sich darum, die Lage der Konvergenzabszissen dieser Reihe aus den analytischen Eigenschaften der Funktion zu bestimmen. Mit dieser Frage verwandt, aber davon wesentlich zu trennen, ist die Frage, welche Bedingungen eine in einer gewissen Halbebene $\sigma > \sigma_0$ *beliebig gegebene* analytische Funktion erfüllen muß, damit sie überhaupt in eine (dort konvergente) *Dirichlet*sche Reihe entwickelt werden kann. Es liegt hierbei nahe, von dem Satze über

53) Der „Grund", weshalb die Zetareihe mit abwechselndem Vorzeichen (die ja der Bedingung $a_m a_l = a_{ml}$ genügt) die absolute Konvergenzabszisse $\sigma_A = 1$ besitzt, ist also, daß die durch die Reihe dargestellte (ganze transzendente) Funktion $\zeta(s)(1-2^{1-s})$ nicht über die Gerade $\sigma = 1$ hinaus beschränkt bleibt.

54) *H. Bohr*, Zur Theorie der allgemeinen Dirichletschen Reihen, Math. Ann. 79 (1919), p. 136—156.

55) *H. Bohr*, Lösung des absoluten Konvergenzproblems einer allgemeinen Klasse Dirichletscher Reihen, Acta Math. 36 (1913), p 197—240. Bei diesem Satze über die Bestimmung der *absoluten* Konvergenzabszisse σ_A ist bemerkenswert, daß — im Gegensatze zu den Sätzen in Nr. 6 über die Konvergenzabszisse σ_B und die gleichmäßige Konvergenzabszisse σ_G — überhaupt keine Bedingung über die „ungefähre" Lage der λ_n (z. B daß sie nicht allzu dicht aufeinander folgen dürfen) nötig ist, sondern nur die angegebene *arithmetische* Bedingung der linearen Unabhängigkeit, welche ja die „genaue" Lage der λ_n betrifft.

die Koeffizientendarstellung in Nr. 4 auszugehen, welcher die Koeffizienten und Exponenten der Reihe von der Funktion aus bestimmt, und zu untersuchen, ob nicht etwa die Konvergenz und streckenweise Konstanz des dort vorkommenden Integrals $J(x) = \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} \frac{e^{xs} f(s)}{s} ds$ für die Entwickelbarkeit einer Funktion $f(s)$ in eine *Dirichlet*sche Reihe genügt. Es zeigt sich nun, daß eine solche unmittelbare Umkehrung des Satzes in Nr. 4 *nicht* gilt[56]), daß sie aber *unter gewissen einschränkenden Bedingungen* gelingt.[57]) Die hierdurch gewonnenen Resultate sind jedoch von einem etwas komplizierten Charakter, und es zeigen überhaupt viele Eigenschaften der *Dirichlet*schen Reihen, daß dieser Reihentypus zur Darstellung von Funktionen allgemeinen Charakters nicht geeignet ist. In diesem Zusammenhange ist vor allem eine schöne Arbeit von *Ostrowski*[58]) zu erwähnen, worin zunächst der Satz bewiesen wird, daß eine durch eine *Dirichlet*sche Reihe (1) dargestellte Funktion $f(s)$ nur in dem sehr speziellen Fall einer *algebraischen „Differenzendifferentialgleichung“* genügen kann, in welchem die Exponentenfolge $\{\lambda_n\}$ eine *endliche* lineare Basis besitzt.[59]) Bei den weiteren Untersuchungen von *Ostrowski* erweist es sich als bequem, die Transformation $e^{-s} = x$ auszuführen, also statt einer *Dirichlet*schen Reihe (1) die entsprechende „irreguläre“ Potenzreihe

$$F(x) = \sum_{n=1}^{\infty} a_n x^{\lambda_n}$$

zu betrachten, die offenbar im Punkte $x = 0$ (welcher $\sigma = +\infty$ ent-

---

56) Vgl *O. Perron*, a. a. O. 13) und *E. Landau*, Handbuch, p. 833.

57) Vgl. *J. Hadamard*, a) Sur les séries de Dirichlet, Palermo Rend. 25 (1908), p. 326—330; b) Rectification à la note „Sur les séries de Dirichlet“, Palermo Rend. 25 (1908), p. 395—396 und insbesondere die Abhandlungen von *W. Schnee*, a. a O 9) und *M. Fujiwara*, Über Abelsche erzeugende Funktion und Darstellbarkeitsbedingung von Funktionen in Dirichletschen Reihen, Tôhoku J. 17 (1920), p 363 bis 383. In anderer Richtung liegt eine Untersuchung von *J. Steffensen*, Eine notwendige und hinreichende Bedingung für die Darstellbarkeit einer Funktion als Dirichletsche Reihe, Nyt Tidskr. f. Mat. 1917, p. 9—11.

58) *A. Ostrowski*, Über Dirichletsche Reihen und algebraische Differentialgleichungen, Math. Ztschr. 8 (1920), p. 241—298.

59) Für den speziellen Fall der *Zetafunktion* war es schon durch *D. Hilbert*, Sur les problèmes futurs des Mathématiques, C. R. du 2 congr. intern. d. math. Paris 1902, p. 58—114, bekannt, daß sie keiner algebraischen Differentialgleichung genügt. Vgl. auch *V. Stadigh*, Ein Satz über Funktionen, die algebraische Differentialgleichungen befriedigen, und über die Eigenschaft der Funktion $\zeta(s)$ keiner solchen Gleichung zu genügen, Dissertation Helsingfors 1902.

spricht) einen Verzweigungspunkt unendlich hoher Ordnung besitzt. Die Frage nach den Funktionen $f(s)$, welche in eine *Dirichlet*sche Reihe entwickelt werden können, tritt dann hier in der Gestalt auf, welche Art von Singularitäten im Punkte $x = 0$ durch eine irreguläre Potenzreihe bewältigt werden können. *Ostrowski* zeigt nun u. a., daß nur in dem oben genannten speziellen Fall, wo die Exponentenfolge eine endliche lineare Basis besitzt, die durch eine solche irreguläre Potenzreihe dargestellte Funktion $F(x)$ einer an der Stelle $x = 0$ *analytischen* Differentialgleichung genügen kann. Durch diesen Satz tritt deutlich zutage, wie „schwer" die Singularität ist, die eine *Dirichlet*sche Reihe im unendlichfernen Punkte besitzt.

9. Der Mittelwertsatz. Aus der Gleichung

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} e^{i\alpha t}dt = \begin{cases} 0 \text{ für reelles } \alpha \neq 0 \\ 1 \text{ für } \alpha = 0 \end{cases}$$

folgt sofort durch formales Rechnen, daß, wenn

$$f(s) = \sum a_n e^{-\lambda_n s},\quad g(s) = \sum b_n e^{-\lambda_n s}$$

zwei beliebige (zur selben λ-Folge gehörige) *Dirichlet*sche Reihen sind, die Gleichung gilt

$$(17)\qquad \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(\sigma_1 + it)g(\sigma_2 - it)dt = \sum a_n b_n e^{-\lambda_n(\sigma_1+\sigma_2)},$$

worin speziell, $b_n = \bar{a}_n$ und $\sigma_1 = \sigma_2$ entsprechend, die Gleichung

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} |f(\sigma_1 + it)|^2 dt = \sum |a_n|^2 e^{-2\lambda_n \sigma_1}$$

enthalten ist. *Hadamard*[60]), der zuerst auf die Gleichung (17) hingewiesen hat, hat ihre Gültigkeit für den Fall bewiesen, in dem die zwei Reihen auf den Geraden $\sigma = \sigma_1$ bzw. $\sigma = \sigma_2$ *absolut* konvergieren, und *Landau*[61]) und *Schnee*[62]) haben später (unter einer gewissen einschränkenden Bedingung über die Dichte der λ-Folge) bewiesen, daß die Formel auch in anderen allgemeinen Fällen gültig bleibt. Als ein für die Anwendungen (z. B. auf die Zetafunktion) besonders wichtiges

60) *J. Hadamard*, Théorème sur les séries entières, Acta Math. 22 (1899), p. 55—63.

61) *E. Landau*, a) a. a. O. 29); b) Neuer Beweis des Schneeschen Mittelwertsatzes über Dirichletsche Reihen, Tôhoku J. 20 (1922), p. 125—130.

62) *W. Schnee*, Über Mittelwertsformeln in der Theorie der Dirichletschen Reihen, Wiener Sitzungsber. (IIa) 118 (1909), p. 1439—1522.

Beispiel der *Landau-Schnee*schen Resultate nennen wir den sogenannten *Schneeschen Mittelwertsatz für gewöhnliche Dirichletsche Reihen* (2), der besagt, daß die Gleichung

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f(\sigma_1+it)|^2dt=\sum\frac{|a_n|^2}{n^{2\sigma_1}}$$

für jedes $\sigma_1 > \frac{1}{2}(\sigma_A+\sigma_B)$ besteht (aber im allgemeinen *nicht* für $\sigma_1 \leqq \frac{1}{2}(\sigma_A+\sigma_B)$).

Aus der Gleichung (17) folgt ferner (indem $g(s)$ gleich $e^{-\lambda_n s}$ und $\sigma_2 = -\sigma_1$ gesetzt wird) die *Koeffizientendarstellungsformel*[63])

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f(\sigma_1+it)e^{\lambda_n(\sigma_1+it)}dt=a_n;$$

diese Formel gilt nach *Landau*[61a]) bei jedem $\sigma > \sigma_B$, und nach *Schnee*[62]) konvergiert der Ausdruck auf der linken Seite sogar *gleichmäßig* in n (unter der oben erwähnten einschränkenden Bedingung über $\{\lambda_n\}$).

10. Über die Nullstellen einer Dirichletschen Reihe. Schon bei Besprechung des Eindeutigkeitssatzes in Nr. 3 wurde die Frage nach der Verteilung der *Nullstellen* einer *Dirichlet*schen Reihe berührt, indem gezeigt wurde, daß gewisse Gebiete der Konvergenzhalbebene nullpunktsfrei sind. Die erste allgemeine Untersuchung des Problems, *wie viel* Nullstellen eine *Dirichlet*sche Reihe in einer Halbebene $\sigma > \sigma_0$ $(> \sigma_B)$ besitzen kann, rührt von *Landau*[64]) her, der mit Hilfe des bekannten *Jensen*schen Satzes bewies, daß für jede *gewöhnliche Dirichletsche* Reihe (2) die Anzahl $n(\sigma_B+\varepsilon, T)$ der im Gebiete $\sigma > \sigma_B+\varepsilon$, $T < t < T+1$ gelegenen Nullstellen gleich $O(\log T)$ und also die Anzahl $N(\sigma_B+\varepsilon, T)$ von Nullstellen im Gebiete $\sigma > \sigma_B+\varepsilon$, $0 < t < T$ gleich $O(T\log T)$ ist. Für *beliebige Dirichlet*sche Reihen (1) bewies *Landau*[64a]) einen entsprechenden Satz, wo nur $\log T$ durch $\log^2 T$ ersetzt ist; später hat *Landau*[64b]) gezeigt, daß in der Formel $n(\sigma_B+\varepsilon, T) = O(\log^2 T)$ der Buchstabe O durch o ersetzt werden kann, während *Wennberg*[33]) bewiesen hat, daß man in der *Landau*schen Formel $N(\sigma_B+\varepsilon, T) = O(T\log^2 T)$ ganz allgemein, d. h. für jede *Dirichlet*sche Reihe (1), $\log^2 T$ durch $\log T$ ersetzen kann, so daß wir also für $N(\sigma_B+\varepsilon, T)$ (aber nicht für $n(\sigma_B+\varepsilon, T)$) genau dieselbe Formel bekommen, wie für die gewöhnlichen Reihen (2).

63) Vgl. Note 17).

64) a) *E. Landau*, Über die Nullstellen Dirichletscher Reihen, Berliner Sitzungsber. 14 (1913), p. 897—907; b) Über die Nullstellen Dirichletscher Reihen, Math. Ztschr. 10 (1921), p. 128—129.

Tiefer — weil auf dem *Schnee*schen Mittelwertsatze beruhend — liegt ein Satz von *Bohr* und *Landau*[65a]) über *gewöhnliche Dirichlet*sche Reihen (2), welcher besagt, daß bei jedem $\sigma_1 > \sigma_B + \frac{1}{2}$ die Relation $N(\sigma_1, T) = O(T)$ besteht.[66]) Dieser Satz läßt sich nicht verbessern; wohl aber gilt[65b]) für gewisse *spezielle*, für die zahlentheoretischen Anwendungen besonders wichtige Reihen (2), daß der Ausdruck $O(T)$ durch $o(T)$ ersetzt werden kann. Im Anschluß an diese Untersuchungen hat *Carlson*[67]) einen allgemeinen Satz über die Anzahl der Nullstellen einer gewöhnlichen *Dirichlet*schen Reihe gefunden, von dem ein (wegen Anwendung auf die Zetafunktion) besonders wichtiger Spezialfall so lautet: In der Reihe $f(s) = \sum \frac{a_n}{n^s}$ sei $a_1 \neq 0$, und es sei σ_B etwa gleich 0; es mögen ferner die Koeffizienten b_n der (formal entwickelten) Reihe $1 : f(s) = \sum \frac{b_n}{n^s}$ die Bedingung $\lim |b_n| : \log n = 0$ erfüllen. Dann ist bei jedem $\varepsilon > 0$ die Anzahl $N(\frac{1}{2} + \varepsilon, T)$ nicht nur gleich $o(T)$, sondern sogar gleich $O(T^{1-4\varepsilon^2+\delta})$, wo δ beliebig klein ist.

Mit Hilfe von Sätzen aus dem *Picard-Landau*schen Satzkreis lassen sich ferner verschiedene interessante Resultate über den Wertvorrat einer *Dirichlet*schen Reihe (1) ableiten. So ergibt sich nach *Lindelöf*[68]), daß, falls $\sigma_b < \infty$ ist, und $f(s)$ für $\sigma > \sigma_b - \varepsilon$ regulär (also dann gewiß nicht beschränkt) bleibt, $f(s)$ in jedem Streifen um die Gerade $\sigma = \sigma_b$ *sämtliche Werte, höchstens mit einer einzigen Ausnahme* annimmt. Dasselbe Resultat gilt in jedem Streifen um die Gerade

65) *H. Bohr* und *E. Landau*, a) Ein Satz über Dirichletsche Reihen mit Anwendung auf die ζ-Funktion und die L-Funktionen, Palermo Rend. 37 (1914), p. 269—272; b) Sur les zéros de la fonction $\zeta(s)$ de Riemann, Paris C. R. 158 (1914), p 106—110.

66) Dieselbe Relation $N(\sigma_1, T) = O(T)$ gilt nach *Wennberg* 33) für eine beliebige *Dirichlet*sche Reihe (1), wenn $\sigma_1 > \sigma_b$ angenommen wird, und sie ist hier (wie *Wennberg* mit Hilfe diophantischer Approximationen beweist) die bestmögliche in dem Sinne, daß, falls die Reihe in der Halbebene $\sigma > \sigma_b + \varepsilon$ überhaupt eine Nullstelle besitzt, die Anzahl $N(\sigma_b + \varepsilon, T) \neq o(T)$ ist.

67) *F. Carlson*, Über die Nullstellen der Dirichletschen Reihen und der Riemannschen ζ-Funktion, Arkiv für Mat., Ast. och Fys. 15 (1920), No. 20. Vgl. auch *E. Landau*, Über die Nullstellen der Dirichletschen Reihen und der Riemannschen ζ Funktion, Arkiv för Mat., Ast. och Fys. 16 (1921), No. 7, der mit Hilfe einer neuen Beweismethode des *Schnee*schen Mittelwertsatzes (vgl. 61 b) einen abgekürzten Beweis des *Carlson*schen Satzes gibt.

68) *E. Lindelöf*, Mémoire sur certaines inégalités dans la théorie des fonctions monogènes, et sur quelques propriétés nouvelles de ces fonctions dans le voisinage d'un point singulier essentiel, Acta soc. sc. Fenn. 35 (1908), No. 7. Vgl. auch *H. Bohr* und *E. Landau*, Über das Verhalten von $\zeta(s)$ und $\zeta_\varkappa(s)$ in der Nähe der Geraden $\sigma = 1$, Gött. Nachr. 1910. p. 303—330.

$\sigma = \sigma_e$, falls $f(s)$ für $\sigma > \sigma_e - \varepsilon$ regulär ist.[69]) Ferner wird, nach *Wennberg*[33]), jede *Dirichlet*sche Reihe mit $\sigma_b = \infty$, in jeder Halbebene $\sigma > \sigma_0$ sämtliche Werte, höchstens mit einer Ausnahme, annehmen. Schließlich sei noch erwähnt, daß jede *Dirichlet*sche Reihe mit *linear unabhängiger* Exponentenfolge (und also mit $\sigma_b = \sigma_A$), falls sie nicht in der ganzen Halbebene $\sigma > \sigma_b$ beschränkt ist, in dieser Halbebene überhaupt *jeden* Wert unendlich oft annimmt.[55])

11. Zusammenhang verschiedener Dirichletscher Reihen. Es seien

(18a) $$f(s) = \sum a_n e^{-\lambda_n s} \qquad (s = \sigma + it)$$

und

(18b) $$F(z) = \sum a_n e^{-\mu_n z} \qquad (z = x + iy)$$

zwei *Dirichlet*sche Reihen mit *denselben Koeffizienten* a_n, deren *Exponenten durch die Relation* $\mu_n = e^{\lambda_n}$ *verbunden sind.* Wie von *Cahen*[5]) gezeigt, besteht ein interessanter Zusammenhang zwischen den beiden Funktionen $f(s)$ und $F(z)$, indem jede von ihnen durch ein bestimmtes Integral dargestellt werden kann, dessen Integrand in einfacher Weise von der anderen der beiden Funktionen abhängt. Formal ergeben sich diese Darstellungen sehr leicht aus der Integraldarstellung der Γ-Funktion

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} dx \qquad (\sigma > 0)$$

und ihrer im *Mellin*schen Sinne „reziproken“ Formel[70])

$$e^{-z} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s) z^{-s} ds \qquad (c > 0,\ x > 0).$$

In der Tat, es lassen sich diese beiden Formeln (nach einer einfachen Transformation) so schreiben:

$$e^{-\lambda_n s}\Gamma(s) = \int_0^\infty x^{s-1} e^{-\mu_n x} dx, \quad e^{-\mu_n z} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s) z^{-s} e^{-\lambda_n s} ds,$$

69) Ein Beweis findet sich (implizite) bei *H. Bohr,* Über die Summabilitätsgrenzgerade der Dirichletschen Reihen, Wiener Sitzungsber. (IIa) 119 (1910), p. 1391—1397.

70) Diese Formel, deren große Bedeutung sich in den Untersuchungen von *H. Mellin* gezeigt hat, ist (nach *Mellin,* Bemerkungen im Anschluß an den Beweis eines Satzes von *Hardy* über die Zetafunction, Ann. Acad. sc. Fenn. (A) 11 (1917), No. 3) schon von *S. Pincherle,* Sulle funzioni ipergeometriche generalizzate, Rend Ac. Linc. 4 (1888), p. 694—700 in etwas anderer Form angegeben. Auch in neueren Arbeiten von *Hardy* und *Littlewood* (vgl. z. B. a. a. O. 31) spielt diese „*Cahen-Mellin*sche Formel“ eine wichtige Rolle.

und hieraus folgen sofort (durch Multiplikation mit a_n und Summation) die gesuchten Integraldarstellungen

(19a) $$f(s) = \frac{1}{\Gamma(s)}\int_0^\infty x^{s-1}F(x)\,dx$$

und

(19b) $$F(z) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s)z^{-s}f(s)\,ds.$$

Bei *Cahen* waren die Konvergenzuntersuchungen noch nicht streng durchgeführt. Dies geschah erst durch *Perron*[71]), der den Satz bewies: Wenn die Reihe (18a) für $\sigma > \sigma_0 > 0$ konvergiert (woraus leicht folgt, daß (18b) mindestens für $x > 0$ konvergiert), so gilt die Formel (19a) für $\sigma > \sigma_0$, und die Formel (19b) bei festem $c > \sigma_0$ für $x > 0$. Im Spezialfalle $\lambda_n = \log n$, $\mu_n = n$ haben wir es mit einer *gewöhnlichen Dirichletschen Reihe* $f(s) = \sum \frac{a_n}{n^s}$ und einer einfachen *Potenzreihe* $F(z) = \sum a_n e^{-nz}$ zu tun.[72]) Ist außerdem noch $a_n = 1$ für alle n, wird $f(s) = \zeta(s)$ und $F(z) = \frac{1}{e^z - 1}$, und wir erhalten aus der obigen Formel (19a) die von *Riemann*[73]) benutzte wichtige Integraldarstellung der Zetafunktion

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{x^{s-1}}{e^x - 1}\,dx \qquad (\sigma > 1).^{74})$$

---

71) *O. Perron*, a. a. O. 13). Vgl. auch *G. H. Hardy*, On a case of term-by-term integration of an infinite series, Mess. of Math. 39 (1910), p. 136—139.

72) Wie aus der Integraldarstellung (19a) ersichtlich ist, hängt das analytische Verhalten der *Dirichlet*schen Reihe $f(s) = \sum \frac{a_n}{n^s}$ mit dem Verhalten der, für $x > 0$ konvergenten, Potenzreihe $F(z) = \sum a_n e^{-nz}$ bei Annäherung an den Punkt $z = 0$, d. h. mit dem Verhalten der (für $|u| < 1$ konvergenten) Potenzreihe $\varphi(u) = \sum a_n u^n$ bei Annäherung an den Punkt $u = 1$, eng zusammen. Vgl. hierüber *G. H. Hardy*, The application to Dirichlet's series of Borel's exponential method of summation, Lond. math. Soc. (2) 8 (1909), p. 277—294 und *M. Fekete*, a. a. O. 36). So besteht z. B. der Satz [*A. Hurwitz*, Über die Anwendung eines funktionentheoretischen Principes auf gewisse bestimmte Integrale, Math. Ann. 53 (1900), p. 220—224], daß, falls $\varphi(u)$ im Punkte $u = 1$ *regulär* ist, $f(s)$ gewiß eine *ganze* Funktion ist. Auch die später zu erwähnende Untersuchung von *Hardy* über *Abel*sche Summabilität *Dirichlet*scher Reihen (2) basiert auf der Verbindung zwischen $f(s)$ und $\varphi(u)$.

73) *B. Riemann*, Über die Anzahl der Primzahlen unter einer gegebenen Größe, Berliner Monatsber 1859, p. 671—680 = Werke (2. Aufl.), p. 145—153.

74) Eine von der Integraldarstellung (19a) wesentlich verschiedene Integraldarstellung einer allgemeinen *Dirichlet*schen Reihe $f(s) = \sum a_n e^{-\lambda_n s}$ ist von

Bei dem oben besprochenen Zusammenhang zweier *Dirichlet*scher Reihen handelte es sich um Reihen mit denselben Koeffizienten, aber verschiedenen Exponenten. Wie von *Cramér*[75]) gezeigt, besteht auch ein gewisser Zusammenhang zwischen zwei *Dirichlet*schen Reihen $f(s) = \sum a_n e^{-\lambda_n s}$ und $g(s) = \sum b_n e^{-\lambda_n s}$ mit *denselben Exponenten*, deren Koeffizienten a_n und b_n aber derart voneinander abhängen, *daß* $b_n = a_n \varphi(\lambda_n)$ *ist, wo* $\varphi(z)$ *eine ganze transzendente Funktion von* z *ist*, welche die Bedingung $|\varphi(z)| < e^{k|z|}$ für alle hinreichend großen $|z|$ erfüllt. *Cramér* beweist nämlich, daß, falls die Funktion $f(s)$, welche durch die erste Reihe definiert wird, in einem Gebiete G_1, das über die Konvergenzgerade $\sigma = \sigma_B$ dieser Reihe hinausreicht, regulär ist, die durch die zweite Reihe definierte Funktion $g(s)$ ebenfalls über die „entsprechende" Gerade $\sigma = \sigma_B + k$ analytisch fortsetzbar sein wird, und zwar auf ein Gebiet G_2, das vom Gebiete G_1 in einfach angebbarer Weise abhängt.

12. Multiplikation Dirichletscher Reihen. Wie leicht zu sehen, wird man durch „gewöhnliches" Rechnen mit *Dirichlet*schen Reihen wieder zu *Dirichlet*schen Reihen geführt; speziell entsteht durch *Multiplikation* zweier beliebiger *Dirichlet*scher Reihen

$$f(s) = \sum a_n e^{-\lambda_n s} \quad \text{und} \quad g(s) = \sum b_n e^{-\mu_n s} \tag{20}$$

wiederum eine *Dirichlet*sche Reihe $\sum c_n e^{-\nu_n s}$, und zwar führt die Multiplikation zweier *gewöhnlicher Dirichlet*scher Reihen $\sum \frac{a_n}{n^s}$ und $\sum \frac{b_n}{n^s}$ wieder zu einer *gewöhnlichen Dirichlet*schen Reihe $\sum \frac{c_n}{n^s}$, deren Koeffizienten c_n durch die Formel $c_n = \sum a_m b_l$ bestimmt werden, wobei m und $l = n : m$ alle Teiler von n durchlaufen.[76])

J. Steffensen, Ein Satz über *Stieltjes*sche Integrale mit Anwendung auf *Dirichlet*sche Reihen, Palermo Rend. 36 (1913), p. 213—219, angegeben; die *Steffensen*sche Formel, die die absolute Konvergenz der Reihe für $\sigma > 0$ voraussetzt, lautet

$$f(s) = \frac{\sin \pi s}{\pi} \int_0^\infty x^{-s} B(x)\, dx \qquad (0 < \sigma < 1),$$

wo $B(x)$ die *Partialbruchreihe*

$$B(x) = \sum \frac{a_n}{x + \mu_n} \qquad (\mu_n = e^{\lambda_n})$$

bezeichnet.

75) *H. Cramér,* a) Sur une classe de séries de Dirichlet, Dissertation Upsala (Stockholm 1917); b) Un théorème sur les séries de Dirichlet et son application, Arkiv f. Mat., Astr. och Fys. 13 (1918), No. 22.

76) Aus diesem Bildungsgesetz der Koeffizienten c_n folgt z. B., wie von *H. Mellin,* Ein Satz über Dirichletsche Reihen, Ann. Ac. sc. Fenn. (A) 11 (1917), No. 1 hervorgehoben, daß die modulo q gebildeten „Partialreihen" der Produkt-

Sind die gegebenen Reihen (20) in einem Punkte s_0 beide *absolut konvergent*, so wird offenbar auch die durch Multiplikation entstandene Reihe im Punkte s_0 absolut konvergieren (und zwar mit der Summe $f(s_0) \cdot g(s_0)$). Einem bekannten *Mertens*schen Satze über Potenzreihen entsprechend (und ihn verallgemeinernd) gilt ferner nach *Stieltjes*[77]) der Satz, daß die Produktreihe konvergiert in jedem Punkt s_0 (mit der Summe $f(s_0) \cdot g(s_0)$), in welchem nur *eine* der Faktorenreihen absolut konvergiert, während die andere nur bedingt konvergiert. Dagegen braucht die Produktreihe in einem Punkte, worin *beide* Faktorenreihen bedingt konvergieren, *nicht* zu konvergieren, und dieses kann nicht nur am Rande der Konvergenzgebiete der Reihen der Fall sein; denn, wie *Landau*[39]) gezeigt hat, gibt es sogar zwei gewöhnliche *Dirichlet*sche Reihen (2), die beide in einer gewissen Halbebene $\sigma > \sigma_0$ konvergieren, deren Produktreihe aber *nicht* in der ganzen Halbebene $\sigma > \sigma_0$ konvergiert. Andererseits gibt es doch, nach *Stieltjes* und *Landau*[78]), wichtige Sätze, welche die Konvergenz der Produktreihe in Gebieten, in welchen die Faktorenreihen *beide nur bedingt konvergieren*, sichern. Als ein charakteristisches Beispiel nennen wir den Satz, daß, falls die Faktorenreihen beide für $\sigma > \alpha$ konvergieren und für $\sigma > \alpha + \beta$ absolut konvergieren, die Produktreihe mindestens für $\sigma > \alpha + \frac{\beta}{2}$ konvergiert. Hierbei läßt sich die Zahl $\alpha + \frac{\beta}{2}$ durch keine bessere (d. h. kleinere) ersetzen, denn wie *Bohr*[38]) gezeigt hat, gibt es eine *Dirichlet*sche Reihe (2) mit $\sigma_A = 1$, $\sigma_B = 0$, deren Quadratreihe die Konvergenzabszisse $\sigma_B = \frac{1}{2}$ besitzt.[79])

---

reihe $\sum \frac{c_n}{n^s}$ in einfacher Weise durch die „Partialreihen" der gegebenen Reihen $\sum \frac{a_n}{n^s}$ und $\sum \frac{b_n}{n^s}$ ausgedrückt werden können.

77) *T. Stieltjes*, Note sur la multiplication de deux séries, Nouv. Ann. de Math. (3) 6 (1887), p. 210—215.

78) *T. Stieltjes* a. a. O. 77) und Sur une loi asymptotique dans la théorie des nombres, Paris C. R. 101 (1885), p. 368—370 gibt ohne Beweise die wesentlichsten dieser Sätze an, doch nur für die gewöhnlichen *Dirichlet*schen Reihen (2). Verallgemeinerungen auf den Fall beliebiger *Dirichlet*scher Reihen (1) (sowie Verallgemeinerungen anderer Art) und Beweise der Sätze sind von *E. Landau*, Über die Multiplikation Dirichletscher Reihen, Palermo Rend. 24 (1907), p. 81—160 und Handbuch, p. 755—762 gegeben.

79) Von etwas anderer Art als die obigen Sätze ist ein Satz von *G. H. Hardy*, On the multiplication of Dirichlet's series, London math. Soc. (2) 10 (1911), p. 396—405, welcher besagt, daß, falls die Faktorenreihen beide im Punkte $s = 0$ konvergieren und $a_n = O\left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n}\right)$, $b_n = O\left(\frac{\mu_n - \mu_{n-1}}{\mu_n}\right)$, auch die Produktreihe im Punkte 0 konvergiert. Vgl. hierzu auch *A. Rosenblatt*, Über einen Satz

Der bekannte *Cesàro*sche Satz über Potenzreihen ($\lambda_n = n$), der ja besagt, daß, wenn $\sum a_n x^n$ und $\sum b_n x^n$ in einem Punkte, etwa $x = 1$, mit den Summen A und B konvergieren, die Produktreihe $\sum c_n x^n$ im Punkte $x = 1$ gewiß summabel $(C, 1)$ mit der Summe $A \cdot B$ ist (d. h. das arithmetische Mittel ihrer Partialsummen strebt gegen $A \cdot B$) wurde von *Phragmén, M. Riesz* und *Bohr*[80]) auf beliebige *Dirichlet*sche Reihen (1) übertragen; der Satz besagt hier, daß, falls $\sum a_n e^{-\lambda_n s}$ und $\sum b_n e^{-\mu_n s}$ in einem Punkte, etwa $s = 0$, mit den Summen A und B konvergieren, die Produktreihe $\sum c_n e^{-\nu_n s}$ im Punkte $s = 0$ in dem Sinne *summabel mit der Summe* AB ist, daß, $C_n = \sum_1^n c_n$ gesetzt,

$$\lim_{n \to \infty} \frac{C_1(\nu_2 - \nu_1) + C_2(\nu_3 - \nu_2) + \cdots + C_{n-1}(\nu_n - \nu_{n-1})}{\nu_n} = AB.$$

Im speziellen Falle *gewöhnlicher Dirichlet*scher Reihen

$$(\lambda_n = \mu_n = \nu_n = \log n)$$

lautet also die Gleichung:

$$\lim_{n \to \infty} \frac{C_1(\log 2 - \log 1) + \cdots + C_{n-1}(\log n - \log(n-1))}{\log n} = AB.^{81})$$

Diese Mittelwertbildung (mit Gewichten) bildet den Ausgangspunkt für die bekannte, von *M. Riesz* ausgearbeitete, allgemeine Summabilitätsmethode für beliebige *Dirichlet*sche Reihen, über die wir im nächsten Paragraphen näher berichten werden. Aus dem oben angegebenen Satze geht speziell hervor, daß, falls die Produktreihe

des Herrn Hardy, Jahresber. d. Deutsch. Math.-Ver. 23 (1914), p. 80—84 (welcher zeigt, daß eine bei *Hardy* der Exponentenfolge auferlegte Bedingung unnötig ist) und *E. Landau,* Über einen Satz des Herrn Rosenblatt, Jahresber d. Deutsch Math.-Ver. 29 (1920), p. 238. Ferner ist ein Satz von *Hardy* und *Littlewood,* a. a. O. 24 b) zu erwähnen, welcher aus der Voraussetzung der Konvergenz gewisser aus den beiden zu multiplizierenden Reihen gebildeten Hilfsreihen die Konvergenz der durch Multiplikation entstandenen Reihe folgert.

80) Der Beweis von *E. Phragmén* wurde brieflich *E. Landau* mitgeteilt und findet sich im Handbuch, p. 762—765. Vgl. auch *M. Riesz,* Sur la sommation des séries de Dirichlet. Paris C. R. 149 (1909), p. 18—21 und *H. Bohr,* a. a. O. 36).

81) Dagegen braucht das einfache [und, *Riesz,* a. a. O. 80), sogar auch das beliebig oft wiederholte] „arithmetische" Mittel $\frac{1}{n}(C_1 + C_2 + \cdots C_n)$ *nicht* für $n \to \infty$ zu konvergieren. [Es ist ein allgemeines Prinzip, daß eine Summabilitätsmethode durch Mittelwertbildungen der Form $\frac{\mu_1 C_1 + \cdots + \mu_n C_n}{\mu_1 + \cdots + \mu_n}$ um so kräftiger ist, je „langsamer" $\mu_1 + \cdots + \mu_n \to \infty$.] Über das nähere Verhältnis der „arithmetischen" Mittelwertbildung zu der „logarithmischen" Mittelwertbildung vgl. Nr. 13, Note 86.

konvergent (und nicht nur summabel) ist, sie gewiß die „richtige" Summe, d. h. die Summe $A \cdot B$ hat. Dieser letzte Satz war schon früher von *Landau*[78]) in dem speziellen Falle, wo mindestens eine der Faktorenreihen eine *absolute* Konvergenzhalbebene besitzt, durch funktionentheoretische Überlegungen bewiesen.

Wir verlassen hiermit die Konvergenztheorie der *Dirichlet*schen Reihen, um uns der Summabilitätstheorie dieser Reihen zuzuwenden. Hierbei werden wir sehen, daß die Erweiterungen des Konvergenzbegriffes für die Theorie der *Dirichlet*schen Reihen eine noch größere Rolle spielt, als es z. B. bei den Potenzreihen der Fall ist. In der Tat, bei den *Dirichlet*schen Reihen können schon die allereinfachsten Summabilitätsmethoden in ganzen *Gebieten* außerhalb der Konvergenzhalbebene verwendet werden, während solche Methoden bei den Potenzreihen nur *auf dem Rande* des Konvergenzgebietes von Bedeutung sind.

13. Summabilität Dirichletscher Reihen. Der in Nr. 2 erwähnte Hauptsatz, daß das Konvergenzgebiet einer *Dirichlet*schen Reihe (1) eine Halbebene $\sigma > \sigma_B$ ist, beruhte auf dem Satze, daß die Zahlenfolge $\{e^{-\lambda_n s}\}$ bei festem s mit $\sigma > 0$ eine „konvergenzerhaltende" war; es ist in derselben Weise klar, daß auch das Gebiet, in welchem eine *Dirichlet*sche Reihe (1) nach einer angegebenen Summabilitätsmethode *summabel* ist, ebenfalls eine *Halbebene* $\sigma > \sigma_0$ sein wird, sobald die betreffende Summabilitätsmethode die Eigenschaft besitzt, daß die Zahlenfolge $\{e^{-\lambda_n s}\}$ für $\sigma > 0$ eine „summabilitätserhaltende" ist. Dies ist nach *Bohr*[82]), der die Summabilität *Dirichlet*scher Reihen in Gebieten der komplexen Ebene zuerst untersucht hat[83]), für die *gewöhnlichen Dirichlet*schen Reihen (2) der Fall, wenn die benutzte Summabilitätsmethode die einfache *Cesàrosche Methode* (C, r) ist, wo r eine beliebige positive ganze Zahl bedeutet (Artikel II C 4, p. 477 u. f.). Es besitzt also jede *Dirichlet*sche Reihe (2) eine Folge von *Summabilitätsabszissen* $\sigma_B = \sigma^{(0)} \geq \sigma^{(1)} \geq \sigma^{(2)} \cdots \geq \sigma^{(r)} \cdots$ derart, daß die Reihe für $\sigma > \sigma^{(r)}$ summabel (C, r) ist, für $\sigma < \sigma^{(r)}$ dagegen nicht. Bezeichnet $\Omega = \lim\limits_{r \to \infty} \sigma^{(r)}$ die *Summabilitätsgrenzabszisse* der Reihe, so ergibt sich ferner, daß die „Summe" der Reihe *in der ganzen Halbebene $\sigma > \Omega$ eine reguläre analytische Funktion* darstellt, so daß wir

82) *H. Bohr*, a) Sur la série de Dirichlet, Paris C. R. 148 (1909), p. 75—80; b) a. a. O. 36); c) Habilitationsschrift, a. a. O. 38); in dieser letzten Arbeit wurde eine zusammenfassende Darstellung der Theorie gegeben.

83) Für *Dirichlet*sche Reihen als Funktionen einer *reellen* Variablen s war die Summabilität schon früher von *G. H. Hardy*, Generalisation of a theorem in the theory of divergent series, London math. Soc. (2) 6 (1908), p. 255—264 untersucht.

also durch die *Cesàro*sche Summabilität die *analytische Fortsetzung* der durch die Reihe in ihrer Konvergenzhalbebene $\sigma > \sigma_B$ bestimmten Funktion über die ganze Summabilitätshalbebene $\sigma > \Omega$ erhalten. Für die in Nr. 1 erwähnten speziellen Reihen (2), deren Koeffizienten den Bedingungen (4) genügen, findet man z. B. $\sigma^{(r)} = -r\,(r = 0, 1, 2 \ldots)$, also $\Omega = -\infty$; jede dieser Reihen ist also in der ganzen Ebene summabel und definiert somit (was übrigens auf anderem Wege schon bekannt war) eine ganze transzendente Funktion. *Bohr*[82c]) gab ferner explizite Ausdrücke der Summabilitätsabszissen $\sigma^{(r)}$ als Funktionen der Koeffizienten und zeigte, daß diese Abszissen den beiden folgenden Ungleichungen genügen

$$\sigma^{(r)} - \sigma^{(r+1)} \leqq 1\,, \quad \sigma^{(r)} - \sigma^{(r+1)} \leqq \sigma^{(r-1)} - \sigma^{(r)},$$

d. h. die Breite jedes Summabilitätsstreifens ist höchstens 1, und diese Breite kann mit wachsender Summabilitätsordnung r niemals zunehmen; diese beiden Ungleichungen sind ferner die für die Verteilung der Summabilitätsabszissen notwendigen und hinreichenden, in dem Sinne, daß es zu jeder monoton abnehmenden Zahlenfolge $\{\sigma^{(r)}\}$, die diesen Ungleichungen genügt, eine *Dirichlet*sche Reihe (2) gibt, die eben diese Zahlen $\sigma^{(r)}$ als Summabilitätsabszissen besitzt.[84])

M. Riesz[85]), der etwas später als *Bohr*, aber unabhängig von ihm,

84) In der bekannten Arbeit von *G. H. Hardy* u. *J. Littlewood*, Contributions to the arithmetic theory of series, London math. Soc. (2) 11 (1912), p. 411—478 wird u. a. die oben referierte Untersuchung über die Verteilung der Summabilitätsabszissen dadurch verfeinert, daß auch das Summabilitätsverhalten der Reihe in Punkten *auf* den Summabilitätsgeraden $\sigma = \sigma^{(r)}$ selbst betrachtet wird. Zur Charakterisierung der gewonnenen Resultate sei der Satz erwähnt, daß eine Reihe (2), falls sie in einem Punkte $s = \sigma_1$ summabel $(C, r+1)$ und in einem Punkte $s = \sigma_2$ summabel $(C, r-1)$ ist, im Mittelpunkte $s = \frac{1}{2}(\sigma_1 + \sigma_2)$ summabel (C, r) ist; in diesem Satze ist die obige Ungleichung $\sigma^{(r)} - \sigma^{(r+1)} \leqq \sigma^{(r-1)} - \sigma^{(r)}$ speziell enthalten. Ferner werden, unter gewissen spezielleren Annahmen über die Größenordnung der Koeffizienten, genauere Sätze über die Lage der Summabilitätsgeraden und das Verhalten der Reihe auf diesen Geraden bewiesen.

85) *M. Riesz*, a) Sur les séries de Dirichlet, Paris C. R. 148 (1909), p. 1658—1660; b) Sur la sommation des séries de Dirichlet, Paris C. R. 149 (1909), p. 18—21. Eine zusammenfassende Darstellung der *Riesz*schen Untersuchungen findet sich in dem a. a. O. 1) zitierten Cambridge tract von *G. H. Hardy* und *M. Riesz*. Vgl. auch die Arbeiten von *P. Nalli*, a) Sulle serie di Dirichlet, Palermo Rend. 40 (1915), p. 44—70; b) Aggiunta alla memoria: „Sulle serie di Dirichlet", Palermo Rend. 40 (1915), p. 167—168, und *M. Kuniyeda*, a) Note on Perron's integral and summability-abscissae of Dirichlet's series, Quart. J 47 (1916), p. 193—219; b) On the abscissa of summability of general Dirichlet's series, Tôhoku J. 9 (1916), p. 245—262, welche sich nahe an die *Riesz*schen Arbeiten anschließen.

die *Cesàro*-Summabilität der *Dirichlet*schen Reihen untersucht hat, beschränkt sich nicht auf Summabilität ganzzahliger Ordnung und — was wesentlicher ist — betrachtet sogleich die *allgemeinen Dirichlet*schen Reihen. *Riesz* mußte daher zunächst das *Cesàro*sche Summabilitätsverfahren so verallgemeinern, daß es auf diesen allgemeineren Reihentypus (1) angewendet werden konnte, und er wurde hierbei auf eine neue bedeutsame Summabilitätsmethode geführt, die von ihm „*Summation nach typischen Mitteln*" genannt wurde. Schon bei dem Multiplikationssatz in Nr. 12 haben wir gesehen, daß es bei den gewöhnlichen *Dirichlet*schen Reihen (2) zweckmäßig sein kann, ein Summabilitätsverfahren zu benutzen, dessen erste Stufe darin besteht, das „logarithmische" Mittel

$$\frac{C_1(\log 2-\log 1)+\cdots+C_{n-1}(\log n-\log(n-1))}{\log n}$$

statt des arithmetischen Mittels

$$\frac{C_1+C_2+\cdots+C_n}{n}$$

zu bilden. Indem *Riesz* diesen Gedanken ausführt und verallgemeinert, führt er, einer gegebenen *Dirichlet*schen Reihe (1) (oder vielmehr einer gegebenen Exponentenfolge $\{\lambda_n\}$) entsprechend, *zwei* verschiedene Summationsmethoden ein, deren eine der logarithmischen, die andere der arithmetischen Mittelwertbildung analog ist. Es sei

$$e^{\lambda_n}=l_n,\quad a_n e^{-\lambda_n s}=a_n l_n^{-s}=c_n,$$
$$C_\lambda(\tau)=\sum_{\lambda_n<\tau}c_n,\quad C_l(t)=\sum_{l_n<t}c_n$$

und

$$C_\lambda^{(k)}(\omega)=\sum_{\lambda_n<\omega}(\omega-\lambda_n)^k c_n=k\int_0^\omega C_\lambda(\tau)(\omega-\tau)^{k-1}d\tau,$$
$$C_l^{(k)}(w)=\sum_{l_n<w}(w-l_n)^k c_n=k\int_0^w C_l(t)(w-t)^{k-1}dt,$$

wobei k eine beliebige positive (ganze oder nicht ganze) Zahl bedeutet. Die Ausdrücke

$$\frac{C_\lambda^{(k)}(\omega)}{\omega^k}\quad\text{und}\quad\frac{C_l^{(k)}(w)}{w^k}$$

heißen dann nach *Riesz* die typischen Mittelwerte der k-Ordnung von der ersten bzw. zweiten Art, welche zu der gegebenen Reihe (1) gehören. Wenn nun

$$\frac{C_\lambda^{(k)}(\omega)}{\omega^k}\to C\quad\text{bzw.}\quad\frac{C_l^{(k)}(w)}{w^k}\to C$$

für $\omega\to\infty$ bzw. $w\to\infty$, wird die Reihe (1) *summabel* (λ, k) *bzw.* (l, k) *mit der Summe* C genannt. Die „Kraft" der Summabilitätsmethode steigt mit wachsender Summabilitätsordnung k, d. h. wenn

eine Reihe summabel (λ, k) bzw. (l, k) ist, so ist sie a fortiori summabel (λ, k') bzw. (l, k') für $k' > k$.

Riesz zeigt nun für eine beliebige Exponentenfolge $\{\lambda_n\}$, daß die Zahlenfolge $\{e^{-\lambda_n s}\}$ bei festem s mit $\sigma > 0$ eine summabilitätserhaltende Faktorenfolge ist, sowohl für die Summabilitätsmethode (λ, k) als für die Methode (l, k), woraus folgt, daß der Gültigkeitsbereich der (einen oder anderen) Summabilitätsmethode eine Halbebene ist. Über die Tragweite der beiden Methoden (λ, k) und (l, k) gegeneinander gilt der Satz: in jedem Punkte s, wo die Reihe (1) summabel (l, k) ist, ist sie gewiß auch summabel (λ, k), so daß (λ, k) die „kräftigere" Methode ist; die Methode (l, k) ist aber „beinahe" ebenso stark, d. h. wenn die Reihe (1) in einem Punkte $s_0 = \sigma_0 + it_0$ summabel (λ, k) ist, braucht sie wohl nicht im Punkte s_0 selbst summabel (l, k) zu sein, ist es aber in jedem Punkte $s = \sigma + it$ mit $\sigma > \sigma_0$.

Aus diesen Sätzen folgt, daß zu jeder *Dirichlet*schen Reihe (1) eine *Summabilitätsabszissenfunktion* $\sigma^{(k)}$ $(0 < k < \infty)$ derart existiert, daß die Reihe bei jedem $k > 0$ für $\sigma > \sigma^{(k)}$ summabel (λ, k) und (l, k) ist, während sie für $\sigma < \sigma^{(k)}$ weder summabel (λ, k) noch (l, k) ist.[86])

Für die *gewöhnlichen Dirichlet*schen Reihen (2) $(\lambda_n = \log n)$ ist die *Riesz*sche Summabilität (l, k) mit der *Cesaro*schen Summabilität (C, k) inhaltsmäßig identisch[87]), und es sind somit für diese Reihen (2), und ganzzahlige k, die hier definierte Summabilitätsabszissen $\sigma^{(k)}$ mit den früher besprochenen identisch. Die dort angegebenen Ungleichungen über die Verteilung der Summabilitätsabszissen werden, sogar für den Fall einer *beliebigen Dirichlet*schen Reihe (1), von *Riesz* dahin verallgemeinert, daß die Summabilitätsabszissenfunktion $\sigma^{(k)}$ eine *konvexe* Funktion von k ist.[88]) Ferner verallgemeinert *Riesz* die expliziten Ausdrücke für die Summabilitätsabszissen $\sigma^{(k)}$ als Funktionen der Koeffizienten und Exponenten auf beliebige *Dirichlet*sche Reihen (1) und beliebiges nicht ganzzahliges k.

86) In Punkten *auf* der Summabilitätsgeraden $\sigma = \sigma^{(k)}$ selbst kann es vorkommen (vgl. eine Bemerkung oben), daß die Reihe summabel (λ, k) aber *nicht* (l, k) ist. So ist nach *Riesz*, a. a. O. 85a) und 85b) die Zetareihe $\sum \frac{1}{n^s}$, für welche $\sigma^{(k)} = 1$ für alle k ist, bei keinem k summabel (l, k) in irgendeinem Punkte der Geraden $\sigma = 1$, während sie in jedem Punkte $s \neq 1$ dieser Geraden summabel $(\lambda, 1)$ ist. (Vgl. Note 81.)

87) *M. Riesz*, Une méthode de sommation équivalente à la méthode des moyennes arithmétiques, Paris C. R. 152 (1911), p. 1651—1654. Die von *Riesz* angegebene Formulierung der *Cesàro*schen Summationsmethode hat sich bei verschiedenen Anwendungen als wesentlich bequemer als die ursprüngliche Formulierung gezeigt.

88) Der Beweis dieses Satzes wird demnächst in den Acta Univ. hung. Francesco-Jos. erscheinen.

Über den Zusammenhang der Summabilitätseigenschaften einer Reihe (1) mit den analytischen Eigenschaften der durch die Reihe dargestellten Funktion $f(s)$ gilt zunächst der folgende leicht beweisbare Satz: *Für* $\sigma > \sigma^{(k)} + \varepsilon$ *ist* $f(s) = O(|t|^{k+1})$. Die Frage nach der *Umkehrung* dieses Satzes ist (wie im Falle $k = 0$, d. h. Konvergenz) viel schwieriger; es zeigt sich, daß eine unmittelbare Umkehrung *nicht* gilt, dagegen eine solche, in welcher *der Exponent* $k + 1$ *durch* k *ersetzt wird*, d. h. wenn die durch eine *Dirichlet*sche Reihe definierte Funktion $f(s)$ für $\sigma > \sigma_0$ regulär und gleich $O(|t|^k)$ ist, so wird die Summabilitätsabszisse $\sigma^{(k)}$ gewiß $\leqq \sigma_0$ sein. Es geben diese *Riesz*schen Sätze einerseits notwendige und andererseits hinreichende Bedingungen für die Summabilität k^{ter} Ordnung, aber (ganz wie im Falle $k = 0$) keine Bedingungen, die zugleich notwendig und hinreichend sind. Betrachten wir aber den *Grenzwert* Ω der abnehmenden Funktion $\sigma^{(k)}$ (für $k \to \infty$), so können wir aus den obigen Sätzen den folgenden Hauptsatz über die funktionentheoretische Bestimmung dieser *Summabilitätsgrenzabszisse* Ω ableiten: Es ist die Reihe genau so weit summabel (von irgendeiner Ordnung) wie die dargestellte Funktion $f(s)$ regulär und von endlicher Ordnung in bezug auf t ist, d. h. *es ist* Ω *gleich der früher eingeführten Abszisse* σ_ε. Dieser Satz wurde, für die gewöhnlichen *Dirichlet*schen Reihen (2), zuerst von *Bohr*[36]) explizite aufgestellt, der ihn aus einigen, den *Riesz*schen ähnlichen, aber nicht so weitreichenden Sätzen herleitete.[89])

Bei einer näheren Untersuchung zeigt es sich, daß die Einführung der *Cesàro-Riesz*schen Summabilität für fast alle Probleme in der Theorie der *Dirichlet*schen Reihen von wesentlicher Bedeutung ist, weil dadurch frühere Resultate aus der Konvergenztheorie sich in wichtiger Weise verallgemeinern lassen. Wegen der allgemeinen Durchführung solcher Untersuchungen und der dabei erhaltenen Resultate sei der Leser auf das *Hardy-Riesz*sche Buch verwiesen.[1]) Hier soll nur noch ein besonders interessantes Resultat über die *Multiplikation Dirichlet*scher Reihen erwähnt werden, welches den klassischen Satz von *Cesàro* über Multiplikation von Potenzreihen ($\lambda_n = n$) auf den allgemeinsten Typus *Dirichlet*scher Reihen (1) verallgemeinert, und so lautet[90]): *Wenn* $\sum a_n e^{-\lambda_n s}$ *im Punkte* $s = s_0$ *summabel* (λ, α) *mit*

89) Vgl. auch eine Arbeit von *W. Schnee*, Über den Zusammenhang zwischen den Summabilitätseigenschaften Dirichletscher Reihen und ihrem funktionentheoretischen Charakter. Acta Math. 35 (1912), p. 357—398, worin der *Landau-Schnee*sche Satz über das Konvergenzproblem (vgl. Nr. 6) von Konvergenz auf Summabilität verallgemeinert wird.

90) *Hardy-Riesz*, a. a. O. 1), p. 64.

der Summe A und $\sum b_n e^{-\mu_n s}$ im selben Punkte s_0 summabel (μ, β) mit der Summe B ist, so ist die Produktreihe $\sum c_n e^{-\nu_n s}$ im Punkte s_0 summabel $(\nu, \alpha + \beta + 1)$ mit der Summe AB. Im speziellen Fall $\alpha = \beta = 0$ erhalten wir den in Nr. 12 erwähnten Satz über die Multiplikation zweier *konvergenter Dirichlet*scher Reihen.

Außer den *Cesàro-Riesz*schen Methoden wurden auch *andere* Summationsmethoden auf die *Dirichlet*schen Reihen angewendet. So hat *Hardy*[91]) die Wirkung der *Borel*schen Summation auf die gewöhnlichen *Dirichlet*schen Reihen (2) geprüft. Auch bei dieser Summationsmethode ist die Zahlenfolge $\left\{\frac{1}{n^s}\right\}$ $(\sigma > 0)$ eine summabilitätserhaltende Faktorenfolge, und das Summabilitätsgebiet also eine *Halbebene* $\sigma > \sigma^{(B)}$. Diese Halbebene $\sigma > \sigma^{(B)}$ kann aber niemals über die *Cesàro-Riesz*sche Summabilitätshalbebne $\sigma > \Omega$ hinausreichen und braucht nicht immer so weit zu reichen. Anders verhält es sich mit einer anderen von *Hardy*[92]) untersuchten Summabilitätsmethode, der sogenannten *Abel*schen Methode, nach welcher eine Reihe $\sum a_n$ summabel mit der Summe A heißt, wenn die Potenzreihe $f(x) = \sum a_n x^n$ für $0 < x < 1$ konvergiert und die Bedingung $f(x) \to A$ für $x \to 1$ erfüllt. *Hardy* beweist, daß auch hier das Summabilitätsgebiet einer gewöhnlichen *Dirichlet*schen Reihe (2) eine *Halbebene* $\sigma > \sigma^{(A)}$ ist, und daß $\sigma^{(A)}$ einfach die untere Grenze aller Abszissen σ_0 ist, für welche die durch die Reihe dargestellte Funktion $f(s)$ in der Halbebene $\sigma > \sigma_0$ *regulär und gleich* $O(e^{k|t|})$ *mit* $k < \frac{\pi}{2}$ *ist.*[93])

Schließlich ist noch eine schöne Arbeit von *M. Riesz*[94]) zu erwähnen, in welcher es ihm gelungen ist, die bekannten *Mittag-Leffler*schen Resultate über die analytische Darstellung der durch eine

91) *G. H. Hardy,* a. a. O. 72). Vgl. auch *Fekete,* a. a. O. 72) und *G. H. Hardy-J. Littlewood,* The relations between Borel's and Cesàro's method of summation, Proc. London math. Soc. (2) 11 (1913), p. 1—16.

92) *G. H. Hardy,* a) Sur la sommation des séries de Dirichlet, Paris C. R. 162 (1916), p. 463—465; b) a. a. O. 51).

93) Einfache Beispiele *Dirichlet*scher Reihen, bei welchen erst die *Abel*sche Summabilität — also nicht die *Cesàro*sche — imstande ist, die durch die Reihe dargestellte Funktion über die Konvergenzhalbebene $\sigma > \sigma_B$ hinaus analytisch fortzusetzen $\left(\text{weil die Funktion für } \sigma < \sigma_B \text{ stärker als } |t|^k, \text{ aber nicht so stark wie } e^{\frac{\pi}{2}|t|} \text{ wächst}\right)$, wurden von *G. H. Hardy,* a) a. a. O. 92) und b) Example to illustrate a point in the theory of Dirichlet's series, Tôhoku J. 8 (1915), p. 59—66, angegeben.

94) *M. Riesz,* Sur la représentation analytique des fonctions définies par des séries de Dirichlet, Acta Math. 35 (1912), p. 253—270.

Potenzreihe ($\lambda_n = n$) definierten Funktion in ihrem Hauptstern auf den allgemeinen Reihentypus (1) zu übertragen. *Riesz* beweist u. a. den folgenden Satz: Es habe die *Dirichlet*sche Reihe (1) ein Konvergenzgebiet, und es sei H der *Hauptstern* der durch die Reihe definierten Funktion $f(s)$, d. h. das Gebiet, welches aus der s-Ebene entsteht, wenn alle mit der negativen reellen Achse parallelen Halbgeraden, die von den singulären Punkten von $f(s)$ ausgehen, entfernt werden. Dann gilt *im ganzen Hauptstern die Darstellung* $f(s) = \lim\limits_{\alpha \to 0} \varphi_\alpha(s)$, *wo* $\varphi_\alpha(s)$ *die* (*ganze transzendente*) *Funktion*

$$\varphi_\alpha(s) = \sum \frac{1}{\Gamma(\alpha\lambda_n + 1)} a_n e^{-\lambda_n s}$$

bezeichnet. Auch die von *Mittag-Leffler* benutzten *Integraldarstellungen* zur analytischen Fortsetzung einer durch eine gegebene Potenzreihe ($\lambda_n = n$) definierten Funktion wurden von *Riesz* auf die *Dirichlet*schen Reihen übertragen.

II. Die Riemannsche Zetafunktion.

14. Die Zetafunktion und ihre Funktionalgleichung. Die Zetafunktion wird (vgl. Nr. **1**) durch die *Dirichlet*sche Reihe

$$\zeta(s) = \sum \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots \tag{21}$$

definiert. Obwohl schon *Euler* diese Funktion betrachtet und ihre zahlentheoretische Bedeutung erkannt hat, wird sie doch gewöhnlich als die „*Riemann*sche" Zetafunktion bezeichnet, weil *Riemann* sie zuerst in seiner berühmten Abhandlung über die Anzahl der Primzahlen[95]), welche auch für die Entwicklung der neueren Funktionentheorie von fundamentaler Bedeutung gewesen ist, einem tiefergehenden Studium unterworfen hat. Über die Bedeutung der Zetafunktion für das Primzahlproblem sei in diesem Kapitel, das sich ausschließlich mit den rein funktionentheoretischen Eigenschaften von $\zeta(s)$ beschäftigen soll, nur bemerkt, daß sie in der *Eulerschen Identität*

$$\zeta(s) = \sum n^{-s} = \prod \frac{1}{1 - p_m^{-s}}, \tag{22}$$

wo p_m die Primzahlen durchläuft, wurzelt; diese *Euler*sche Produktdarstellung spielt übrigens auch (vgl. z. B. Nr. **17**) bei manchen funktionentheoretischen Untersuchungen von $\zeta(s)$ eine bedeutsame Rolle.

Die die Zetafunktion definierende Reihe (21) konvergiert nur in der Halbebene $\sigma > 1$, und auch das Produkt (22) ist für $\sigma < 1$ divergent und gibt somit keinen Aufschluß über die Möglichkeit analyti-

95) *B. Riemann*, Über die Anzahl der Primzahlen unter einer gegebenen Größe, Monatsber. Akad. Berlin 1859, p. 671—680 = Werke (2. Aufl.), p. 145—153.

scher Fortsetzung über die Gerade $\sigma = 1$ hinaus. Anders verhält es sich mit der in Nr. **11** erwähnten Integraldarstellung

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx;$$

in der Tat, es läßt sich dieses zunächst ebenfalls nur für $\sigma > 1$ brauchbare Integral als ein komplexes Kurvenintegral

$$(23) \qquad \zeta(s) = \frac{1}{\Gamma(s)} \frac{1}{e^{-\pi s i} - e^{\pi s i}} \int_W \frac{(-x)^{s-1}}{e^x - 1} dx = \frac{i\Gamma(1-s)}{2\pi} \int_W \frac{(-x)^{s-1}}{e^x - 1} dx$$

schreiben, wo der Integrationsweg W eine Schleife ist, die vom Punkte $x = +\infty$ ausgeht und nach einem einmaligen Umkreisen des Punktes $x = 0$ zum Punkte $x = +\infty$ zurückkehrt. Aus dieser Integraldarstellung, die offensichtlich für jedes s konvergiert, schloß *Riemann*, daß die Funktion $\zeta(s)\Gamma(s)\sin \pi s$ eine ganze Transzendente ist, und hieraus weiter, *daß* $\zeta(s)$ *in der ganzen Ebene als eine eindeutige Funktion existiert, die überall regulär ist mit Ausnahme des einzigen Punktes* $s = 1$, *wo sie einen Pol erster Ordnung* (*mit dem Residuum* 1) *besitzt.*

Aus der Darstellung (23) leitete *Riemann* des weiteren durch eine Deformation des Integrationsweges und Anwendung des *Cauchy*schen Satzes eine fundamentale Eigenschaft der Zetafunktion ab, nämlich daß sie der *Funktionalgleichung*

$$(24) \qquad \zeta(1-s) = \frac{2}{(2\pi)^s} \cos\frac{s\pi}{2}\, \Gamma(s)\zeta(s)$$

genügt[96]), oder anders ausgedrückt, daß die Funktion

$$\eta(s) = \zeta(s)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}$$

ungeändert bleibt, wenn die Variable s durch $1 - s$ ersetzt wird. Für die Funktionalgleichung in dieser letzten Form: $\eta(s) = \eta(1-s)$ und gleichzeitig auch für die Existenz von $\zeta(s)$ in der ganzen Ebene[97])

96) Nach *E. Landau*, Euler und die Funktionalgleichung der Riemannschen Zetafunktion, Bibl. Math. (3) 7 (1906—7), p. 69—79 war diese Funktionalgleichung schon *Euler* bekannt.

97) Außer den beiden, von *Riemann* selbst herrührenden, Beweisen des Satzes, daß $\zeta(s)$ von der Definitionshalbebene $\sigma > 1$ aus in die ganze Ebene fortgesetzt werden kann, gibt es eine Menge anderer Beweise dieses Satzes. So hat z. B. *J. L. W. V. Jensen*, Interméd. math. 1 (1895), p. 346—347 verschiedene Integraldarstellungen für $\zeta(s)$ angegeben, aus welchen die Existenz von $\zeta(s)$ in der ganzen Ebene unmittelbar ersichtlich ist; vgl. hierzu auch *E. Lindelöf*, Le calcul des résidus et ses applications à la théorie des fonctions (Collection Borel), Paris 1905, p. 1—141. Ein anderer Beweis von *Jensen*, Sur la fonction $\zeta(s)$ de

hat *Riemann*[95]) auch einen anderen Beweis gegeben, welcher sich bei der Anwendung auf mit der Zetafunktion verwandte Funktionen als sehr verallgemeinerungsfähig erwiesen hat. *Riemann* geht hierbei vom Integral

$$\frac{1}{n^s}\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}} = \int_0^\infty e^{-n^2\pi x} x^{\frac{s}{2}-1}\,dx$$

aus und erhält durch Summation die Formel

$$(25) \qquad \zeta(s)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}} = \int_0^\infty x^{\frac{s}{2}-1}\,\omega(x)\,dx, \qquad (\sigma > 1)$$

wo $\omega(x)$ die Reihe

$$\omega(x) = \sum_{n=1}^\infty e^{-n^2\pi x}$$

bezeichnet. Nun ist aber, nach einer bekannten Formel aus der Theorie der elliptischen Thetafunktionen,

$$1 + 2\omega(x) = \frac{1}{\sqrt{x}}\left(1 + 2\omega\left(\frac{1}{x}\right)\right), \qquad (x > 0)$$

woraus sich duich Einsetzen in (25) und eine leichte Rechnung die Formel

$$(26) \qquad \zeta(s)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}} - \frac{1}{s(s-1)} = \int_1^\infty \left(x^{\frac{1-s}{2}-1} + x^{\frac{s}{2}-1}\right)\omega(x)\,dx$$

Riemann, Paris C. R. 104 (1887), p. 1156—1159 beruht auf einer Relation zwischen den unendlich vielen Gliedern der Folge $\zeta(s)$, $\zeta(s+1)$, $\zeta(s+2), \ldots$; ähnliche Relationen, welche überdies die Eigenschaft besitzen, in sich als Definitionsgleichungen der Zetafunktion gelten zu können, wurden später von *J. Hadamard,* Sur une propriété fonctionelle de la fonction $\zeta(s)$ de Riemann, Bull. Soc. math. France 37 (1909), p. 59—60, angegeben. *Ch. de la Vallée Poussin,* Démonstration simplifiée du théorème de Dirichlet sur la progression arithmétique, Mém. Acad. Belgique 53 (1895—96), No. 6, p. 1—32, beweist den Satz durch Vergleich der Reihe $\sum \frac{1}{n^s} = \zeta(s)$ mit dem entsprechenden Integral $\int_1^\infty \frac{du}{u^s} = \frac{1}{s-1}$, indem er (durch partielle Integrationen) nachweist, daß die Differenz $\zeta(s) - \frac{1}{s-1}$ eine ganze Transzendente ist; die Idee dieser Beweismethode ist von *H. Cramér,* Sur une classe de séries de Dirichlet, Diss. Upsala (Stockholm 1917), p. 1—51, zur Untersuchung beliebiger *Dirichlet*scher Reihen verallgemeinert. Setzt man die Theorie der *Cesàro*schen Summabilität *Dirichlet*scher Reihen (vgl. Nr. 13) als bekannt voraus, dürfte der einfachste Beweis für die Existenz von $\zeta(s)$ in der ganzen Ebene wohl derjenige sein, daß man die Funktion $\zeta(s)(1-2^{1-s})$ betrachtet, welche durch die in der ganzen Ebene summable Reihe $\sum \frac{(-1)^{n+1}}{n^s}$ dargestellt wird und somit sich sofort als eine ganze Transzendente erweist.

ergibt, welche sofort erkennen läßt, daß die auf der linken Seite stehende Funktion eine ganze Transzendente ist, die ungeändert bleibt, wenn s durch $1 - s$ ersetzt wird.[98])

Die Funktionalgleichung (24) verbindet die Werte der Zetafunktion in zwei Punkten s und $1 - s$, welche in bezug auf den Punkt $\frac{1}{2}$ symmetrisch gelegen sind. Hieraus folgt, daß man das Studium der Zetafunktion wesentlich auf die Halbebene $\sigma \geqq \frac{1}{2}$ beschränken kann (übrigens sogar auf die Viertelebene $\sigma \geqq \frac{1}{2}$, $t \geqq 0$, da $\zeta(s)$ in konjugierten Punkten konjugierte Werte annimmt); denn die Funktionalgleichung erlaubt ja das Verhalten von $\zeta(s)$ in der Halbebene $\sigma < \frac{1}{2}$ aus dem Verhalten der Funktion für $\sigma > \frac{1}{2}$ abzulesen. So können wir z. B. aus der aus der *Euler*schen Identität (22) unmittelbar folgenden Tatsache, daß $\zeta(s)$ *in der Halbebene* $\sigma > 1$ *überall von* 0 *verschieden* ist, mittels der Funktionalgleichung (24) sofort die Nullstellen von $\zeta(s)$ in der Halbebene $\sigma < 0$ bestimmen; in der Tat, es folgt ja aus (24), daß diese Nullstellen mit den Nullstellen der Funktion $1 : \left\{\cos\frac{s\pi}{2}\Gamma(s)\right\}$ für $\sigma < 0$ übereinstimmen, d. h. daß $\zeta(s)$ *in der besprochenen Halbebene* $\sigma < 0$ *Nullstellen* (*einfache*) *in den Punkten* $s = -2, -4, -6, \ldots$ und nur in diesen Punkten besitzt. Es werden diese Nullstellen gewöhnlich als die „trivialen" Nullstellen von $\zeta(s)$ bezeichnet, im Gegensatze zu den (in Nr. **15** zu erwähnenden) „nichttrivialen" Nullstellen im Streifen $0 \leqq \sigma \leqq 1$. Diese letzten Nullstellen liegen übrigens alle im *Innern* des Streifens $0 < \sigma < 1$; denn wie *de la Vallée Poussin*[99]) und *Hadamard*[100]) unabhängig von ein-

---

98) *H. Hamburger*, a) Über die Riemannsche Funktionalgleichung der ζ-Funktion, Math. Ztschr. 10 (1921), p. 240—254; 11 (1922), p. 224—245; 13 (1922), p. 283—311; b) Über einige Beziehungen, die mit der Funktionalgleichung der Riemannschen ζ-Funktion äquivalent sind, Math. Ann. 85 (1922), p. 129—140, beweist, daß $\zeta(s)$ bis auf einen konstanten Faktor *eindeutig* bestimmt ist durch die folgenden Eigenschaften: sie ist eine meromorphe Funktion mit nur endlich vielen Polen, die 1. der Funktionalgleichung (24) genügt, 2. für $|s| \to \infty$ gleich $O(e^{|s|^k})$ und 3. für $\sigma > 1$ durch eine absolut konvergente *Dirichlet*sche Reihe $\sum \frac{a_n}{n^s}$ darstellbar ist. Im Laufe des Beweises dieses Satzes [durch welchen eine Fragestellung von *J. Hadamard*, a. a. O. 60) behandelt wird] gibt *Hamburger* einen neuen Beweis der *Riemann*schen Funktionalgleichung. Vgl. auch *C. Siegel*, Bemerkung zu einem Satz von Hamburger über die Funktionalgleichung der Riemannschen Zetafunktion, Math. Ann. 86 (1922), p. 276—279.

99) *Ch. de la Vallée Poussin*, Recherches analytiques sur la théorie des nombres premiers, Ann. soc. sc. Bruxelles 20[2] (1896), p. 183—256 und p. 281—397.

100) *J. Hadamard*, Sur la distribution des zéros de la fonction $\zeta(s)$ et ses conséquences arithmétiques, Bull. Soc. math. France 24 (1896), p. 199—220.

ander durch sinnreiche Überlegungen bewiesen haben, ist *die Gerade* $\sigma = 1$ (*und daher auch die Gerade* $\sigma = 0$) *nullpunktsfrei*. In diesem Zusammenhange sei noch erwähnt, daß *Mertens*[101]) schon früher durch eine interessante Abschätzung bewiesen hatte, daß das *Euler*sche Produkt (22) in jedem Punkte $s \neq 1$ auf der Geraden $\sigma = 1$, in welchem $\zeta(s) \neq 0$ ist (also nach dem obigen Satze in den sämtlichen Punkten $s \neq 1$) noch konvergiert; mit Hilfe des in Nr. 5 besprochenen *Riess*schen Konvergenzsatzes (auf $\log \zeta(s)$ verwendet) läßt sich dieses Resultat[102]), oder was damit gleichbedeutend ist, die Konvergenz der Reihe $\sum \frac{1}{p_m^{1+it}}$ für alle $t \gtrless 0$, unmittelbar ohne jede spezielle Abschätzung aus dem Nichtverschwinden von $\zeta(s)$ auf der Geraden $\sigma = 1$ ableiten.

15. Die Riemann-Hadamardsche Produktentwicklung. Betrachten wir mit *Riemann* die Funktion[103])

$$\xi(s) = \frac{s(s-1)}{2}\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s),$$

welche eine ganze Transzendente ist, deren Nullstellen mit den nichttrivialen Nullstellen von $\zeta(s)$ übereinstimmen (indem die trivialen Nullstellen weggeschafft sind), und die der Gleichung $\xi(s) = \xi(1-s)$ genügt. Die durch diese Funktionalgleichung ausgedrückte Eigenschaft kann auch dadurch zum Ausdruck gebracht werden, daß die Funktion $\Xi(z)$, welche aus $\xi(s)$ durch die Transformation $s = \frac{1}{2} + iz$ entsteht, eine *gerade* Funktion von z ist, d. h. eine Funktion von z^2, die wir mit $g(z^2)$ bezeichnen werden, wo also $g(x)$ eine ganze Funktion von x ist. Jedem Nullstellenpaar $\pm\lambda$ von $\Xi(z)$, d. h. jedem Nullstellenpaar ϱ und $1-\varrho$ von $\zeta(s)$ entspricht also nur eine einzige Nullstelle $\mu = \lambda^2$ von $g(x)$. Über diese Funktion $g(x)$ behauptete *Riemann*[95]), daß sie unendlich viele Nullstellen μ_n hat — also daß die Zetafunktion unendlich viele nichttriviale Nullstellen besitzt — und ferner, daß sie durch das Produkt

$$g(x) = g(0)\prod_{n=1}^{\infty}\left(1 - \frac{x}{\mu_n}\right) \tag{27}$$

101) *F. Mertens*, Über die Konvergenz einer aus Primzahlpotenzen gebildeten unendlichen Reihe, Gött. Nachr. 1887, p. 265—269.

102) Vgl. *E. Landau*, a. a. O. 21).

103) Die folgenden Bezeichnungen der Funktionen sind die von *E. Landau* benutzten (und jetzt üblichen), welche von den *Riemann*schen etwas abweichen. Die von *Riemann* mit ξ bezeichnete Funktion ist die unten erwähnte Funktion Ξ.

dargestellt werden kann. Die Richtigkeit dieser Behauptung wurde bekanntlich zuerst von *Hadamard*[104]) durch seine grundlegenden Untersuchungen über ganze transzendente Funktionen endlichen Geschlechtes bewiesen. Es ist nämlich, wie leicht zu zeigen,

$$g(x) = O\left(e^{|x|^{\frac{1}{2}+\varepsilon}}\right) \qquad (\varepsilon > 0)$$

und nach einem allgemeinen Satz der *Hadamard*schen Theorie folgt aus dieser Abschätzung sofort die Richtigkeit der obigen Behauptung.[105]) Wie oben erwähnt, entsprechen jeder Nullstelle μ_n von $g(x)$ zwei Nullstellen von $\zeta(s)$, nämlich $\frac{1}{2} \pm i\sqrt{\mu_n}$, welche symmetrisch in bezug auf den Punkt $\frac{1}{2}$ liegen. Wird die Produktentwicklung (27) von $g(x)$ zu einer Produktentwicklung der Funktion $\zeta(s)$ selbst umgeschrieben, so findet man — indem der Bequemlichkeit halber Konvergenzfaktoren hinzugefügt werden, die das Produkt von der Reihenfolge der Faktoren (d. h. von dem paarweisen Zusammennehmen zweier „entsprechender" Nullstellen ϱ und $1 - \varrho$) unabhängig machen — die grundlegende Formel

$$(28) \qquad (s-1)\zeta(s) = \frac{1}{2} e^{bs} \frac{1}{\Gamma\left(\frac{s}{2}+1\right)} \prod_{\varrho} \left(1 - \frac{s}{\varrho}\right) e^{\frac{s}{\varrho}},$$

wo ϱ die sämtlichen nichttrivialen Nullstellen durchläuft, und b eine Konstante ($b = \log 2\pi - 1 - \frac{1}{2} C$, wo C die *Euler*sche Konstante ist) bezeichnet. In der Primzahlentheorie kommt diese Formel (28) meistens in der Form

$$(29) \qquad \frac{\zeta'}{\zeta}(s) = b - \frac{1}{s-1} - \frac{1}{2} \frac{\Gamma'}{\Gamma}\left(\frac{s}{2}+1\right) + \sum_{\varrho} \left(\frac{1}{s-\varrho} + \frac{1}{\varrho}\right)$$

zur Anwendung.

Es sei schon hier erwähnt, daß *Riemann*[95]) des weiteren die Vermutung ausgesprochen hat — aber mit ausdrücklicher Hervorhebung, daß er diese Vermutung nicht beweisen konnte — daß die Nullstellen von $g(x)$ alle *reell* sind, d. h. *daß die nichttrivialen Nullstellen von $\zeta(s)$ alle auf der Geraden $\sigma = \frac{1}{2}$ liegen.* Ob diese berühmte „*Riemann*sche Vermutung" richtig ist oder nicht, ist bekanntlich noch heute unentschieden, und man weiß auch nicht, durch welche Ar-

---

104) *J. Hadamard,* Étude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann, J. de math. (4) 9 (1893), p. 171—215.

105) In dem ursprünglichen *Hadamard*schen Beweise wird übrigens nicht die Größenordnung der Funktion $g(x)$ selbst, sondern — was nach *Hadamard* auf genau dasselbe hinauskommt — die Größenordnung der Koeffizienten a_n der Potenzreihe $g(x) = \sum a_n x^n$ abgeschätzt.

gumente (abgesehen von der *symmetrischen Lage der Nullstellen in bezug auf die Gerade* $\sigma = \frac{1}{2}$) *Riemann* auf diese Vermutung geführt worden ist.

16. Die Riemann-v. Mangoldtsche Formel für die Anzahl der Nullstellen. Über die nähere Verteilung der Ordinaten der nichttrivialen Nullstellen von $\zeta(s)$ hat *Riemann*[95]) ohne Beweis eine Formel angegeben, die viel präziser ist als diejenigen Resultate, welche man aus der *Hadamard*schen Theorie direkt entnehmen kann, nämlich die Formel

$$(30) \qquad N(T) = \frac{1}{2\pi} T \log T - \frac{1 + \log 2\pi}{2\pi} T + O(\log T),$$

wo $N(T)$ die Anzahl der Nullstellen von $\zeta(s)$ im Rechteck $0 < \sigma < 1$, $0 < t \leqq T$ bezeichnet. Es gelang erst *v. Mangoldt*[106]), diese Formel streng zu beweisen. Betreffs des Beweises sei nur erwähnt, daß man von dem Ausdruck $N(T) = \frac{1}{2\pi i}\int \frac{\zeta'(s)}{\zeta(s)} ds$ ausgehend, wo das Integral längs des Randes eines Rechteckes mit den Eckpunkten 2, $2 + iT$, $-1 + iT$, -1 erstreckt ist, durch einfache Rechnungen (unter Benutzung der Funktionalgleichung der Zetafunktion und bekannter Eigenschaften der Gammafunktion) leicht findet, daß

$$(31) \qquad N(T) = \frac{1}{2\pi} T \log T - \frac{1 + \log 2\pi}{2\pi} T + R(T)$$

ist, wo das Restglied $R(T) = O(1) + \frac{1}{2\pi i}\int\limits_{2+iT}^{-1+iT} \frac{\zeta'(s)}{\zeta(s)} ds$, und daß die ganze, erst von *v. Mangoldt* überwundene Schwierigkeit darin liegt, dies letzte Integral, welches den „kritischen" Streifen $0 < \sigma < 1$ durchsetzt, abzuschätzen. Der *v. Mangoldt*sche Beweis der Ungleichung $R(T) = O(\log T)$, wie auch ein später vereinfachter von *Landau*[107]), stützt sich wesentlich auf die *Hadamard*schen Resultate, d. h. auf die Produktentwicklung von $\zeta(s)$. Vor einigen Jahren wurde ein sehr eleganter Beweis dieser Ungleichung von *Backlund*[108]) gefunden, der

106) *H. v. Mangoldt,* Zur Verteilung der Nullstellen der Riemannschen Funktion $\xi(t)$, Math. Ann. 60 (1905), p. 1—19. Schon früher hatte *v. Mangoldt* [zu *Riemanns* Abhandlung „Über die Anzahl der Primzahlen unter einer gegebenen Größe", Crelles J. 114 (1895), p. 255—305] die Formel (30) mit einem Restgliede $O(\log^2 T)$ (statt $O(\log T)$) bewiesen.

107) *E. Landau,* Über die Verteilung der Nullstellen der Riemannschen Zetafunktion und einer Klasse verwandter Funktionen, Math. Ann. 66 (1909), p. 419—445.

108) *R. Backlund,* a) Sur les zéros de la fonction $\zeta(s)$ de Riemann, Paris C. R. 158 (1914), p. 1979—1981; b) Über die Nullstellen der Riemannschen Zeta-

nicht die Produktentwicklung (28), sondern nur eine ganz grobe Abschätzung von $\zeta(s)$ benutzt.

Ob die Abschätzung $R(T) = O(\log T)$ verbessert werden kann, weiß man nicht (vgl. jedoch Nr. 20, wo über Folgerungen der „*Riemann*schen Vermutung" berichtet wird); dagegen weiß man nach *Cramér*[109]), daß der *Mittelwert* $\frac{1}{T}\int\limits_0^T R(t)\,dt$ beschränkt ist, sogar daß er für $T \to \infty$ einem bestimmten Grenzwert, nämlich dem Werte $\frac{7}{8}$, zustrebt.[110]) Verfeinerte Resultate dieser Art sind neulich von *Littlewood*[111]) angegeben.

17. Über die Werte von $\zeta(s)$ auf einer vertikalen Geraden $\sigma = \sigma_0 (> \frac{1}{2})$. Betrachten wir zunächst die Halbebene $\sigma > 1$; da $\zeta(s)$ hier $\neq 0$ ist, ist es ein natürlicheres (und allgemeineres) Problem, nach den Werten von $\log \zeta(s)$, statt nach den Werten von $\zeta(s)$ selbst zu fragen, wo $\log \zeta(s)$ z. B. denjenigen (für $\sigma > 1$ regulären) Zweig des Logarithmus der Zetafunktion bezeichnet, der für reelles $s > 1$ reell ist. Dieser Zweig ist, nach der *Euler*schen Identität (22) durch

$$\log \zeta(s) = -\sum_{m=1}^{\infty} \log\left(1 - p_m^{-s}\right) \qquad (\sigma > 1) \tag{32}$$

gegeben, also (wenn $\log(1 - p_m^{-s})$ in eine Potenzreihe entwickelt wird) durch eine *Dirichlet*sche Reihe, in welcher die einzelnen Primzahlen separiert sind. Durch diesen Umstand wird es möglich, das in Nr. 7 angegebene Verfahren, welches auf der Theorie diophantischer Approximationen beruht, in einfacher Weise durchzuführen, indem die dort mit $M(\sigma_0)$ bezeichnete Wertmenge explizite bestimmt werden

---

funktion, Dissertation Helsingfors 1916, p. 1—24; c) Über die Nullstellen der Riemannschen Zetafunktion, Acta Math. 41 (1918), p. 345—375.

109) *H. Cramér*, Studien über die Nullstellen der Riemannschen Zetafunktion, Math. Ztschr. 4 (1919), p. 104—130.

110) Daß der Grenzwert gerade den Wert $\frac{7}{8}$ hat, hängt damit zusammen, daß $R(T)$ auf die Form $R(T) = \frac{7}{8} + O\left(\frac{1}{T}\right) + \frac{1}{\pi}\,\Delta \arg \zeta(s)$ gebracht werden kann (*Backlund*, a. a. O. 108c), wo die Konstante $\frac{7}{8}$ von der Gammafunktion und den anderen in der Funktionalgleichung eingehenden elementaren Funktionen herrührt, während $\Delta \arg \zeta(s)$ den Zuwachs von $\arg \zeta(s)$ angibt, wenn s den gebrochenen Linienzug $2,\ 2 + iT,\ \frac{1}{2} + iT$ durchläuft.

111) *J. E. Littlewood*, Researches in the theory of the Riemann ζ-function, Proc. London math. Soc. 20 (1922), (Records et cet. p. XXII—XXVIII). In dieser kurzen Mitteilung wird, ohne Beweise, eine Reihe sehr tiefgehender Sätze über $\zeta(s)$ angegeben.

kann. Es ergibt sich, daß diese Menge $M(\sigma_0)$ ein endliches Gebiet (in der komplexen Ebene) ist, das je nach der Lage der Abszisse $\sigma_0 (> 1)$ von einer oder von zwei konvexen Kurven begrenzt wird. Bei festem σ_0 läuft nun (nach Nr. 7) die von $\log \zeta(\sigma_0 + it)$ $(-\infty < t < \infty)$ beschriebene Kurve im Gebiete $M(\sigma_0)$ überall dicht herum, und es ist die Menge der Werte, welche $\log \zeta(s)$ in unendlicher Nähe der Geraden $\sigma = \sigma_0$ annimmt, mit $M(\sigma_0)$ identisch. Für σ_0 nahe an 1 ist $M(\sigma_0)$ ein einfach zusammenhängendes Gebiet (d. h. nur von einer Kurve begrenzt), das sich für $\sigma_0 \to 1$ nach und nach über die ganze Ebene ausbreitet[112]); hiermit ist speziell gefunden, daß $\log \zeta(s)$ in der Halbebene $\sigma > 1$ jeden Wert unendlich oft annimmt, also a fortiori, *daß* $\zeta(s)$ *in der Halbebene* $\sigma > 1$ (übrigens sogar in jedem Streifen $1 < \sigma < 1 + \delta$) *sämtliche Werte außer* 0 *unendlich oft annimmt.*[113])

Wesentlich schwieriger ist die Bestimmung der Werte von $\zeta(s)$ auf einer vertikalen Geraden $\sigma = \sigma_0$, welche im Streifen $\frac{1}{2} < \sigma \leqq 1$ liegt, weil das *Euler*sche Produkt, das ja die Quelle der obigen Untersuchung war, für $\sigma < 1$ divergiert (und für $\sigma = 1$, $t \neq 0$ nur bedingt konvergiert). Die auf der Theorie der diophantischen Approximationen beruhende Untersuchungsmethode läßt sich aber, obwohl in einer wesentlich modifizierten Form, auch hier verwenden[114]), und es ergibt sich, daß $\zeta(s)$ auf jeder festen Geraden $\sigma = \sigma_0$ im Streifen $\frac{1}{2} < \sigma \leqq 1$

---

112) *H. Bohr*, Sur la fonction $\zeta(s)$ dans le demi-plan $\sigma > 1$, Paris C. R. 154 (1912), p. 1078—1081. Die genaue Ausführung der betreffenden geometrischen Überlegungen findet sich in der Abhandlung: Om Addition af uendelig mange konvekse Kurve, Overs. Vidensk. Selsk. Köbenhavn 1913, p. 326—366. Die entsprechende Untersuchung der (bei den zahlentheoretischen Anwendungen wichtigen) Funktion $\frac{\zeta'}{\zeta}(s)$ findet sich bei *H. Bohr*, Über die Funktion $\frac{\zeta'}{\zeta}(s)$, Crelles J. 141 (1912), p. 217—234; die Untersuchung gestaltet sich hier wesentlich einfacher, weil die (konvexen) Begrenzungskurven der Gebiete $M(\sigma_0)$ einfach Kreise werden.

113) Über einen Beweis dieses letzteren (spezielleren) Resultates siehe *H. Bohr*, Über das Verhalten von $\zeta(s)$ in der Halbebene $\sigma > 1$, Gött. Nachr. 1911, p. 409—428. Schon früher hatten *H. Bohr* und *E. Landau*, Über das Verhalten von $\zeta(s)$ und $\zeta_x(s)$ in der Nähe der Geraden $\sigma = 1$, Gött. Nachr. 1910, p. 303—330 mit Hilfe allgemeiner funktionentheoretischer Methoden den weniger aussagenden Satz bewiesen, daß $\zeta(s)$ in jedem Streifen $1 - \delta < \sigma < 1 + \delta$ alle Werte, höchstens mit einer einzigen Ausnahme, annimmt.

114) Hierbei spielt eine von *H. Weyl* (Über die Gleichverteilung von Zahlen mod. Eins, Math. Ann. 77 (1916), p. 313—352) herrührende Verschärfung des in Nr 7 erwähnten *Kronecker*schen Satzes über diophantische Approximationen eine wesentliche Rolle.

Werte annimmt, die *in der ganzen Ebene überall dicht* liegen[115]), und ferner, daß die Menge der Werte von $\zeta(s)$ in unendlicher Nähe einer solchen Geraden gewiß *sämtliche Werte, höchstens mit Ausnahme des einen Wertes* 0, *enthält.*[116]) Bei der besonders wichtigen Frage, ob auch der „kritische" Wert 0 angenommen wird oder nicht — also ob die „*Riemann*sche Vermutung" falsch oder richtig ist — versagt aber die Methode, und sie vermag nur (weil sie im Grunde eine Wahrscheinlichkeitsmethode ist) zu zeigen, daß, falls 0 in einem Streifen $\sigma_0 - \delta < \sigma < \sigma_0 + \delta$ überhaupt angenommen wird, 0 jedenfalls „unendlich seltener" angenommen wird als jeder andere Wert a, d. h. wenn $N_0(T)$ und $N_a(T)$ die Anzahl von 0-Stellen bzw. a-Stellen im Rechteck $\sigma_0 - \delta < \sigma < \sigma_0 + \delta,\ 0 < t < T$ bezeichnen, so gilt für $T \to \infty$ die Gleichung $\lim N_0(T) : N_a(T) = 0$ [116])

18. Über die Größenordnung der Zetafunktion auf vertikalen Geraden. Man findet sehr leicht, daß $\zeta(s)$ in jeder Halbebene $\sigma > \sigma_0$ von endlicher Größenordnung in bezug auf t ist, und es läßt sich daher im ganzen Intervalle $-\infty < \sigma < \infty$ eine endliche *Größenordnungsfunktion* $\mu(\sigma)$ definieren (vgl. Nr. 6) als die untere Grenze aller Zahlen α, für welche $\zeta(\sigma + it)$ bei festem σ gleich $O(|t|^\alpha)$ ist. Die Funktionalgleichung (24) liefert die Relation $\mu(\sigma) = \mu(1-\sigma) + \frac{1}{2} - \sigma$, und es genügt somit $\mu(\sigma)$ für $\sigma \geqq \frac{1}{2}$ zu untersuchen. Für $\sigma > 1$ ist $\mu(\sigma) = 0$ (es ist sogar $|\zeta(s)|$ und $\frac{1}{|\zeta(s)|}$ beschränkt auf jeder vertikalen Geraden $\sigma = \sigma_0 > 1$). Die Schwierigkeit besteht darin, $\mu(\sigma)$ für $\frac{1}{2} \leqq \sigma \leqq 1$ zu bestimmen. Nachdem zuerst *Mellin* und später *Landau* gewisse Abschätzungen der μ-Funktion gewonnen hatten[117]), gelang es

---

115) *H. Bohr* und *R. Courant,* Neue Anwendungen der Theorie der diophantischen Approximationen auf die Riemannsche Zetafunktion, Crelles J. 144 (1914), p. 249-274.

116) *H. Bohr,* a) Sur la fonction $\zeta(s)$ de Riemann, Paris C. R. 158 (1914), p. 1986—1988; b) Zur Theorie der Riemannschen Zetafunktion im kritischen Streifen, Acta Math. 40 (1915), p. 67—100.

Schon früher hatten *H. Bohr* und *E. Landau,* Beiträge zur Theorie der Riemannschen Zetafunktion, Math. Ann. 74 (1913), p. 3—30 durch Überlegungen ganz anderer Art gezeigt, daß unter Annahme der Richtigkeit der „Riemannschen Vermutung" die Wertmenge von $\zeta(s)$ im Streifen $\sigma_0 - \delta < \sigma < \sigma_0 + \delta$ $(\frac{1}{2} < \sigma_0 < 1)$ alle Werte außer 0 enthält.

117) *H. Mellin,* Eine Formel für den Logarithmus transcendenter Funktionen von endlichem Geschlechte, Acta Soc. Sc. Fenn. 29 (1900), No. 4, p. 1—50, bewies, daß $\mu(\sigma) \leqq 1 - \sigma$ für $\frac{1}{2} \leqq \sigma \leqq 1$, und *E. Landau,* Sur quelques inégalités dans la théorie de la fonction $\zeta(s)$ de Riemann, Bull. Soc. math. France 33 (1905), p. 229—241 verschärfte dieses Resultat zu $\mu(\sigma) \leqq \frac{3}{4}(1-\sigma)$ für $\frac{1}{2} \leqq \sigma \leqq 1$.

Lindelöf[118]) durch allgemeine funktionentheoretische Betrachtungen zu beweisen (vgl Nr. 6), *daß $\mu(\sigma)$ eine stetige konvexe Funktion von σ ist,* woraus sofort folgt, wegen $\mu(\sigma) = 0$ für $\sigma > 1$ und $\mu(\sigma) = \frac{1}{2} - \sigma$ für $\sigma < 0$), daß die μ-Kurve für $0 \leqq \sigma \leqq 1$ im Dreieck mit den Endpunkten $(0, \frac{1}{2})$ $(\frac{1}{2}, 0)$ $(1, 0)$ verläuft, also speziell, daß $\mu(\sigma) \leqq \frac{1}{2}(1 - \sigma)$ für $\frac{1}{2} \leqq \sigma \leqq 1$. Ganz neuerdings ist es *Hardy* und *Littlewood*[119]) gelungen, über das *Lindelöf*sche Resultat hinauszukommen, und zwar u. a. zu beweisen, daß die μ-Kurve im Punkte $\sigma = 1$ die Abszissenachse berührt. Der genaue Verlauf der μ-Funktion für $0 < \sigma < 1$ ist noch heute unbekannt (vgl. jedoch Nr. 20).

Ein besonderes Interesse bietet die Untersuchung der Größenverhältnisse von $\zeta(s)$ $\left(\text{und } \frac{1}{\zeta(s)}\right)$ *auf der Geraden* $\sigma = 1$, die den kritischen Streifen von der „trivialen" Halbebene $\sigma > 1$ trennt, und wo $\zeta(s)$ (nach Nr. 17) „zum ersten Mal" Werte annimmt, die in der ganzen Ebene überall dicht liegen. Über das Resultat $\mu(1) = 0$, d. h. $\zeta(1 + it) = O(t^{\varepsilon})$ hinaus, bewies *Mellin*[120]) das viel schärfere Resultat $\zeta(1 + it) = O(\log t)$, und mit Hilfe tiefgehender Untersuchungen über diophantische Approximationen haben später *Hardy-Littlewood*[121]) und *Weyl*[122]) die *Mellin*sche Abschätzung zu $\zeta(1 + it) = o(\log t)$ und *Weyl* sogar zu

$$\zeta(1 + it) = O\left(\frac{\log t}{\log\log t}\right) \tag{33}$$

verschärfen können. Andererseits haben *Bohr* und *Landau*[123]) ebenfalls mit Hilfe diophantischer Approximationen bewiesen, daß

$$\zeta(1 + it) \neq o(\log\log t) \tag{34}$$

ist. Die Frage nach der „wahren" Größenordnung von $\zeta(1 + it)$ ist aber hiermit noch lange nicht gelöst (vgl. jedoch Nr. 20), denn es besteht ja noch eine beträchtliche Lücke zwischen (33) und (34). Die entsprechende Frage über $1 : \zeta(1 + it)$ ist noch weniger aufgeklärt. Nachdem es zuerst *Mertens*[124]), durch eine neue Beweisanordnung des

118) *E. Lindelöf,* Quelques remarques sur la croissance de la fonction $\zeta(s)$, Bull. Sc. math. (2) 32 I (1908), p. 341—356.

119) Vgl. *J. Littlewood,* a. a. O. 111).

120) *H. Mellin,* a. a. O. 117).

121) *G. H. Hardy* und *J. Littlewood,* Some problems of diophantine approximation, Internat. Congr. of math. Cambridge 1 (1912), p. 223—229.

122) *H. Weyl,* Zur Abschätzung von $\zeta(1 + it)$, Math. Ztschr. 10 (1921), p. 88—100.

123) *H. Bohr* und *E. Landau,* a. a. O. 113).

124) *F. Mertens,* Über eine Eigenschaft der Riemannschen ζ-Funktion, Sitzungsber. Acad. Wien 107, IIa (1898), p. 1429—1434.

*Hadamard-de la Vallée Poussin*schen Satzes: $\zeta(1+it) \neq 0$, gelungen war, eine obere Grenze $\varphi(t)$ für $\frac{1}{|\zeta(1+it)|}$ explizite anzugeben, fand *Landau*[125]) die für seine zahlentheoretischen Zwecke wichtige Abschätzung $1 : \zeta(1+it) = O\{(\log t)^c\}$, die von *Landau* selbst[29]) auf $O(\log t \cdot \log\log t)$, dann von *Gronwall*[126]) auf $1 : \zeta(1+it) = O(\log t)$ und neuerdings von *Littlewood*[111]) auf

$$1 : \zeta(1+it) = O\left(\frac{\log t}{\log\log t}\right)$$

verschärft wurde; andererseits weiß man aber nur, nach *Bohr*, daß

$$1 : \zeta(1+it) \neq O(1)$$

ist, also daß $\zeta(s)$ auf der Geraden $\sigma = 1$ beliebig kleine Werte annimmt[127])), ohne daß man bis jetzt imstande gewesen ist, irgendeine mit t ins Unendliche wachsende Funktion $\psi(t)$ explizite anzugeben, für die $1 : \zeta(1+it) \neq O(\psi(t))$ ist.

Obwohl $\zeta(s)$ auf keiner Geraden $\sigma = \sigma_0 \leqq 1$ beschränkt bleibt, ist sie doch bei jedem σ_0 im Intervalle $\frac{1}{2} < \sigma < 1$ *im Mittel* beschränkt, ja es ist sogar ihr Quadrat im Mittel beschränkt; denn aus dem *Schnee*schen Mittelwertsatze (Nr. 9) ergibt sich leicht[128]), daß

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |\zeta(\sigma+it)|^2\, dt = \sum \frac{1}{n^{2\sigma}} = \zeta(2\sigma). \qquad (\tfrac{1}{2} < \sigma < 1)$$

Hieraus folgt mit Hilfe der Funktionalgleichung, daß für $\sigma < \frac{1}{2}$ der Mittelwert $\frac{1}{2T}\int_{-T}^{T} |\zeta(\sigma+it)|^2 dt$ sich asymptotisch wie

$$\tfrac{1}{2}(2\pi)^{2\sigma-1}\frac{\zeta(2-2\sigma)}{2-2\sigma}\cdot T^{1-2\sigma}$$

verhält. Viel schwieriger ist das Problem des Verhaltens von $\frac{1}{2T}\int_{-T}^{T} |\zeta(\sigma+it)|^2 dt$ für $\sigma = \frac{1}{2}$; dieses wurde von *Hardy* und *Little-*

125) *E. Landau,* Neuer Beweis des Primzahlsatzes und Beweis des Primidealsatzes, Math. Ann. 56 (1903), p. 645—670.

126) *T. Gronwall,* Sur la fonction $\zeta(s)$ de Riemann au voisinage de $\sigma = 1$, Palermo Rend. 35 (1913), p. 95—102.

127) Ein direkter Beweis dieses Satzes (der ja als Spezialfall in dem in Nr. 17 erwähnten Resultate über $\zeta(1+it)$ enthalten ist) findet sich bei *H. Bohr,* Sur l'existence de valeurs arbitrairement petites de la fonction $\zeta(s) = \zeta(\sigma+it)$ de Riemann pour $\sigma > 1$, Oversigt. Vidensk. Selsk. Köbenhavn 1911, p. 201—208. Vgl. auch *H. Bohr,* Note sur la fonction Zéta de Riemann $\zeta(s) = \zeta(\sigma+it)$ sur la droite $\sigma = 1$, Oversigt Vidensk. Selsk. Köbenhavn 1913, p. 3—11.

128) Vgl. z. B. *E. Landau,* Handbuch, a. a. O. 1), § 228.

wood gelöst[81b]), und zwar mit dem Ergebnis

$$\frac{1}{2T}\int_{-T}^{T}|\zeta(\tfrac{1}{2}+it)|^2 dt \sim \log T.$$

Neuerdings haben *Hardy* und *Littlewood*[129]) bewiesen, daß auch $\frac{1}{2T}\int_{-T}^{T}|\zeta(\sigma+it)|^4 dt$ bei festem σ im Intervalle $\frac{1}{2}<\sigma<1$ beschränkt bleibt und sogar für $T\to\infty$ einem bestimmten Grenzwerte zustrebt. Der Beweis basiert auf der von *Hardy* und *Littlewood*[129]) entdeckten sogenannten „*approximativen Funktionalgleichung*", welche besagt, daß für $|\sigma|<k$, $x>K$, $y>K$ und $2\pi xy=|t|$

$$\zeta(s)=\sum_{n<x}\frac{1}{n^s}+\chi\sum_{n<y}\frac{1}{n^{1-s}}+R,$$

wo $\chi=2(2\pi)^{s-1}\sin\frac{\pi s}{2}\Gamma(1-s)$ und $R=O(x^{-\sigma})+O(y^{\sigma-1}|t|^{\frac{1}{2}-\sigma})$. Diese Formel, welche bei Abschätzungen der Zetafunktion im kritischen Streifen $0<\sigma<1$ von der größten Bedeutung ist, ist[129]) eine Art „Kompromiß zwischen der für $\sigma>1$ gültigen Formel $\zeta(s)=\sum\frac{1}{n^s}$ und der für $\sigma<0$ gültigen Formel $\zeta(s)=\chi\sum\frac{1}{n^{1-s}}$".

Die oben erwähnten Mittelwertsformeln und andere ähnliche, die sich durch Anwendung des *Schnee*schen Mittelwertsatzes auf mit der Zetareihe beschlechtete *Dirichlet*sche Reihen abgeleitet werden, spielen bei neueren Untersuchungen über die Zetafunktion eine immer wichtigere Rolle.

19. Näheres über die Nullstellen im kritischen Streifen. Aus dem Satze (Nr. **14**), daß keine Nullstelle von $\zeta(s)$ *auf* der Geraden $\sigma=1$ gelegen ist, folgt sofort, daß für eine mit $t\to\infty$ „hinreichend" schnell zu 0 abnehmende Funktion $\varphi(t)$ der asymptotisch unendlich schmale Streifen $1\geqq\sigma>1-\varphi(t)$ ebenfalls nullpunktsfrei ist. Mit Hilfe der *Hadamard*schen Produktentwicklung von $\zeta(s)$ gelang es *de la Vallée Poussin*[130]), und später durch elementarere Mittel *Landau*[125]), eine solche Funktion $\varphi(t)$ explizite anzugeben; das *de la Vallée Poussin*-

129) *G. H. Hardy* und *J. Littlewood*, The approximate functional equation in the theory of the zeta-function, with applications to the divisor-problems of Dirichlet and Piltz, Proc. London math. Soc. 21 (1922), p. 39—74.

130) *Ch. de la Vallée Poussin*, Sur la fonction $\zeta(s)$ de Riemann et le nombre des nombres premiers inférieurs à une limite donnée, Mém. Acad. Belgique 59, No. 1 (1899—1900), p. 1—74.

sche Resultat, welches das genauere war, besagt, daß $\frac{k}{\log t}$ (bei passender Wahl von $k > 0$) eine zulässige Funktion $\varphi(t)$ ist. Ganz neuerdings hat *Littlewood*[111]) dieses dahin verschärft, daß $\varphi(t) \sim \frac{k \log\log t}{\log t}$ angenommen werden darf. Ob es aber eine Konstante $\sigma_0 < 1$ gibt mit der Eigenschaft, daß $\zeta(s) \neq 0$ für $\sigma > \sigma_0$, ist immer noch unentschieden.

Wie in Nr. **16** erwähnt, ist die Anzahl $N(T)$ von Nullstellen im Rechteck $0 < \sigma < 1$, $0 < t < T$ für $T \to \infty$ asymptotisch gleich $k \cdot T \log T$. Über die Verteilung dieser Nullstellen haben *Bohr* und *Landau*[65a]) bewiesen, daß *ihre Mehrzahl in nächster Nähe der Geraden* $\sigma = \frac{1}{2}$ *gelegen ist*, d. h. bei jedem festen $\delta > 0$ ist die Anzahl $N_1(T)$ von Nullstellen, welche innerhalb des obigen Rechtecks, aber außerhalb des dünnen Streifens $\frac{1}{2} - \delta < \sigma < \frac{1}{2} + \delta$ liegen, gleich $o(T \log T)$; dies folgt aus einem allgemeinen, in Nr. **10** erwähnten Satz über *Dirichlet*sche Reihen (auf die Zetareihe mit abwechselndem Vorzeichen angewendet), nach welchem die besprochene Anzahl $N_1(T)$ sogar gleich $O(T)$ ist. Durch eine weitergehende Untersuchung wurde dieses Resultat zuerst[65b]) zur Gleichung $N_1(T) = o(T)$ und später von *Carlson*[131]) mit Hilfe seines in Nr. **10** erwähnten Satzes über *Dirichlet*sche Reihen zu

$$N_1(T) = O(T^{1-4\delta^2+\varepsilon}) \qquad (\varepsilon > 0)$$

verschärft. Ferner gelang es *Littlewood*[111]) eine mit $t \to \infty$ zu 0 abnehmende Funktion $\varphi(t)$ explizite anzugeben mit der Eigenschaft, daß das Hauptresultat $N_1(T) = o(T \log T)$ noch gültig bleibt, wenn $N_1(T)$ die Anzahl der Nullstellen im Gebiete $\sigma > \frac{1}{2} + \varphi(t)$, $0 < t < T$ angibt.

Ein sehr bedeutsamer Fortschritt in den Untersuchungen über die Nullstellen von $\zeta(s)$ wurde von *G. H. Hardy*[132]) gemacht, dem es zuerst zu beweisen gelang, daß *auf der Geraden* $\sigma = \frac{1}{2}$ *tatsächlich unendlich viele Nullstellen liegen.* Daß es überhaupt auf dieser „kritischen" Geraden Nullstellen gibt, war schon früher durch numerische Untersuchungen festgestellt.[133]) Der ursprüngliche *Hardy*sche Beweis

---

131) *F. Carlson*, a. a. O. 67). Vgl. auch eine frühere Arbeit von *S. Wennberg*, a. a. O. 33), worin die weniger genaue Relation $N_1(T) = O\left(T : (\log T)^{\frac{\delta}{1-\delta}}\right)$ bewiesen wird.

132) *G. H. Hardy*, Sur les zéros de la fonction $\zeta(s)$ de Riemann, Paris C. R. 158 (1914), p. 1012—1014.

133) Vgl. *J. Gram*, a) Note sur le calcul de la fonction $\zeta(s)$ de Riemann, Oversigt Vidensk. Selsk. Köbenhavn 1895, p. 303—308; b) Note sur les zéros de la fonction $\zeta(s)$ de Riemann, Acta Math. 27 (1903), p. 289—304; *Ch. de la Vallée Poussin*, a. a. O. 130); *E. Lindelöf*, a) Quelques applications d'une formule som-

dieses Satzes nahm seinen Ausgangspunkt in der folgenden von *Mellin*[134]) herrührenden Integraldarstellung einer Thetareihe durch die Zetafunktion

$$\sum_{n=-\infty}^{n=\infty} e^{-n^2 y} = 1 + \sqrt{\frac{\pi}{y}} + \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \Gamma\left(\frac{s}{2}\right) y^{-\frac{s}{2}} \zeta(s)\, ds, \qquad (\Re(y) > 0)$$

welche zu den in Nr. **11** erwähnten Typen von Integraldarstellungen einer *Dirichlet*schen Reihe durch eine andere *Dirichlet*sche Reihe gehört. Wird unter dem Integralzeichen statt $\zeta(s)$ die in Nr. **14** erwähnte Funktion $\eta(s) = \Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s)$ eingeführt, und $\eta(\frac{1}{2} + it) = \varrho(t)$ gesetzt, so geht die *Mellin*sche Formel, wenn $y = \pi e^{2\alpha i}\left(-\frac{\pi}{4} < \alpha < \frac{\pi}{4}\right)$ gewählt wird, in die Formel

$$(35) \qquad \int_0^\infty (e^{\alpha t} + e^{-\alpha t})\varrho(t)\, dt = -4\pi\cos\frac{\alpha}{2} + 2\pi e^{\frac{\alpha i}{2}} \sum_{n=-\infty}^{n=\infty} e^{-n^2\pi e^{2\alpha i}}$$

über, wobei noch benutzt ist, daß (wegen der Funktionalgleichung $\eta(s) = \eta(1-s)$) die Funktion $\varrho(t)$ eine gerade ist. Es handelt sich darum zu beweisen, daß $\varrho(t)$ unendlich viele *reelle* Nullstellen besitzt, also daß der Integrand in (35) in unendlich vielen Punkten des Integrationsintervalles $0 < t < \infty$ verschwindet. Der Beweis beruht nun darauf, daß die Funktion $\eta(s)$ (wie z. B. aus der *Riemann*schen Integraldarstellung (26) ersichtlich) auf der betrachteten Geraden $s = \frac{1}{2} + it$ *reell* ist, d. h. daß $\varrho(t)$ für reelles t reell ist, und daß daher, wenn $\varrho(t)$ nur *endlich viele* Nullstellen auf der reellen Achse besäße, für alle t von einer gewissen Stelle t_0 an durchweg die Gleichung $\varrho(t) = |\varrho(t)|$ oder durchweg die Gleichung $\varrho(t) = -|\varrho(t)|$ stattfände. Die ursprüng-

---

matoire générale, Acta soc. sc. Fenn. 31, No. 3 (1903), p. 1—46; b) Sur une formule sommatoire générale, Acta Math. 27 (1903), p. 305—311; *R. Backlund*, Einige numerische Rechnungen, die Nullpunkte der Riemann'schen ζ-Funktion betreffend, Öfversigt Finska Vetensk. Soc. (A) 54 (1911—12), No. 3, p. 1—7 und a. a. O. 108a), b). Das weitestgehende Resultat rührt von *Backlund* her, welcher in der letztzitierten Arbeit beweist, 1. daß auf der Strecke $\sigma = \frac{1}{2}$, $0 < t < 200$ genau 79 Nullstellen von $\zeta(s)$ liegen, und 2. daß es außer diesen 79 Nullstellen keine einzige Nullstelle von $\zeta(s)$ im Rechtecke $0 < \sigma < 1$, $0 < t < 200$ gibt. Dies Resultat gehört zu den kräftigsten Argumenten für den Glauben an die Richtigkeit der „*Riemann*schen Vermutung".

134) *H. Mellin*, Über eine Verallgemeinerung der Riemannschen Funktion $\zeta(s)$, Acta soc. sc. Fenn. 24, No. 10 (1899), p. 1—50. Ein anderes Integral der Funktion $\zeta(\frac{1}{2} + it)$, welches (nach *Hardy*) ebenfalls zum Beweis des *Hardy*schen Satzes verwendbar ist, ist von *S. Ramanujan*, New expressions for Riemann's functions $\xi(s)$ and $\Xi(t)$, Quart. J. 183 (1915), p. 253—261, angegeben.

liche Fassung des *Hardy*schen Beweises, daß diese Annahme mit der Gleichung (35) in Widerspruch steht, wurde bald von *Landau*[135]) etwas vereinfacht. In der vereinfachten Form kommt dieser Widerspruch einfach so heraus, daß einerseits aus (35), unter der (falschen) Annahme $\varrho(t) = |\varrho(t)|$ (oder $\varrho(t) = -|\varrho(t)|$), durch den Grenzübergang $\alpha \to \frac{\pi}{4}$ (bei welchem die Thetareihe verschwindet) die Konvergenz des Integrales

$$\int_0^\infty e^{\frac{\pi}{4}t}|\varrho(t)|\,dt$$

geschlossen wird, woraus, bei Wiedereinführung der Zetafunktion selbst, die *Konvergenz* von

$$(36) \qquad \int_1^\infty t^{-\frac{1}{4}}|\zeta(\tfrac{1}{2}+it)|\,dt$$

sich ergibt, während andererseits mit Hilfe des *Cauchy*schen Satzes leicht gezeigt wird, daß für hinreichend große T das Integral $\left|\int_1^T \zeta(\frac{1}{2}+it)\,dt\right|$, und also um so mehr das Integral $\int_1^T |\zeta(\frac{1}{2}+it)|\,dt$, größer als kT ist, woraus durch eine grobe Abschätzung die Ungleichung

$$\int_1^T t^{-\frac{1}{4}}|\zeta(\tfrac{1}{2}+it)|\,dt > kT^{\frac{3}{4}},$$

also gewiß die *Divergenz* von (36) erfolgt.

Der *Hardy*sche Beweis wurde bald so umgeformt, daß er nicht nur die Relation $M(T) \to \infty$, wo $M(T)$ die Anzahl der Nullstellen *auf* der Geraden $\sigma = \frac{1}{2}$ mit Ordinaten zwischen 0 und T bezeichnet, sondern zugleich auch eine untere Abschätzung von $M(T)$ liefern konnte. Nachdem zuerst *Landau*[135]) die Ungleichung $M(T) > K \log\log T$ bewiesen hatte, wurden wesentlich weitergehende Abschätzungen von *de la Vallée Poussin*[136]) und unabhängig davon von *Hardy* und *Littlewood*[31b]) gegeben; die letzteren, welche die genaueren Resultate erhielten, bewiesen u. a., daß $M(T) > KT^{\frac{3}{4}-\varepsilon}$. In den Beweisen dieser weitergehenden Sätze wurden verschiedene wesentliche Änderungen der ursprünglichen *Hardy*schen Beweismethode vorgenommen (vor allem konnten *Hardy* und *Littlewood* in ihrem Beweis der Ungleichung $M(T) > K \cdot T^{\frac{3}{4}-\varepsilon}$ den Gebrauch der *Mellin*schen Formel (35) und da-

135) *E. Landau*, Über die Hardysche Entdeckung unendlich vieler Nullstellen der Zetafunktion mit reellem Teil $\frac{1}{2}$, Math. Ann 76 (1915), p. 212—248.

136) *Ch. de la Vallée Poussin*, Sur les zéros de $\zeta(s)$ de Riemann, Paris C. R. 163 (1916), p. 418—421 und p. 471—473.

durch die Einführung der elliptischen Thetafunktionen gänzlich vermeiden); die wesentliche Idee der Beweismethode ist aber immer dieselbe geblieben. Neuerdings ist es *Hardy* und *Littlewood*[137]) durch eine sehr verfeinerte Analyse gelungen, sogar die Abschätzung

$$M(T) > KT$$

zu beweisen, und damit festzustellen, daß die Anzahl $M(T)$ von Nullstellen auf der Geraden $\sigma = \frac{1}{2}$ jedenfalls „fast" von derselben Größenordnung ist als die Anzahl $N(T)\,(\sim \frac{1}{2\pi}\, T \log T)$ von Nullstellen im ganzen Streifen $0 < \sigma < 1$.

20. Folgerungen aus der „Riemannschen Vermutung". Es wird in dieser Nummer über einige Untersuchungen referiert, deren Resultate nicht auf gesicherte Wahrheit Anspruch erheben dürfen, weil sie auf der *Annahme* der Richtigkeit der *Riemann*schen Vermutung, daß alle nicht-trivialen Nullstellen auf der Geraden $\sigma = \frac{1}{2}$ liegen, beruhen.[138]) Der Weg zu solchen Untersuchungen wurde von *Littlewood*[138]) geöffnet, der bei dem Problem der Bestimmung der μ-Funktion zuerst gezeigt hat, in welcher Weise die Annahme $\zeta(s) \neq 0$ für $\sigma > \frac{1}{2}$ für das funktionentheoretische Studium der Zetafunktion ausgenützt werden kann. Die *Littlewood*sche Methode, welche auf der Anwendung des sogenannten *Hadamard*schen Dreikreisensatzes (vgl. Art. II C 4, Nr. 62) auf die (unter Annahme der Richtigkeit der *Riemann*schen Vermutung) für $\sigma > \frac{1}{2}$, $t > 0$ *reguläre* Funktion $\log \zeta(s)$ beruht, lieferte die *genaue Bestimmung der μ-Funktion für alle σ*, und zwar mit dem Resultat $\mu(\sigma) = 0$ für $\sigma \geqq \frac{1}{2}$, also (vgl. Nr. 18) $\mu(\sigma) = \frac{1}{2} - \sigma$

137) *G. H. Hardy* und *J. Littlewood*, The zeros of Riemann's Zeta-Funktion on the critical line, Math. Ztschr. 10 (1921), p. 283—317. Die Verfasser beweisen übrigens noch mehr, nämlich daß bei jedem $a > 0$ und $U = T^a$ die Ungleichung $M(T+U) - M(T) > KU$ für alle hinreichend großen T besteht. Der Beweis dieses letzten Satzes basiert auf der „approximativen Funktionalgleichung" (Nr. 18), welche in dieser Abhandlung zum ersten Mal (obwohl nicht in ihrer weitestgehenden Form) bewiesen wird.

138) Im Laufe der vielen (bisher mißglückten) Versuche, die „*Riemann*sche Vermutung" zu beweisen, haben verschiedene Forscher das Problem in mannigfacher Weise umgeformt. Vor allem hat *J. Littlewood*, Quelques conséquences de l'hypothèse que la fonction $\zeta(s)$ de Riemann n'a pas de zéros dans le demiplan $R(s) > \frac{1}{2}$, Paris C. R. 154 (1912), p. 263—266, entdeckt, daß die *Riemann*sche Hypothese gleichwertig ist mit der Hypothese, daß die *Dirichlet*sche Reihe für $1 : \zeta(s)$ (siehe Nr. 22) die Konvergenzabszisse $\sigma_B = \frac{1}{2}$ besitze. Vgl. auch eine Arbeit von *M. Riesz*, Sur l'hypothèse de Riemann, Acta Math. 40 (1916), p. 185—190, in welcher die Umformung des Problems auf einer von *Riesz* gefundenen interessanten Integraldarstellung der Funktion $1 : \zeta(s)$ beruht.

für $\sigma \leqq \frac{1}{2}$.[139]) Über das Resultat $\mu(\sigma) = 0$ für $\sigma \geqq \frac{1}{2}$ hinaus bewies *Littlewood*[138]), daß bei jedem σ des Intervalles $\frac{1}{2} < \sigma < 1$ und jedes $a > 2$

(37) $$\log \zeta(s) = O((\log t)^{a(1-\sigma)}),$$

und er konnte ferner $\zeta(s)$ *auf der Geraden* $\sigma = 1$ viel genauer abschätzen, als es ohne Benutzung der „*Riemann*schen Vermutung" möglich gewesen war (vgl. Nr. 18). Neuerdings hat *Littlewood*[111]) seine Abschätzung von $\zeta(1 + it)$ noch verbessert, und zwar die Relation

$$\zeta(1 + it) = O(\log\log t)$$

bewiesen; hiermit ist das Problem, die Größenordnung von $\zeta(1 + it)$ zu bestimmen, zu einem gewissen Abschluß gebracht, weil ja andererseits bekannt ist (Nr. 18), daß $\zeta(1 + it) \neq o(\log\log t)$.

An die erste *Littlewood*sche Arbeit schloß sich eine Arbeit von *Bohr* und *Landau*[140]) an, worin (unter Annahme der *Riemann*schen Vermutung) die Relation

(38) $$\log \zeta(s) \neq O((\log t)^{b(1-\sigma)}) \qquad (\tfrac{1}{2} < \sigma < 1)$$

bei passender Wahl einer Konstanten $b > 0$ bewiesen wurde. Hiermit wurde (unter Berücksichtigung des *Littlewood*schen Resultates (37)) auch die Größenordnung von $\log \zeta(s)$ im kritischen Streifen einigermaßen genau bestimmt.

Mit der Frage nach der Größenordnung von $\zeta(s)$ eng verbunden ist die Frage nach der „feineren" Verteilung der Ordinaten der nichttrivialen Nullstellen von $\zeta(s)$, d. h. die Frage nach dem Verhalten des Restgliedes $R(T)$ in der *Riemann-v. Mangoldt*schen Formel (31)[141]), und auch bei diesem Problem ist es möglich gewesen, unter Heranziehen der *Riemann*schen Hypothese recht genaue Aufschlüsse zu erhalten. Einerseits hat *Landau*[142]) bewiesen, daß $R(T) \neq O(1)$ (also daß $R(T)$ nicht beschränkt bleibt), und später haben *Bohr* und *Lan-*

139) Nach *R. Backlund,* Über die Beziehung zwischen Anwachsen und Nullstellen der Zetafunktion, Öfversigt Finska Vetensk. Soc. 61 (1918—19). No. 9, ist es, um den Beweis der Gleichung $\mu(\sigma) = 0$ für $\sigma \geqq \frac{1}{2}$ zu führen, nicht nötig, die *Riemann*sche Vermutung in ihrem vollen Umfange zu benutzen. Vielmehr ist die Annahme: $\mu(\sigma) = 0$ für $\sigma \geqq \frac{1}{2}$ mit der Annahme, das bei jedem festen $\delta > 0$ die Anzahl $A(T)$ von Nullstellen im Rechtecke $\frac{1}{2} + \delta < \sigma < 1$, $T < t < T + 1$ gleich $o(\log T)$ ist, gleichwertig. Sichergestellt (d. h. ohne irgendeine Annahme bewiesen) ist nur die Abschätzung $A(T) = O(\log T)$.

140) *H. Bohr* und *E. Landau,* Beiträge zur Theorie der Riemannschen Zetafunktion, Math. Ann. 74 (1913), p. 3—30.

141) Der Zusammenhang dieser beiden Probleme ist neuerdings von *J. Littlewood,* a. a. O. 111) einem tiefgehenden Studium unterworfen worden.

142) *E. Landau,* Zur Theorie der Riemannschen Zetafunktion, Vierteljahrschr. Naturf. Ges. Zürich 56 (1911), p. 125—148.

dau[140]) aus der Ungleichung (38) gefolgert, daß sogar

$$R(T) \neq O((\log T)^c)$$

bei passender Wahl einer Konstanten $c > 0$. Andererseits verbesserte *Bohr*[143]) die *v. Mangoldt*sche Abschätzung $R(T) = O(\log T)$ zu $R(T) = o(\log T)$; dieses Resultat wurde dann von *Cramér*[144]), *Landau*[145]) und *Littlewood*[111]) noch etwas verschärft; letzterer bewies, daß

$$R(T) = O\left(\frac{\log T}{\log\log T}\right)$$

und außerdem (vgl. Nr. **16**), daß

$$\frac{1}{T}\int_0^T |R(t) - \tfrac{7}{8}|\, dt = O(\log\log T).$$

Schließlich sei noch ein interessanter Satz von *Littlewood*[143]) über das Verhalten von $\zeta(s)$ in der unmittelbaren Nähe der kritischen Geraden $\sigma = \frac{1}{2}$ erwähnt, welcher (unter Annahme der *Riemann*schen Vermutung) das Resultat (vgl. Nr. **17** und **19**), daß $\zeta(s)$ in jedem Streifen $\frac{1}{2} - \delta < \sigma < \frac{1}{2} + \delta$ sämtliche Werte unendlich oft annimmt, dahin verschärft, daß $\zeta(s)$ bei festem $K > 0$, $\delta > 0$ in jedem Kreise $|s - (\frac{1}{2} + i\tau)| < \delta$ $(\tau > \tau_0 = \tau_0(K, \delta))$ sämtliche Werte vom absoluten Betrage $< K$ annimmt.

21. Verallgemeinerte Zetafunktionen. An die *Riemann*sche Zetafunktion schließen sich mehrere Klassen anderer „Zetafunktionen" an, welche ebenfalls durch *Dirichlet*sche Reihen definiert werden und Eigenschaften besitzen, die in vielen Hinsichten mit denjenigen der *Riemann*schen Zetafunktion übereinstimmen. Die interessantesten Klassen solcher Funktionen werden, wegen des zahlentheorischen Charakters der Koeffizienten ihrer Reihenentwicklung, erst im zweiten Teil des Artikels besprochen, wo sie im Zusammenhange mit den zahlentheoretischen Problemen, für deren Behandlung sie erfunden sind, eingeführt werden. In diesem Paragraphen sollen nur von rein analytischem Gesichtspunkte aus gewisse „verallgemeinerte" Zetafunktionen

143) Vgl. *H. Bohr*, *E. Landau* und *J. Littlewood*, Sur la fonction $\zeta(s)$ dans le voisinage de la droite $\sigma = \frac{1}{2}$, Bull. Acad. Belgique 15 (1913), p. 1144—1175.

144) *H. Cramér*, Über die Nullstellen der Zetafunktion, Math. Ztschr. 2 (1918), p. 237—241. In dieser Abhandlung wird u. a. bewiesen, daß zur Herleitung der Abschätzung $R(T) = o(\log T)$ nicht die volle „*Riemann*sche Vermutung" nötig ist, sondern nur die (vgl. Note 139) weniger aussagende sogenannte „*Lindelöf*sche Vermutung" $\mu(\sigma) = 0$ für $\sigma \geq \frac{1}{2}$.

145) *E. Landau*, Über die Nullstellen der Zetafunktion, Math. Ztschr. 6 (1920), p. 151—154. Vgl. hierzu auch *H. Cramér*, Bemerkung zu der vorstehenden Arbeit des Herrn E. Landau, Math. Ztschr. 6 (1920), p. 155—157.

kurz besprochen werden, deren Definitionen kein zahlentheoretisches Moment enthalten.

Betrachten wir zunächst die Reihe

$$(39) \qquad \zeta(w, s) = \sum_{n=0}^{\infty} \frac{1}{(w+n)^s} \text{ [146]}$$

oder allgemeiner die von *Lipschitz*[147] und *Lerch*[148] untersuchte Reihe

$$\sum_{n=0}^{\infty} \frac{e^{2\pi i n x}}{(w+n)^s},$$

wo x eine komplexe Zahl bedeutet, deren reeller Teil $\Re(x)$ etwa dem Intervalle $0 \leqq x < 1$ angehört, während ihr imaginärer Teil $\Im(x) \geqq 0$ ist. Diese Reihe, als Funktion von s betrachtet, ist offenbar im Falle $\Im(x) > 0$ in der ganzen s-Ebene konvergent und stellt eine ganze Transzendente dar; für $\Im(x) = 0$ ist sie, abgesehen vom Fall $x = 0$, in der Halbebene $\sigma > 0$ konvergent, definiert aber auch hier eine ganze Transzendente, und im speziellen Falle $x = 0$ (d. h. im Falle der Reihe (39)) konvergiert sie für $\sigma > 1$ und stellt, wie die Zetareihe selbst, eine meromorphe Funktion dar, die überall regulär ist mit Ausnahme des einzigen Punktes $s = 1$, wo sie einen Pol erster Ordnung besitzt. Dies ersieht man in ganz ähnlicher Weise, wie *Riemann* die Fortsetzbarkeit von $\zeta(s)$ bewies, d. h. es wird die Reihe zunächst

146) Diese Funktion ist besonders von *H. Mellin*, Über eine Verallgemeinerung der Riemannschen Funktion $\zeta(s)$, Acta soc. sc. Fenn. 24, No. 10 (1899), im Zusammenhange mit seinen Studien über Umkehrformeln (vgl. Note 70) näher untersucht. Vgl. auch *A. Piltz*, Über die Häufigkeit der Primzahlen in arithmethischen Progressionen und über verwandte Gesetze, Habilitationsschrift, Jena 1884, p. 1—48; *E. Lindelöf*, a. a. O. 97) und *E. W. Barnes*, a) The theory of the Gamma function, Mess. of Math. (2) 29 (1899), p. 64—128; b) The theory of the double Gamma function, London Phil. Trans. (A) 196 (1901), p. 265—387; c) On the theory of the multiple Gamma function, Cambridge Phil. Trans. 19 (1904), p. 374—425; *Barnes* untersucht auch Reihen der Form

$$\sum_{n_1=0}^{\infty} \cdots \sum_{n_p=0}^{\infty} \frac{1}{(w + n_1 \omega_1 + \cdots n_p \omega_p)^s}.$$

147) *R. Lipschitz*, a) Untersuchung einer aus vier Elementen gebildeten Reihe, Crelles J. 54 (1857), p. 313—328; b) Untersuchung der Eigenschaften einer Gattung von unendlichen Reihen, Crelles J. 105 (1889), p. 127—156.

148) *M. Lerch*, Note sur la fonction $K(w, x, s) = \sum_{k=0}^{\infty} \frac{e^{2k\pi i x}}{(w+k)^s}$, Acta Math. 11 (1887), p. 19—24.

durch ein einfaches bestimmtes Integral dargestellt und dieses wieder in ein komplexes Kurvenintegral umgeformt. Aus dieser letzten Integraldarstellung folgt weiter, wie bei *Riemann*, durch Deformation des Integrationsweges und Anwendung des *Cauchy*schen Satzes[149]), daß unsere Funktion einer der *Riemann*schen ähnlichen *Funktionalgleichung* genügt, welche in der Form

$$\sum_{n=0}^{\infty} \frac{e^{2\pi i n x}}{(w+n)^s} = \frac{\Gamma(1-s)}{(2\pi i)^{1-s}} \sum_{m=-\infty}^{\infty} \frac{e^{2\pi i w(m-x)}}{(-x+m)^{1-s}}$$

geschrieben werden kann.

Eine wesentlich weitergehende Verallgemeinerung der *Riemann*schen Zetafunktion ist von *Epstein*[150]) gegeben, dessen Untersuchungen an den zweiten *Riemann*schen Beweis der Funktionalgleichung von $\zeta(s)$, d. h. an die Darstellung der Zetafunktion durch ein bestimmtes Integral mit Hilfe elliptischer Thetafunktionen anknüpfen. *Epstein* betrachtet Reihen der Form

$$\sum_{m_1} \cdots \sum_{m_p} \frac{e^{2\pi i \sum_{\mu=1}^{p} m_\mu x_\mu}}{\{\varphi(y+m)\}^{\frac{s}{2}}}, \tag{40}$$

wo $x_1, \ldots x_p, y_1, \ldots y_p$ Konstanten sind und $\varphi(\alpha+\beta)$ ein symbolischer Ausdruck für die quadratische Form $\sum_\mu \sum_\nu (\alpha_\mu + \beta_\mu)(\alpha_\nu + \beta_\nu)$ der $2p$ Variabeln $\alpha_1, \ldots \beta_p$ ist; die durch eine solche Reihe (40) definierte Funktion wird eine *Zetafunktion* p^{ter} *Ordnung* genannt. *Epstein* zeigt nun, daß die Reihe (40) durch ein bestimmtes Integral mit Hilfe allgemeiner Thetafunktionen dargestellt werden kann, und durch Anwendung von Transformationsformeln dieser Thetafunktionen beweist er, daß auch diese allgemeine Zetafunktion einer Funktionalgleichung von ähnlichem Charakter wie die *Riemann*sche für $\zeta(s)$ genügt.

---

149) Vgl. *M. Lerch*, a. a. O. 148); die Funktionalgleichung wurde zuerst von *R. Lipschitz*, a. a. O. 147a), gefunden, welcher sie mit Hilfe der Theorie der *Fourier*schen Integrale herleitete.

150) *P. Epstein*, a) Zur Theorie allgemeiner Zetafunktionen, Math. Ann. 56 (1903), p. 615—644; b) Zur Theorie allgemeiner Zetafunktionen II, Math. Ann. 63 (1907), p. 205—216.

Zweiter Teil.

22. Einleitung. Bezeichnungen. Dieser Teil handelt von den zahlentheoretischen Anwendungen der im Vorhergehenden entwickelten Theorien. Die *gewöhnlichen Dirichlet*schen Reihen $\sum a_n n^{-s}$ sind als Hilfsmittel für analytisch-zahlentheoretische Untersuchungen besonders wertvoll; einen „Grund" hierfür kann man in ihrer Multiplikationsregel (vgl. Nr. 12) sehen, wonach bei der Koeffizientenbildung die „multiplikativen" Eigenschaften der Zahlen zur Geltung kommen. Die wichtigsten zahlentheoretischen Funktionen treten als Koeffizienten gewisser *Dirichlet*scher Reihen auf, die mit der *Riemann*schen Zetafunktion in einfacher Weise zusammenhängen. Indem man auf diese Reihen die Sätze über Beziehungen zwischen den Koeffizienten einer *Dirichlet*schen Reihe und der von der Reihe dargestellten Funktion (vgl. Nr. 4 und 5) anwendet, gelangt man mit Hilfe der Theorie der Zetafunktion zu neuen Ergebnissen über die Natur der zahlentheoretischen Funktionen. Manche Probleme erfordern die Einführung neuer erzeugender Funktionen, die alle der *Riemann*schen $\zeta(s)$ mehr oder weniger ähnlich sind. Verschiedene Probleme lassen sich auch durch Methoden angreifen, die von der Theorie der *Dirichlet*schen Reihen gänzlich unabhängig sind.

Es dürfte zweckmäßig sein, einige der im folgenden gebrauchten Bezeichnungen hier zusammenzustellen; die unten gegebenen Definitionen werden also im Texte nicht wiederkehren.

A. Die folgenden zahlentheoretischen Funktionen seien für ganze $n \geqq 1$ definiert:

$$\Lambda(n) = \begin{cases} \log p & \text{für } n = p^m \ (p \text{ Primzahl, } m \geqq 1 \text{ ganz}). \\ 0 & \text{sonst;} \end{cases}$$

$$\mu(n) = \begin{cases} (-1)^\nu & \text{für } n = p_1 p_2 \cdots p_\nu \ (\text{die } p_i \textit{ verschiedene } \text{Primzahlen}), \\ 1 & \text{für } n = 1, \\ 0 & \text{sonst;} \end{cases}$$

$$\lambda(n) = \begin{cases} (-1)^{\alpha_1 + \alpha_2 + \cdots \alpha_\nu} & \text{für } n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_\nu^{\alpha_\nu}, \\ 1 & \text{für } n = 1; \end{cases}$$

$d(n)$ = Anzahl der Teiler von n;

$\sigma(n)$ = Summe der Teiler von n;

$\varphi(n)$ = Anzahl der zu n teilerfremden positiven ganzen Zahlen $\leqq n$.

Diese Funktionen sind alle mit $\zeta(s)$ nahe verbunden; es gilt in der Tat für hinreichend große σ

$$\sum_1^\infty \frac{\Lambda(n)}{n^s} = -\frac{\zeta'}{\zeta}(s), \qquad \sum_1^\infty \frac{\Lambda(n)}{\log n \cdot n^s} = \log \zeta(s),$$

$$\sum_1^\infty \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}, \qquad \sum_1^\infty \frac{\lambda(n)}{n^s} = \frac{\zeta(2s)}{\zeta(s)},$$

$$\sum_1^\infty \frac{d(n)}{n^s} = (\zeta(s))^2, \qquad \sum_1^\infty \frac{\sigma(n)}{n^s} = \zeta(s)\zeta(s-1),$$

$$\sum_1^\infty \frac{\varphi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)}.$$

B. Die folgenden summatorischen Funktionen der obigen und einiger verwandten *Dirichlet*schen Reihen seien für jeden positiven Wert von x definiert:

$$\pi(x) = \text{Anzahl der Primzahlen} \leqq x$$
$$= \sum_{p \leqq x} 1,$$

$$\Pi(x) = \sum_{p^m \leqq x} \frac{1}{m} \quad (p \text{ durchläuft die Primzahlen, } m \text{ die ganzen positiven Zahlen})$$
$$= \sum_{n=1}^{x} \frac{\Lambda(n)}{\log n}$$
$$= \pi(x) + \tfrac{1}{2}\pi(x^{\frac{1}{2}}) + \tfrac{1}{3}\pi(x^{\frac{1}{3}}) + \cdots,$$

$$\vartheta(x) = \sum_{p \leqq x} \log p,$$

$$\psi(x) = \sum_{p^m \leqq x} \log p,$$
$$= \sum_{n=1}^{x} \Lambda(n),$$
$$= \vartheta(x) + \vartheta(x^{\frac{1}{2}}) + \vartheta(x^{\frac{1}{3}}) + \cdots,$$

$$M(x) = \sum_{n=1}^{x} \mu(n), \qquad \Phi(x) = \sum_{n=1}^{x} \varphi(n),$$

$$D(x) = \sum_{n=1}^{x} d(n), \qquad S(x) = \sum_{n=1}^{x} \sigma(n).$$

C. Es sei $g(x)$ irgendeine der unter B. eingeführten summatorischen Funktionen. Aus $g(x)$ werde die Funktion $\bar{g}(x)$ dadurch ab-

geleitet, daß für alle $x > 0$

$$\bar{g}(x) = \lim_{\varepsilon \to 0} \frac{g(x+\varepsilon) + g(x-\varepsilon)}{2}$$

gesetzt wird. $\bar{g}(x)$ ist also nur in den Unstetigkeitspunkten (d. h. für gewisse ganzzahlige x) von $g(x)$ verschieden.

III. Die Verteilung der Primzahlen.

23. Der Primzahlsatz. Ältere Vermutungen und Beweisversuche. Schon früh entstand das Problem, die Anzahl der Primzahlen zwischen zwei gegebenen Grenzen zu bestimmen, also insbesondere für die Funktion $\pi(x)$ einen (angenäherten oder exakten) Ausdruck aufzustellen. Bei dem höchst unregelmäßigen Verlauf dieser Funktion schien es von vornherein unmöglich, sie durch eine einfache analytische Funktion genau darzustellen; man mußte also zunächst darauf ausgehen, ein *asymptotisches* Resultat, etwa von der Form $\pi(x) \sim f(x)$, zu erhalten. Hierdurch ist schon die Fragestellung angebahnt, die zu dem berühmten *Primzahlsatz*[151]) führte: es gilt für unendlich wachsendes x:

$$\pi(x) \sim Li(x), \tag{41}$$

wo

$$Li(x) = \lim_{\varepsilon \to 0} \left(\int_0^{1-\varepsilon} \frac{du}{\log u} + \int_{1+\varepsilon}^{x} \frac{du}{\log u} \right)$$

gesetzt ist. — Dieser Satz kann wohl als das wichtigste Ergebnis der analytischen Zahlentheorie bezeichnet werden; durch die Anstrengungen, ihn zu beweisen, wurden ihre feinsten Methoden geschaffen und ausgebildet.

In (41) kann man, ohne den Sinn der Formel zu verändern, $Li(x)$ durch jede der Bedingung $f(x) \sim Li(x)$ genügende Funktion, z. B. durch $\frac{x}{\log x}$, ersetzen. Eine zu (41) äquivalente Behauptung wurde zuerst von *Legendre*[152]) ohne Beweis ausgesprochen: es werde $\pi(x)$ angenähert durch die Funktion $\frac{x}{\log x - 1{,}08366}$ dargestellt. Schon vor *Legendre* war *Gauß*[153]), wie aus einem viel später geschriebenen Briefe ersichtlich ist, auf die Vermutung $\pi(x) \sim \int_2^x \frac{du}{\log u}$ gekommen. Von

151) Die Benennung rührt von *H. v. Schaper* her: Über die Theorie der *Hadamard*schen Funktionen und ihre Anwendung auf das Problem der Primzahlen, Diss. Göttingen 1898.

152) *A. M. Legendre*, a) Essai sur la théorie des nombres (2. Aufl.), Paris 1808, p. 394; b) Théorie des nombres (3. Aufl.), Paris 1830, Bd. 2, p. 65.

153) *C. F. Gauß*, Werke 2, 2. Aufl., p. 444—447.

Dirichlet[154]) wurde gelegentlich behauptet, $\sum_{n=1}^{x}\frac{1}{\log n}$ sei eine bessere Vergleichsfunktion als diejenige von *Legendre*.

Einen präzisen Sinn erhielten diese Andeutungen erst durch die Arbeiten von *Tschebyschef*.[155]) In moderner Ausdrucksweise können seine wichtigsten Resultate etwa folgendermaßen zusammengefaßt werden: Er betrachtet die Funktionen $\pi(x)$, $\vartheta(x)$ und $\psi(x)$; zwischen ihnen besteht ein Zusammenhang, der durch die Beziehungen[156])

$$(42)\qquad \begin{aligned} \pi(x) &= \frac{\vartheta(x)}{\log x} + \int_2^x \frac{\vartheta(u)}{u \log^2 u}\,du \\ \vartheta(x) &= \pi(x)\log x - \int_2^x \frac{\pi(u)}{u}\,du \\ \psi(x) &= \vartheta(x) + O(\sqrt{x}) \end{aligned}$$

ausgedrückt wird (in den beiden ersten Gleichungen kann man übrigens π bzw. ϑ durch Π bzw. ψ ersetzen). Hieraus läßt sich unmittelbar ablesen, daß für unendlich wachsendes x alle drei Quotienten

$$(43)\qquad \frac{\pi(x)}{Li(x)},\quad \frac{\vartheta(x)}{x},\quad \frac{\psi(x)}{x}$$

dieselben oberen bzw. unteren Unbestimmtheitsgrenzen haben. Werden diese durch l (lim inf) und L (lim sup) bezeichnet, so findet *Tschebyschef*

$$a \leqq l \leqq 1 \leqq L \leqq \tfrac{6}{5}a,$$

mit $a = 0{,}92129$.

Insbesondere folgt hieraus[157]): existiert für irgendeinen der Quotienten (43) ein Grenzwert, so haben alle drei Quotienten den Grenzwert 1. Außer durch (41) läßt sich also der Primzahlsatz durch irgendeine der Gleichungen

$$(44)\qquad \vartheta(x) \sim x$$

$$(45)\qquad \psi(x) \sim x$$

ausdrücken.

154) Vgl. *G. Lejeune-Dirichlet*, Werke 1, p. 372, Fußnote 2).

155) *P. Tschebyschef*, a) Sur la fonction qui détermine la totalité des nombres premiers inférieurs à une limite donnée, Mém. présentés Acad. Pétersb. 6 (1851), p. 141—157; J. math. pures appl. (1) 17 (1852), p. 341—365; Œuvres 1, St. Pétersbourg 1899, p. 27—48; b) Mémoire sur les nombres premiers, J math. pures appl. (1) 17 (1852), p 366—390; Mém. présentés Acad. Pétersb. 7 (1854), p. 15—33; Œuvres 1, p. 49—70.

156) Die beiden ersten Gleichungen werden einfach durch partielle Summation aus den Definitionsgleichungen für $\pi(x)$ und $\vartheta(x)$ abgeleitet.

157) Das kann jetzt unmittelbar aus elementaren Sätzen über *Dirichlet*sche Reihen gefolgert werden. Vgl. Nr. 5, insbesondere die Fußnoten 27) und 28), vgl. auch *E. Landau*, Handbuch, § 31.

Die *Tschebyschef*schen Resultate wurden teils durch Betrachtung der Funktionen $\zeta(s)$ und $\log \zeta(s)$ für reelle, gegen 1 abnehmende Werte von s, teils durch elementare Summenabschätzungen mit Hilfe der Identität[158])

$$\psi(x) + \psi\left(\frac{x}{2}\right) + \psi\left(\frac{x}{3}\right) + \cdots = \log([x]!)$$

abgeleitet. Die Schranken für l und L wurden später von anderen[159]) mit analogen Methoden verengert; es ist jedoch bisher niemand gelungen, auf diesem Wege die Existenz eines Grenzwertes, d. h. den Primzahlsatz, zu beweisen.

24. Die Beweise von Hadamard und de la Vallée Poussin. Der Weg, der zu einem strengen Beweis des Primzahlsatzes führen konnte, wurde erst geöffnet durch die Erscheinung der grundlegenden *Riemann*schen Arbeit[95]) vom Jahre 1859, wo zum ersten Male die komplexe Funktionentheorie auf das Problem angewandt und die Zetafunktion völlig allgemein untersucht wurde. Das Endziel dieser Arbeit war allerdings nicht der Beweis des Primzahlsatzes, doch findet man hier schon die Integralformel für die Koeffizientensumme einer *Dirichlet*schen Reihe (vgl. Nr. 4), auf

$$\log \zeta(s) = \sum_{p,\,m} \frac{1}{m p^{ms}} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{\log n \cdot n^s}$$

angewandt. *Halphen*[160]) und *Cahen*[161]) versuchten, diesen *Riemann*schen Ansatz für den Beweis des Primzahlsatzes zu benutzen, ein vollständiger Beweis wurde jedoch erst im Jahre 1896 gegeben, und zwar fast gleichzeitig von *Hadamard*[162]) und *de la Vallée Poussin.*[163])

Die früheren Versuche waren hauptsächlich an den folgenden zwei Schwierigkeiten gescheitert: 1. die Eigenschaften der komplexen Null-

158) Diese Identität wurde unabhängig von *Tschebyschef*[156]) und *de Polignac,* Recherches nouvelles sur les nombres premiers, Paris 1851, entdeckt.

159) Betreffs der an *Tschebyschef* in dieser Richtung anschließenden Arbeiten sei auf *G. Torelli,* Sulla totalità dei numeri primi fino a un limite assegnato, Neapel 1901 (Atti Accad. sc. fis. mat. (2) 11 No. 1), Cap. 4—5 verwiesen. In dieser Monographie wird die Geschichte des Gegenstandes ausführlich dargestellt.

160) *G. H. Halphen,* Sur l'approximation des sommes de fonctions numériques, Paris C. R. 96 (1883), p. 634—637. Auch *T. J. Stieltjes* gibt an, einen Beweis gefunden zu haben: Correspondance d'Hermite et de Stieltjes, Paris 1905, verschiedene Stellen, vgl. z. B. Lettre 314.

161) *E. Cahen,* Sur la somme des logarithmes des nombres premiers qui ne dépassent pas x, Paris C. R. 116 (1893), p. 85—88.

162) *J. Hadamard,* Sur la distribution des zéros de la fonction $\zeta(s)$ et ses conséquences arithmétiques, Bull. soc. math. France 24 (1896), p. 199—220.

163) *Ch. de la Vallée Poussin,* Recherches analytiques sur la théorie des nombres premiers, Première partie, Ann. soc. sc. Bruxelles 20:2 (1896), p. 183—256.

stellen von $\zeta(s)$ waren noch nicht hinreichend bekannt; 2. die Integrale

$$(46)\qquad -\frac{1}{2\pi i}\int\limits_{a-i\infty}^{a+i\infty}\frac{x^s}{s}\frac{\zeta'}{\zeta}(s)\,ds$$

und

$$(47)\qquad \frac{1}{2\pi i}\int\limits_{a-i\infty}^{a+i\infty}\frac{x^s}{s}\log\zeta(s)\,ds,$$

die für $a>1$ die Funktionen $\overline{\psi}(x)$ bzw. $\overline{\Pi}(x)$ darstellen (vgl. Nr. **4**; in einem Unstetigkeitspunkt muß man nach den dortigen Ausführungen die Hauptwerte der Integrale nehmen), sind nur bedingt konvergent.

Die erste Schwierigkeit wurde von *Hadamard* und *de la Vallée Poussin* dadurch überwunden, daß sie zeigten: jede Nullstelle von $\zeta(s)$ liegt links von der Geraden $\sigma = 1$ (vgl. Nr. **14**). Dieser Satz wird bei allen bisher bekannten Beweisen des Primzahlsatzes als wesentliche Grundlage benutzt. — Um unbedingt konvergente Ausdrücke zu erhalten, benutzen die beiden Verfasser an der Stelle von (46) und (47) Integralausdrücke für gewisse mit $\overline{\psi}$ und $\overline{\Pi}$ zusammenhängende Funktionen.

Hadamard betrachtet das für $\mu>1$ unbedingt konvergente Integral (vgl. (12) Nr **4**).

$$(48)\qquad -\frac{1}{2\pi i}\int\limits_{a-i\infty}^{a+i\infty}\frac{x^s}{s^\mu}\frac{\zeta'}{\zeta}(s)\,ds = \frac{1}{\Gamma(\mu)}\sum_{n=1}^{x}\Lambda(n)\log^{\mu-1}\frac{x}{n}$$

$$= \frac{1}{\Gamma(\mu-1)}\int\limits_2^x\frac{\psi(t)}{t}\log^{\mu-2}\frac{x}{t}\,dt.$$

Durch eine Verschiebung des Integrationsweges folgert er, unter Benutzung der Tatsache, daß die ganze Funktion $(s-1)\,\zeta(s)$ vom Geschlechte 1 ist (vgl. Nr. **15**),

$$\frac{1}{\Gamma(\mu-1)}\int\limits_2^x\frac{\psi(t)}{t}\log^{\mu-2}\frac{x}{t}\,dt$$

$$= x - \sum_{\varrho}{}'\frac{x^\varrho}{\varrho^\mu} - \frac{1}{2\pi i}\int\limits_{ABCDEF}\frac{x^s}{s^\mu}\frac{\zeta'}{\zeta}(s)\,ds.$$

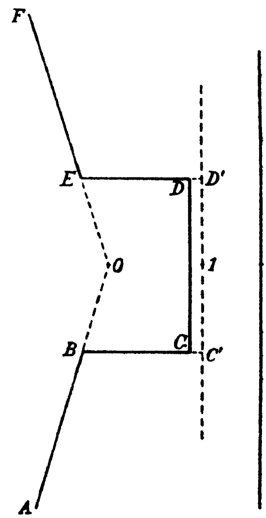


Fig. 1.

Rechts durchläuft ϱ in Σ' nur die oberhalb $D'E$ oder unterhalb BC' gelegenen Nullstellen von $\zeta(s)$; da auf $\sigma = 1$ keine Nullstellen liegen, kann DD' so klein gewählt werden, daß auch noch das Rechteck $CDD'C'$ nullstellenfrei wird (vgl. Figur 1).

Da der neue Integrationsweg ganz in der Halbebene $\sigma < 1$ verläuft, schließt man hieraus

$$(49) \qquad \frac{1}{\Gamma(\mu-1)}\int_2^x \frac{\psi(t)}{t} \log^{\mu-2}\frac{x}{t}\,dt \sim x$$

und speziell für $\mu = 2$

$$(50) \qquad \int_2^x \frac{\psi(t)}{t}\,dt = \sum_{n=1}^{x} \Lambda(n)\log\frac{x}{n} \sim x.$$

Hadamard zeigt, daß hieraus unmittelbar zu (44) oder (45) übergegangen werden kann (vgl. auch Nr. 25), womit der Primzahlsatz bewiesen ist.

Auch *de la Vallée Poussin* nimmt als Ausgangspunkt ein unbedingt konvergentes Integral, nämlich

$$\int_{a-i\infty}^{a+i\infty} \frac{x^s}{(s-u)(s-v)}\frac{\zeta'}{\zeta}(s)\,ds.$$

Durch Anwendung der Gleichung (29), Nr. 15, erhält er, da $\Re(\varrho) < 1$ ist,

$$(51) \qquad \int_2^x \frac{\psi(t)}{t^2}\,dt = \sum_{n=1}^{x} \Lambda(n)\left(\frac{1}{n} - \frac{1}{x}\right)$$

$$= \log x + K - \sum_{\varrho} \frac{x^{\varrho-1}}{\varrho(\varrho-1)} + \frac{a}{x} + O\left(\frac{1}{x^2}\right) = \log x + K + o(1),$$

und zeigt, wie man hieraus zu (50) und (45) übergehen kann.

25. Die Beweismethoden von Landau. Bei den ersten Beweisen des Primzahlsatzes traten als wichtige Hilfsmittel Sätze auf, die durch die Anwendung der *Hadamard*schen Theorie der ganzen Funktionen auf $(s-1)\zeta(s)$ gefunden wurden und also die Existenz der Zetafunktion in der ganzen Ebene und gewisse Eigenschaften ihrer Nullstellen voraussetzen.

Landau hat aber gezeigt, daß der Beweis in weitgehendem Maße von diesen Voraussetzungen befreit werden kann, was für die Anwendung der Methode auf allgemeinere Fälle wichtig ist (vgl. Nr. 42).

Durch Benutzung der elementar nachweisbaren Ungleichung

$$(52) \qquad \left|\frac{\zeta'}{\zeta}(s)\right| < K(\log t)^A \quad \text{für} \quad \sigma > 1 - \frac{1}{(\log t)^B}, \quad t > t_0,$$

mit konstanten A, B, K, t_0, gelang es ihm[164]) den Primzahlsatz zu beweisen, indem er mit dem *Hadamard*schen Integral (48) für $\mu = 2$ und mit einem in jenem Gebiete verlaufenden Integrationsweg arbeitete.

164) *E. Landau*, a. a. O. 125) und Handbuch, § 51—54.

Später[165]) zeigte er, daß man die Voraussetzungen sogar noch mehr verringern kann: für den Beweis des Primzahlsatzes ist in der Tat nur wesentlich, daß $\frac{\zeta'}{\zeta}$ auf der Geraden $\sigma = 1$ (abgesehen vom Pole $s = 1$) regulär ist und für $\sigma \geqq 1$, $|t| \to \infty$ gleichmäßig von der Form $O(|t|^k)$ ist. Der am Ende von Nr. **5** genannte Satz von *Landau*[29]) über *Dirichlet*sche Reihen mit positiven Koeffizienten ist nämlich unmittelbar auf $-\frac{\zeta'}{\zeta}(s) = \sum \Lambda(n) n^{-s}$ anwendbar und liefert gerade die Beziehung (45). Für den Beweis dieses Satzes werden gewisse allgemeine Grenzwertsätze herangezogen, die speziell den Übergang von (50) oder (51) zu (45) ermöglichen (vgl. auch Nr. **33**). Wenn beispielsweise die Funktion $f(t)$ für $t > a$ nirgends abnimmt, so kann man von

$$\int_a^x \frac{f(t)}{t}\, dt \sim x$$

auf die asymptotische Gleichheit der Ableitungen schließen: $f(x) \sim x$.

Durch Benutzung des Integranden $\frac{x^s}{s^2} \log \zeta(s)$ anstatt $\frac{x^s}{s^2} \frac{\zeta'}{\zeta}(s)$ kann man, wie *Landau*[166]) zeigt, den Satz (41) über $\pi(x)$ direkt, d. h. ohne den Umweg über $\psi(x)$ oder $\vartheta(x)$, beweisen; auch gelingt es ihm[167]) mit Hilfe des nur bedingt konvergenten Integrals (46) direkt zu (45) — und sogar zur Gleichung (53) von Nr. **27** — ohne den Umweg über (50) zu gelangen.

26. Andere Beweise. Der Beweis von *H. v. Koch*[168]) weicht von den vorhergehenden dadurch ab, daß er gar nicht mit Integralen von der Form $\int \frac{x^s}{s^u} f(s)\, ds$ arbeitet. Er gibt für die summatorischen Funktionen der *Dirichlet*schen Reihen für $\frac{\zeta'}{\zeta}$ und $\log \zeta$ unter Benutzung gewisser Diskontinuitätsfaktoren Ausdrücke, die in der folgenden Darstellungsformel für die Koeffizientensumme einer beliebigen *Dirichlet*schen Reihe $f(s) = \sum a_n e^{-\lambda_n s}$ (mit absolutem Konvergenzbereich) zu-

165) *E. Landau*, a. a. O. 29) und 21) sowie Zwei neue Herleitungen für die asymptotische Anzahl der Primzahlen unter einer gegebenen Grenze, Sitzungsber. Akad. Berlin 1908, p. 746—764 und Handbuch, § 66.

166) *E. Landau*, a. a. O. 165) (Zwei neue Herleitungen . . .) und Handbuch, § 64.

167) *E. Landau*, Über den Gebrauch bedingt konvergenter Integrale in der Primzahltheorie, Math. Ann. 71 (1912), p. 368—379.

168) *H. v. Koch*, Sur la distribution des nombres premiers, Acta Math. 24 (1901), p. 159—182.

sammengefaßt werden können[169]):

$$\sum_{\lambda_n < x} a_n = \lim_{c \to \infty} \sum_{\nu=1}^{\infty} \frac{(-1)^{\nu-1}}{\nu!} e^{c\nu x} f(c\nu), \qquad (x \neq \lambda_n).$$

Diese Formel erscheint dadurch bemerkenswert, daß $f(s)$ darin nur mit einem reellen und positiven Argument auftritt.

Hardy und *Littlewood*[170]) beweisen, wie schon in Nr. **5** erwähnt wurde, mit Hilfe des „*Cahen-Mellin*schen Integrals" (vgl. Nr. **11**) einen Satz über *Dirichlet*sche Reihen, der den *Landau*schen, in Nr. **5** und **25** erwähnten, Satz — und damit den Primzahlsatz — als Spezialfall enthält.

Steffensen[171]) zeigt, daß eine von ihm und schon früher von *Mellin*[172]) gefundene Integraldarstellung für die Koeffizientensumme einer *Dirichlet*schen Reihe zum Beweis des Primzahlsatzes benutzt werden kann.

27. Die Restabschätzung. Schon durch die Resultate von *Tschebyschef*[155]) wurde die Vermutung nahe gelegt, daß unter allen asymptotisch gleichwertigen Funktionen, die man als Vergleichsfunktionen für $\pi(x)$ benutzt hatte, dem Integrallogarithmus eine besonders ausgezeichnete Stellung zukommt. Streng entschieden wurde diese Frage erst durch *de la Vallée Poussin*[173]), der aus seiner Gleichung (51) mit Hilfe seines Satzes (vgl. Nr. **19**)

$$\zeta(s) \neq 0 \quad \text{für} \quad \sigma > 1 - \frac{k}{\log t}, \quad t > t_0$$

die Folgerung

$$\pi(x) = Li(x) + O\left(x e^{-\alpha\sqrt{\log x}}\right) \tag{53}$$

für jedes $\alpha < \sqrt{k}$ zog. Gleichzeitig folgt, daß auch die Differenzen

$$\Pi(x) - Li(x), \quad \vartheta(x) - x, \quad \psi(x) - x$$

alle drei von der Größenordnung $O\left(xe^{-\alpha\sqrt{\log x}}\right)$ sind. Der Integral-

169) Vgl. auch *Hj. Mellin,* Die Dirichletschen Reihen, die zahlentheoretischen Funktionen und die unendlichen Produkte von endlichem Geschlecht, Acta Math. 28 (1904), p. 37—64.

170) *G. H. Hardy* und *J. E. Littlewood,* a. a. O. 31).

171) *J. F. Steffensen,* Analytiske studier med anvendelser paa taltheorien, Diss. Kopenhagen 1912; Über eine Klasse von ganzen Funktionen und ihre Anwendung auf die Zahlentheorie, Acta Math. 37 (1914), p. 75—112; vgl. auch: Über Potenzreihen, im besonderen solche, deren Koeffizienten zahlentheoretische Funktionen sind, Palermo Rend. 38 (1914), p. 376—386.

172) a. a. O. 169).

173) *Ch. de la Vallée Poussin,* Sur la fonction $\zeta(s)$ de *Riemann* et le nombre des nombres premiers inférieurs à une limite donnée, Mém. couronnés et autres mém. Acad. Belgique 59 (1899—1900), No. 1.

logarithmus stellt demnach $\pi(x)$ in einem ganz präzisen Sinne *besser* dar als $\frac{x}{\log x}$ oder irgendeine der Funktionen

$$f_q(x) = \frac{x}{\log x} + \frac{1!\,x}{\log^2 x} + \cdots \frac{(q-1)!\,x}{\log^q x} = Li(x) + O\left(\frac{x}{\log^{q+1} x}\right),$$

die bei der asymptotischen Entwicklung von $Li(x)$ auftreten.[174] Nach (53) gilt nämlich für $q = 1, 2, \ldots$

$$\pi(x) - f_q(x) \sim \frac{q!\,x}{\log^{q+1} x}, \tag{54}$$

aber

$$\pi(x) - Li(x) = o\left(\frac{x}{\log^{q+1} x}\right). \tag{55}$$

Bei *Landau*[164]) wird mit Hilfe von (52), also ohne Benutzung der Fortsetzbarkeit von $\zeta(s)$ oder der Existenz ihrer Nullstellen, die Abschätzung

$$\pi(x) = Li(x) + O\left(x e^{-\sqrt[13]{\log x}}\right) \tag{56}$$

bewiesen; diese ist weniger scharf als (53), reicht aber doch für die Folgerungen (54) und (55) aus. *Landau*[175]) hat übrigens auch den Beweis von (53) wesentlich vereinfacht; diese Gleichung, mit dem von ihm angegebenen Werte von α, stellt die schärfste bisher mit Sicherheit bekannte Abschätzung von $\pi(x)$ dar.[176])

Nimmt man dagegen an, die Riemannsche Vermutung über die Nullstellen der Zetafunktion sei richtig (vgl. Nr. **20**), so erhält man noch schärfere Resultate, nämlich

$$\left.\begin{array}{l}\pi(x) - Li(x)\\ \Pi(x) - Li(x)\end{array}\right\} = O(\sqrt{x}\log x), \tag{57}$$

$$\left.\begin{array}{l}\vartheta(x) - x\\ \psi(x) - x\end{array}\right\} = O(\sqrt{x}\log^2 x). \tag{58}$$

174) Hieraus folgt die Richtigkeit einer von *Lionnet*, Question 1075. Nouv. ann. math. (2) 11 (1872), p. 190, ausgesprochenen Vermutung, daß für große x mehr Primzahlen im Intervalle $(1, x)$ als in $(x, 2x)$ liegen. Es gilt nämlich $2\pi(x) - \pi(2x) \sim 2\log 2\,\frac{x}{\log^2 x}$; vgl. *E. Landau*, Solutions de questions proposées, 1075, Nouv. ann. math. (4) 1 (1901), p. 281—282.

175) *E. Landau*, a) a. a. O. 29); b) Neue Beiträge zur analytischen Zahlentheorie, Palermo Rend. 27 (1909), p. 46—58; c) Handbuch § 81; *Landau* zeigt, daß (53) für alle $\alpha < \frac{1}{\sqrt{18{,}53}}$, also z. B. für $\alpha = \frac{1}{5}$, gilt.

176) Der von *Littlewood*, a. a. O. 111) ohne Beweis ausgesprochene Satz $\zeta(s) \neq 0$ für $\sigma > 1 - \frac{c \log\log t}{\log t}$ würde eine Verbesserung von (53) zulassen, indem er ein Restglied von der Form $O\left(x e^{-\alpha\sqrt{\log x \log\log x}}\right)$ liefern würde.

Diese Gleichungen sind zuerst von *v. Koch*[168]) mit seiner in der vorigen Nummer erwähnten Methode bewiesen; sie können auch aus der *de la Vallée Poussin*schen Gleichung (51) erhalten werden, durch ein Verfahren, das von *Holmgren*[177]) und in einem analogen Fall von *Landau*[178]) benutzt wurde. *Landau*[179]) hat diese Abschätzungen auch auf anderem Wege bewiesen; die etwas unschärfere Abschätzung $O\left(x^{\frac{1}{2}+\varepsilon}\right)$ folgt nach den *Littlewood*schen Ergebnissen über die μ-Funktion (vgl. Nr. 20) direkt aus dem Konvergenzsatz von *Landau-Schnee* (vgl. Nr. 6).

Bezeichnet man allgemein durch Θ die obere Grenze der reellen Teile der Nullstellen von $\zeta(s)$, wobei also $\frac{1}{2} \leqq \Theta \leqq 1$ ist, so bleiben (57) und (58) jedenfalls richtig, wenn $\sqrt{x}$ durch x^{Θ} ersetzt wird.[179]) (Im Falle $\Theta = 1$ ist dies natürlich trivial.) Die *Dirichlet*sche Reihe

$$(59) \qquad \sum_{1}^{\infty} \frac{\Lambda(n)-1}{n^s} = -\left(\frac{\zeta'}{\zeta}(s) + \zeta(s)\right)$$

konvergiert also für $\sigma > \Theta$. Aus (53) folgt, daß sie jedenfalls auf der ganzen Geraden $\sigma = 1$ konvergiert.

Auch wenn die *Riemann*sche Vermutung bewiesen wird, kann man nicht hoffen, die durch (57) und (58) gegebenen Abschätzungen wesentlich zu verbessern. Jedenfalls kann für kein $\eta < \Theta$ z. B.

$$\psi(x) - x = O(x^{\eta})$$

sein[180]), denn daraus würde die Konvergenz der linken Seite von (59) — und also die Regularität der rechten Seite — für $\sigma > \eta$ folgen. Weitere Sätze in dieser Richtung gaben *Phragmén*[181]), *Schmidt*[182]) und *Landau*[183]), der die Frage in Beziehung zu seinem Satze über *Dirichlet*-

177) *E. Holmgren*, Om primtalens fördelning, Öfvers. af Kgl. Vetensk. Förh. 59, Stockholm 1902—1903, p 221—225.

178) *E. Landau*, Über einige ältere Vermutungen und Behauptungen in der Primzahltheorie, Math. Ztschr. 1 (1918), p. 1—24.

179) *E. Landau*, a. a. O. 175b) und Handbuch, § 93—94.

180) Dies wurde schon von *A. Piltz* behauptet: Über die Häufigkeit der Primzahlen in arithmetischen Progressionen und über verwandte Gesetze, Habilitationsschrift, Jena 1884. Vgl. auch *T. J. Stieltjes*, a. a. O. 160), Lettre 299.

181) *E. Phragmén*, Sur le logarithme intégral et la fonction $f(x)$ de *Riemann*, Öfvers. af Kgl. Vetensk. Förh. Stockholm 48 (1891—1892), p. 599—616 und Sur une loi de symétrie relative à certaines formules asymptotiques, ibid. 58 (1901—1902), p. 189—202.

182) *E. Schmidt*, Über die Anzahl der Primzahlen unter gegebener Grenze, Math. Ann. 57 (1903), p. 195—204.

183) *E. Landau*, Über einen Satz von *Tschebyschef*, Math. Ann. 61 (1905), p. 527—550, und Handbuch, § 201—204.

sche Reihen mit positiven Koeffizienten (vgl. Nr. 6) setzte. Ein bedeutender Fortschritt wurde von *Littlewood*[184]) gemacht; durch eine Methode, auf die wir in der nächsten Nummer zurückkommen, bewies er die Existenz einer positiven Konstanten K derart, daß alle vier Ungleichungen

$$(60)\qquad \begin{cases} \pi(x) - Li(x) > \quad K\dfrac{\sqrt{x}}{\log x}\,\text{logloglog}\, x \\ \pi(x) - Li(x) < -K\dfrac{\sqrt{x}}{\log x}\,\text{logloglog}\, x \\ \vartheta(x) - x \quad > \quad K\sqrt{x}\,\text{logloglog}\, x \\ \vartheta(x) - x \quad < -K\sqrt{x}\,\text{logloglog}\, x \end{cases}$$

beliebig große Lösungen besitzen. Das gleiche gilt für die entsprechenden Ungleichungen mit Π an der Stelle von π und ψ an der Stelle von ϑ. Dies ist das beste bisher bekannte Resultat über die wirklich stattfindenden Unregelmäßigkeiten der Primzahlfunktionen; wäre es aber gelungen, die *Falschheit* der *Riemann*schen Vermutung (d. h. $\Theta > \frac{1}{2}$) zu beweisen, so könnte nach *Schmidt*[182]) der Faktor von K in (60) sogar durch $x^{\Theta-\epsilon}$ ersetzt werden. — Das Resultat von *Littlewood* ist besonders darum bemerkenswert, weil man früher die Beziehung

$$(61)\qquad \pi(x) < Li(x)$$

als höchst wahrscheinlich betrachtet hat[185]); diese Beziehung gilt insbesondere für alle $x < 10.000.000$. Nach (60) kann sie aber nicht allgemein gelten.

Nach (60) ist z. B. die Funktion $\frac{\psi(x)-x}{\sqrt{x}}$ sicher nicht beschränkt. Wenn die *Riemann*sche Vermutung richtig ist, so hat sie trotzdem, wie *Cramér*[186]) zeigt, einen beschränkten quadratischen Mittelwert, d. h.

$$\frac{1}{x}\int_{2}^{x}\left(\frac{\psi(t)-t}{\sqrt{t}}\right)^2 dt$$

ist beschränkt, strebt aber für $x \longrightarrow \infty$ keinem bestimmten Grenzwert

184) *J. E. Littlewood*, Sur la distribution des nombres premiers, Paris C. R. 158 (1914), p. 1869—1872; *G. H. Hardy* und *J. E. Littlewood*, a. a. O. 31b).

185) Vgl. *Gauß*, a. a. O. 153), Bemerkung von *E. Schering* in *Gauß'* Werke 2, p. 520; *Phragmén*, a. a. O. 189); *Lehmer*, List of prime numbers from 1 to 10.006.721, Washington 1914.

186) *H. Cramér*, Some theorems concerning prime numbers, Arkiv f. Mat., Astr. och Fys 15 (1920), No. 5.

zu, was dagegen von

$$\frac{1}{\log x}\int_2^x \left(\frac{\psi(t)-t}{t}\right)^2 dt$$

gilt.[187])

28. Die Riemannsche Primzahlformel. Das Hauptziel der *Riemann*schen Primzahlarbeit[95]) war die Aufstellung eines *exakten* Ausdrucks für die Funktion $\overline{\Pi}(x)$; durch die Betrachtung des Integrals (47) wurde *Riemann* nämlich auf die Formel

$$(62)\quad \overline{\Pi}(x) = Li(x) - \sum_{\gamma>0}\left(Li(x^{\varrho}) + Li(x^{1-\varrho})\right) + \int_x^{\infty}\frac{dt}{(t^2-1)t\log t} - \log 2$$

geführt. (Ein unwesentlicher Schreib- oder Rechenfehler im letzten, konstanten Gliede wurde von *Genocchi*[188]) berichtigt.) Die Summe ist hier über alle Nullstellen $\varrho = \beta + \gamma i$ von $\zeta(s)$ zu erstrecken, die der oberen Halbebene angehören, und es ist

$$Li(x^{a+bi}) = \int_{-\infty+bi\log x}^{(a+bi)\log x}\frac{e^z}{z}\,dz \pm \pi i$$

gesetzt, je nachdem $b\log x \gtrless 0$ gilt.[189]) Über seinen Beweis der Konvergenz dieser Reihe gab *Riemann* nur eine unbestimmte Andeutung, und auch aus anderen Gründen war die Formel als nur heuristisch begründet anzusehen. Wegen der äußerst verwickelten Natur der auftretenden Funktionen wurde sogar an der Möglichkeit gezweifelt, die Formel überhaupt beweisen oder jedenfalls daraus irgendwelche Schlüsse ziehen zu können.[190]) Es hat auch lange gedauert, bis ein vollständiger Beweis gegeben wurde; nach verschiedenen Versuchen[191]) gelang dies zuerst *v. Mangoldt*[192]), der eine entsprechende Formel für die

---

187) *H. Cramér,* Ein Mittelwertsatz in der Primzahltheorie, Math. Ztschr. 12 (1922), p. 147—153; vgl. auch Sur un problème de *M. Phragmén,* Arkiv f. Mat., Astr. och Fys. 16 (1922), No. 27.

188) *A. Genocchi,* Formole per determinare quanti siano i numeri primi fino ad un dato limite, Ann. Mat. pura appl. (1) 3 (1860), p. 52—59.

189) Über den Sinn der Formel und ihre Verwendung für numerische Rechnungen vgl. *E. Phragmén,* Über die Berechnung der einzelnen Glieder der *Riemann*schen Primzahlformel, Öfvers. af Kgl. Vetensk. Förh. 48, Stockholm 1891—1892, p. 721—744.

190) Vgl. z. B. *Ch. de la Vallée Poussin,* a. a. O. 168), p. 252—256.

191) Vgl. z. B. *A. Piltz,* a. a. O. 180); *J. P. Gram,* Undersögelser angaaende Mængden af Primtal under en given Grænse, Kgl. Danske Vidensk. Selsk. Skrifter, naturv. og math. Afd. (6) 2 (1881—1886), p. 183—308.

192) *H. v. Mangoldt,* a. a. O. 16) und Zu *Riemanns* Abhandlung „Über die Anzahl der Primzahlen unter einer gegebenen Größe", Crelles J. 114 (1895), p. 255—305.

Funktion

$$F(x, r) = \sum_{n=1}^{x} \frac{\Lambda(n)}{n^r} - \frac{\Lambda(x)}{2x^r} \quad \text{(für nicht ganze } x \text{ bedeutet } \Lambda(x) \text{ Null)}$$

aufstellte, um dann durch Integration nach dem Parameter r zur *Riemann*schen Formel überzugehen.[193]) $F(x, 0)$ ist mit $\overline{\psi}(x)$ identisch, und in diesem Falle lautet die Formel

$$(63) \qquad \overline{\psi}(x) = x - \sum_{\varrho} \frac{x^\varrho}{\varrho} - \frac{1}{2} \log\left(1 - \frac{1}{x^2}\right) - \log 2\pi,$$

wo jetzt die Summe über alle komplexen ϱ, nach absolut wachsenden Ordinaten geordnet, erstreckt wird. Diese Formel, deren einzelne Glieder elementare Funktionen sind, ist für die spätere Entwicklung sogar wichtiger als (62) geworden. Formal kommt sie bei der Betrachtung des Integrals (46) unmittelbar heraus, da rechts die Summe der Residuen des Integranden links vom Integrationswege steht.

Da $\overline{\psi}(x)$ in den Punkten $x = p^m$ unstetig ist, kann $\sum \frac{x^\varrho}{\varrho}$ dort nicht gleichmäßig konvergieren. *Landau*[194]), der den *v. Mangoldt*schen Beweis vereinfacht und auch (62) direkt aus (47) abgeleitet hat, zeigt aber, daß die Reihe in jedem Intervall, das rechts von $x = 1$ liegt und von den $x = p^m$ frei ist, gleichmäßig konvergiert. Er dehnt seine Untersuchungen auch auf die allgemeinere Reihe

$$(64) \qquad \sum_{\gamma > 0} \frac{x^\varrho}{\varrho^k} \qquad (0 < k \leqq 1)$$

aus[195]), welche analoge Konvergenzeigenschaften besitzt[196]), die auf das Verhalten der endlichen Summe $\sum_{0 < \gamma \leqq T} x^\varrho$ zurückgeführt werden können. *Cramér*[197]) betrachtet diese Reihen auch für komplexe Werte der

193) Zu diesem Übergang vgl. *H. Cramér,* Über die Herleitung der *Riemann*schen Primzahlformel, Arkiv f. Mat., Astr. och Fys 13 (1918), No. 24.

194) *E. Landau,* Neuer Beweis der Riemannschen Primzahlformel, Sitzungsber Akad. Berlin 1908, p. 737—745; Nouvelle démonstration pour la formule de Riemann sur le nombre des nombres premiers inférieurs à une limite donnée, et démonstration d'une formule plus générale pour le cas des nombres premiers d'une progression arithmétique, Ann. Éc. Norm (3) 25 (1908), p. 399—442.

195) *E. Landau,* Über die Nullstellen der Zetafunktion, Math. Ann. 71 (1912), p. 548—564.

196) In den Unstetigkeitspunkten $x = p^m$ ist jedoch (64) divergent, während $\sum_{\varrho} \frac{x^\varrho}{\varrho}$ für alle $x > 0$ konvergiert.

197) *H. Cramér,* Studien über die Nullstellen der Riemannschen Zetafunktion, Math. Ztschr. 4 (1919), p. 104—130.

Veränderlichen, indem er insbesondere die Funktion

$$V(s) = \sum_{\gamma > 0} e^{\varrho s}$$

untersucht. Wird die s-Ebene längs der negativen imaginären Achse aufgeschnitten, so ist $V(s)$ im Innern der aufgeschnittenen Ebene meromorph und hat nur die singulären Stellen $s = \pm \log p^m$, welche Pole erster Ordnung sind. Hierdurch wird es möglich, auf die Reihe (64) den Konvergenzsatz von *M. Riesz* anzuwenden (vgl. Nr. 5). — Alle diese Erscheinungen deuten auf irgendeinen arithmetischen Zusammenhang zwischen den Nullstellen ϱ und den Primzahlen p hin.

Die Formeln (62) und (63) setzen die Hauptglieder der Funktionen $\overline{\Pi}(x)$ bzw. $\overline{\psi}(x)$ in Evidenz; wegen der nur bedingten Konvergenz der auftretenden Reihen läßt sich aus ihnen jedoch nicht einmal der Primzahlsatz *unmittelbar* erschließen. Zwar ist z. B. in $\sum \frac{x^\varrho}{\varrho}$ jedes Glied von der Form $o(x)$ — wenn die *Riemann*sche Vermutung wahr ist, sogar von der Form $O\left(x^{\frac{1}{2}}\right)$ — wegen der Divergenz von $\sum \left|\frac{x^\varrho}{\varrho}\right|$ ist es aber nicht zulässig, unmittelbar hieraus $\sum \frac{x^\varrho}{\varrho} = o(x)$ zu folgern.[198]) *v. Koch*[199]) hat diese Formeln dadurch für asymptotische Zwecke verwerten können, daß er in die unendlichen Reihen konvergenzerzeugende Faktoren einführt und die Reihen dann durch endliche Summen ersetzt. Auf diese Weise ist es ihm gelungen, $\Pi(x)$ als Summe einer absolut konvergenten Reihe und eines beschränkten Fehlergliedes darzustellen, für $\psi(x)$ erhält er z. B. den Ausdruck

$$\psi(x) = x - \sum_{|\varrho| < \sqrt{x}} \frac{x^\varrho}{\varrho} \Gamma\left(1 - \frac{\varrho \log x}{3\sqrt{x}}\right) + O(\sqrt{x} \log^2 x). \tag{65}$$

Landau[200]) zeigt, daß diese Gleichung auch dann richtig bleibt, wenn

198) Die Ausführungen von *H. v. Mangoldt,* Über eine Anwendung der Riemannschen Formel für die Anzahl der Primzahlen unter einer gegebenen Grenze, Crelles J. 119 (1898), p. 65—71, enthalten nur einen Übergang von (45) zu (41).

199) *H. v. Koch,* Über die Riemannsche Primzahlfunction, Math. Ann. 55 (1902), p. 441—464; Contribution à la théorie des nombres premiers, Acta Math. 33 (1910), p. 293—320.

200) *E. Landau,* Über einige Summen, die von den Nullstellen der Riemannschen Zetafunktion abhängen, Acta Math. 35 (1911), p. 271—294. Vgl. auch *A. Hammerstein,* Zwei Beiträge zur Zahlentheorie, Diss., Göttingen 1919. — *Littlewood,* a. a. O. 111) hat sogar (ohne Beweis) die Formel

$$\psi(x) = x - \sum_{|\varrho| \leq v} \frac{x^\varrho}{\varrho} + O(\sqrt{x} \log x),$$

man den Γ-Faktor wegläßt. *Cramér*[186]) gibt die Formel

$$(66)\qquad \psi(x) = x - \sum_{\varrho} \frac{x^{\varrho}}{\varrho} e^{-\frac{|\gamma|}{x^{3}}} + O(\log^2 x),$$

wo die Reihe absolut konvergiert.

Wenn die Riemannsche Vermutung richtig ist, so kann aus (65), der entsprechenden *Landau*schen Formel, oder (66) sofort (58) erhalten werden. Unter derselben Voraussetzung folgt aus der *v. Mangoldt*schen Formel (63)

$$\psi(x) = x - 2\sqrt{x} \sum_{\gamma > 0} \frac{\sin(\gamma \log x)}{\gamma} + O(\sqrt{x}).$$

Die hier auftretende Reihe stellt den „kritischen Teil" von $\psi(x)$ dar; wird jedes Glied mit dem entsprechenden $e^{-\gamma\sigma}$ multipliziert und $\log x = t$ gesetzt, so erhält man den imaginären Teil der Funktion von $s = \sigma + it$

$$\sum_{\gamma > 0} \frac{1}{\gamma} e^{-\gamma s}.$$

Durch Betrachtung dieser Funktion beweist *Littlewood*[184]) unter Benutzung eines Satzes über diophantische Approximationen sein oben erwähntes, durch (60) ausgedrücktes Resultat.

Aus (62) erhält man mit Hilfe der Beziehung (vgl. Nr. **32**)

$$(67)\qquad \bar{\pi}(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \overline{\Pi}\left(x^{\frac{1}{n}}\right)$$

eine explizite Formel für $\bar{\pi}(x)$. Diese Formel hat früher die theoretische Stütze der (falschen) Vermutung (61) geliefert[201]), es läßt sich jedoch zur Zeit daraus nicht wesentlich mehr über $\bar{\pi}(x)$ folgern, als schon aus der einfacheren Beziehung

$$\bar{\pi}(x) = \overline{\Pi}(x) + O\left(\frac{\sqrt{x}}{\log x}\right)$$

folgt.

29. Theorie der L-Funktionen. Es sei $k > 1$ eine gegebene ganze Zahl; dann muß jede Primzahl, mit Ausnahme der endlich vielen in k aufgehenden, irgendeiner der $\varphi(k)$ zu k teilerfremden Restklassen

---

gleichmäßig für $y \geq \sqrt{x}$, angegeben, deren Gültigkeit aber nur unter Voraussetzung der *Riemann*schen Vermutung behauptet wird.

201) *Riemann* (a. a. O. 95) sagt z. B. bei der Besprechung der Formel (67): „Die bekannte Näherungsformel $F(x) = Li(x)$ (sein $F(x)$ ist unser $\bar{\pi}(x)$) ist also nur bis auf Größen von der Ordnung $x^{\frac{1}{2}}$ richtig und gibt einen etwas zu großen Wert."

modulo k angehören. Schon von *Legendre*[202]) wurde (mit falschem Beweis) die Behauptung ausgesprochen, daß jede dieser Restklassen unendlich viele Primzahlen — und sogar asymptotisch gleich viele wie jede andere — enthält. Für die erste Behauptung gab *Dirichlet*[203]) einen strengen Beweis, die zweite aber wurde erst von *Hadamard*[162]) und *de la Vallée Poussin*[204]) bewiesen. Bei diesen Untersuchungen treten als Hilfsmittel gewisse Funktionen auf, die auch bei verschiedenen anderen Fragen der analytischen Zahlentheorie eine Rolle spielen (vgl. Nr. **35, 40, 41**), und die deshalb jetzt besprochen werden müssen.

Die obengenannten $\varphi(k)$ Restklassen bilden in bezug auf die gewöhnliche Multiplikation eine *Abel*sche Gruppe. Es sei $X(K)$ irgendeiner der $\varphi(k)$ Charaktere der Gruppe (vgl. I A 6, Nr. **20**); diese Funktion nimmt für jede der fraglichen Restklassen K einen bestimmten Wert an, der übrigens immer eine $\varphi(k)$-te Einheitswurzel ist. Es sei nun die zahlentheoretische Funktion $\chi(n)$ für $n = 0, \pm 1, \pm 2, \ldots$ folgendermaßen erklärt: für jedes n einer mit k gemeinteiligen Restklasse sei $\chi(n) = 0$; für jedes n einer zu k teilerfremden Restklasse K sei $\chi(n) = X(K)$. Unter den so eingeführten $\varphi(k)$ verschiedenen *Charakteren modulo* k zeichnet sich besonders der *Hauptcharakter* aus, der für jedes zu k teilerfremde n den Wert 1 hat. Um die verschiedenen Charaktere zu unterscheiden, bezeichnet man sie durch $\chi_1(n), \chi_2(n), \ldots \chi_{\varphi(k)}(n)$, wobei $\chi_1(n)$ immer der Hauptcharakter ist. Die Charaktere besitzen die folgenden vier Fundamentaleigenschaften:

$$\text{a)}\quad \chi(n) = \chi(n') \quad \text{für} \quad n \equiv n' \pmod{k},$$

$$\text{b)}\quad \chi(n)\cdot\chi(n') = \chi(nn'),$$

$$\text{c)}\quad \sum_{n=1}^{k}\chi_\nu(n) = \begin{cases}\varphi(k) & \text{für} \quad \nu = 1,\\ 0 & \text{sonst}, \quad ^{205}),\end{cases}$$

202) *A.-M. Legendre,* a. a. O. 152a) p. 404; 152b) p. 77 und 99. Vgl. auch *A. Dupré*, Examen d'une proposition de Legendre relative à la théorie des nombres, Paris 1859; *C. Moreau,* Extrait d'une lettre, Nouv Ann math. (2) 12 (1873), p. 322—324; *A. Piltz,* a. a. O. 180).

203) *G. Lejeune-Dirichlet,* Beweis des Satzes, daß jede unbegrenzte arithmetische Progression, deren erstes Glied und Differenz ganze Zahlen ohne gemeinschaftlichen Faktor sind, unendlich viele Primzahlen enthält, Abh. Akad. Berlin 1837, math. Abhandl, p. 45—71 und Werke 1, p. 313—342

204) *Ch. de la Vallée Poussin,* Recherches analytiques sur la théorie des nombres premiers, Deuxième partie, Ann. soc. sc. Bruxelles 20: 2 (1896), p. 281—362.

205) Hieraus folgt speziell, daß $\left|\sum_{1}^{N}\chi(n)\right|$ für jeden Nicht-Hauptcharakter

d) $\sum_{\nu=1}^{\varphi(k)} \chi_\nu(n) = \begin{cases} \varphi(k) & \text{für } n \equiv 1 \pmod{k}, \\ 0 & \text{sonst.} \end{cases}$

Die *Dirichlet*sche[203] Reihe

$$(68) \qquad L_\nu(s) = L(s, \chi_\nu) = \sum_{n=1}^{\infty} \frac{\chi_\nu(n)}{n^s}$$

ist für $\sigma > 1$ absolut konvergent; wegen b) gilt auch dort

$$(69) \qquad L_\nu(s) = \prod_p \left(1 - \frac{\chi_\nu(p)}{p^s}\right)^{-1}.$$

Diese L-Funktionen können als Verallgemeinerungen von $\zeta(s)$ — die dem Falle $k = 1$ entspricht — angesehen werden und besitzen auch durchaus analoge Eigenschaften.[206] Für den Fall des Hauptcharakters folgt unmittelbar

$$(70) \qquad L_1(s) = \sum_{n=1}^{\infty} \frac{\chi_1(n)}{n^s} = \zeta(s) \cdot \prod_{p|k} \left(1 - \frac{1}{p^s}\right).$$

($p|k$ bedeutet: p geht in k auf.) Die Funktion $L_1(s)$ läßt sich somit direkt auf $\zeta(s)$ zurückführen; sie besitzt wie diese in $s = 1$ einen Pol erster Ordnung und ist sonst überall im Endlichen regulär. Für $\nu > 1$ folgt dagegen aus c), daß (68) für $\sigma > 0$ konvergiert und sogar für jeden Wert von s durch die *Cesàro*sche Methode summabel ist (vgl. Nr. **13**); für jedes vom Hauptcharakter verschiedene χ ist also $L(s)$ eine ganze transzendente Funktion.[207]

Dirichlet untersuchte die L-Funktionen nur für reelle s; verschiedene andere Verfasser[208] haben dann auch komplexe s berücksichtigt und

unter einer nur von k abhängenden Schranke liegt. In der Tat gilt sogar $\left|\sum_{1}^{N} \chi(n)\right| < c\sqrt{k}\log k$, wo c eine absolute Konstante bedeutet. Vgl. *G. Pólya,* Über die Verteilung der quadratischen Reste und Nichtreste, Göttinger Nachr 1918, p 21—29; *J. Schur,* Einige Bemerkungen zu der vorstehenden Arbeit des Herrn G. Pólya, Gött. Nachr. 1918, p. 30—36; *E. Landau,* Abschätzungen von Charaktersummen, Einheiten und Klassenzahlen, Gött. Nachr. 1918, p. 79—97.

206) Eine ausführliche Darstellung der Theorie gibt *E. Landau,* Handbuch, § 95—140.

207) Dies folgt auch aus der Identität

$$L(s) = \sum_{m=1}^{k-1} \chi(m) \sum_{n=0}^{\infty} \frac{1}{(m + nk)^s} = k^{-s} \sum_{m=1}^{k-1} \chi(m)\, \zeta\left(\frac{m}{k}, s\right)$$

und den in Nr. **21** erwähnten Untersuchungen über $\zeta(w, s)$

208) Vgl. *C. J. Malmstén,* Specimen analyticum etc., Diss., Upsala 1842 und De integralibus quibusdam definitis, seriebusque infinitis, Crelles J. 38 (1849), p. 1—39; *R. Lipschitz,* a. a O. 147); *H. Kinkelin,* Allgemeine Theorie der har-

die L-Funktionen auf die verallgemeinerten Zetafunktionen von Nr. **21** zurückgeführt. Aus diesen Untersuchungen geht vor allem hervor, daß jede L-Funktion eine Funktionalgleichung besitzt, die derjenigen von $\zeta(s)$ (vgl. Nr. **14**) analog gebaut ist. Wenn $\chi(n)$ einem sog. eigentlichen Charakter[209]) entspricht, so gilt in der Tat

$$(71) \qquad L(s) = \Theta \frac{\sqrt{k}}{\pi} \left(\frac{2\pi}{k}\right)^s \sin\frac{\pi(s+\alpha)}{2}\, \Gamma(1-s)\,\bar{L}(1-s),$$

wo $\bar{L}(s)$ mit dem konjugiert komplexen Charakter $\bar{\chi}(n)$ gebildet ist, Θ eine Konstante vom absoluten Betrage 1 und $\alpha = 0$ oder 1 ist.[210]) Dies wird z. B. dadurch bewiesen, daß $L(s)$ durch die Funktion $\sum\chi(n)e^{-n^2 z}$ (für $\alpha = 0$) oder durch $\sum\chi(n)ne^{-n^2 z}$ für ($\alpha = 1$) analog wie bei $\zeta(s)$ (vgl. Nr. **14**) ausgedrückt wird[211]), wonach die Funktionalgleichung aus der Transformationstheorie der Thetafunktionen folgt. — Gehört $L(s)$ dagegen zu einem uneigentlichen Charakter, so lassen sich immer ein echter Teiler k' von k und ein eigentlicher Charakter $\chi'(n)$ modulo k' derart angeben, daß für $\sigma > 1$

$$(72) \qquad L(s) = \sum_{n=1}^{\infty} \frac{\chi'(n)}{n^s} \cdot \prod_{p \mid k} \left(1 - \frac{\chi'(p)}{p^s}\right)$$

gilt.

In diesem Falle unterscheidet sich $L(s)$ also nur um einen trivialen Faktor von einer zu einem eigentlichen Charakter gehörigen L-Funktion (modulo k'); (70) stellt offenbar einen Spezialfall hiervon dar.

Jetzt können die Eigenschaften von $L(s)$ genau wie bei $\zeta(s)$ abgeleitet werden. In der Halbebene $\sigma > 1$ ist $L(s) \neq 0$, für $\sigma < 0$ gibt es nur die vom Faktor $\sin\frac{\pi(s+\alpha)}{2}$ in (71) herrührenden „tri-

monischen Reihen, mit Anwendung auf die Zahlentheorie, Progr. d. Gewerbeschule, Basel 1862; *A. Piltz*, a. a. O. 180); *A. Hurwitz*, Einige Eigenschaften der Dirichletschen Funktionen $F(s) = \sum \left(\frac{D}{n}\right) \frac{1}{n^s}$, die bei der Bestimmung der Klassenanzahl binärer quadratischer Formen auftreten; *M. Lerch*, a. a. O. 148). Bei *Hadamard*, a. a. O. 162) und *de la Vallée Poussin*, a. a O. 204), werden die früheren Resultate zusammengestellt und die Hilfsmittel der modernen Funktionentheorie zum erstenmal auf die L-Funktionen angewandt.

209) Ein Charakter $\chi(n)$ modulo k heißt *uneigentlich*, wenn es einen echten Teiler k' von k und einen Charakter $\chi'(n)$ modulo k' gibt, so daß für jedes n entweder $\chi(n) = 0$ oder $\chi(n) = \chi'(n)$ gilt. Sonst heißt $\chi(n)$ *eigentlich*. Der Hauptcharakter ist für $k > 1$ immer uneigentlich. Vgl. z. B. *Landau*, Handbuch, Bd. 1, p. 478.

210) Nämlich $\alpha = 0$ im Falle $\chi(-1) = 1$, $\alpha = 1$ im Falle $\chi(-1) = -1$.

211) *de la Vallée Poussin*, a. a. O. 204). Seine Darstellung wurde von *Landau*, a. a. O. 206) vereinfacht.

vialen" Nullstellen, auch der Punkt $s = 0$ kann unter Umständen Nullstelle sein. Im Streifen $0 \leqq \sigma \leqq 1$ liegen unendlich viele von Null verschiedene Nullstellen $\varrho = \beta + \gamma i$, und die Anzahl $N(T)$ der ϱ, deren Ordinaten dem Intervall $0 < \gamma \leqq T$ angehören, ist gleich

$$N(T) = \frac{1}{2\pi} T \log T - cT + O(\log T),$$

wo c von k und χ abhängt.[212]) (Vgl. Nr. **16**.) Für jeden Nicht-Hauptcharakter gibt es eine Produktentwicklung[211])

$$(73) \qquad L(s) = a s^{\mu} e^{bs} \frac{1}{\Gamma\left(\frac{s+\alpha}{2}\right)} \prod_{\varrho} \left(1 - \frac{s}{\varrho}\right) e^{\frac{s}{\varrho}},$$

wo μ ganz und $\geqq 0$ ist; beim Hauptcharakter muß auf der linken Seite das Produkt $(s-1)L(s)$ stehen (vgl. Nr. **15**).

Der für die Primzahltheorie besonders wichtige Satz, daß der Punkt $s = 1$ bei keiner L-Funktion eine Nullstelle ist, wurde schon von *Dirichlet*[203]) gefunden. Der Beweis ist ganz verschieden, je nachdem der Charakter ein *reeller* (d. h. ein für alle n reeller) oder ein *komplexer* (d. h. ein für wenigstens ein n nicht-reeller) ist. Im letzteren Falle wäre gleichzeitig mit $L(1)$ auch $\bar{L}(1)$ gleich Null, und die Funktionen $L(s)$ und $\bar{L}(s)$ wären nicht identisch. Dies wäre aber nicht mit der Identität

$$\prod_{\iota=1}^{\varphi(k)} L_{\iota}(s) = e^{\varphi(k) \sum\limits_{p^m \equiv 1 \pmod{k}} \frac{1}{m p^{ms}}}, \qquad (\sigma > 1)$$

verträglich, da die linke Seite für $s = 1$ eine Nullstelle hätte, während die rechte Seite für reelle $s > 1$ immer $\geqq 1$ ist. Für jeden komplexen Charakter gilt sogar[213])

$$\frac{1}{|L(1)|} < M \log k \, (\log \log k)^{\frac{3}{8}}$$

212) *E. Landau*, a. a. O. 107)

213) Vgl. *H. Gronwall*, Sur les séries de Dirichlet correspondant à des caractères complexes, Palermo Rend. 35 (1913), p. 145—159; *E. Landau*, a) Über das Nichtverschwinden der Dirichletschen Reihen, welche komplexen Charakteren entsprechen, Math. Ann. 70 (1911), p. 69—78; b) Über die Klassenzahl imaginär-quadratischer Zahlkörper, Gött. Nachr. 1918, p 285—295; c) Zur Theorie der Heckeschen Zetafunktionen, welche komplexen Charakteren entsprechen, Math. Ztschr. 4 (1919), p. 152—162. Bei diesen Abschätzungen von $\frac{1}{L(1)}$ als Funktion von k zeigt sich eine eigenartige Analogie mit der Abschätzung von $\zeta(1+ti)$ als Funktion von t (vgl. Nr. **18**). Für die reellen Charaktere wird das entsprechende Ergebnis (mit $\frac{1}{2}$ statt $\frac{3}{8}$) nur unter einer gewissen unbewiesenen Voraussetzung erhalten (vgl. Nr. **40**).

mit absolut konstantem M. — Bei den *reellen* Charakteren war der Beweis viel schwieriger; es war eben die Hauptleistung von *Dirichlet*[208]), $L(1)$ als Produkt von einer positiven Konstanten und einer gewissen Klassenzahl quadratischer Formen darzustellen (vgl. Nr. **40**); eo ipso war $L(1) \neq 0$. Vereinfachte Beweisanordnungen gaben *Mertens*[214]), *de la Vallée Poussin*[215]), *Teege*[216]) und *Landau*[217]), die den Beweis durch reihen- oder funktionentheoretische Überlegungen, ohne Benutzung der Theorie der quadratischen Formen, führten.[218])

Für jeden von $s = 1$ verschiedenen Punkt der Geraden $\sigma = 1$ läßt sich wie bei $\zeta(s)$ (vgl. Nr. **14**) $L(s) \neq 0$ nachweisen.[219]) Auch die entsprechenden schärferen Sätze gelten hier, es gibt z. B. eine absolute Konstante $a > 0$ derart, daß im Gebiete $\sigma > 1 - \frac{a}{\log|t|}$, $|t| > t_0$ keine Nullstellen von $L(s)$ liegen (vgl. Nr. **19**).[220]) Das Gegenstück der *Riemann*schen Vermutung wurde für die L-Funktionen von *Piltz*[180]) ausgesprochen. Da man im allgemeinen nicht weiß, ob Nullstellen auf der Strecke $0 < s < 1$ der reellen Achse liegen, und

214) *F. Mertens*, Über Dirichletsche Reihen, Sitzungsber. Akad. Wien 104 Abt. 2a (1895), p. 1093—1153; Über das Nichtverschwinden Dirichletscher Reihen mit reellen Gliedern, ebenda 104 Abt. 2a, p. 1158—1166; Über Multiplikation und Nichtverschwinden Dirichletscher Reihen, Crelles J. 117 (1897), p. 169—184; Über Dirichlets Beweis usw Sitzungsber. Akad. Wien 106 Abt 2a (1897), p. 254—286; Eine asymptotische Aufgabe, ebenda 108, Abt. 2a (1899), p. 32—37.

215) *Ch. de la Vallée Poussin*, a. a. O. 204) und Démonstration simplifiée du théorème de Dirichlet sur la progression arithmétique, Mém. couronnés et autres mém. Acad. Belgique 53 (1895—1896), No. 6

216) *H. Teege*, Beweis, daß die unendliche Reihe $\sum \left(\frac{P}{n}\right) \frac{1}{n}$ einen positiven von Null verschiedenen Wert hat, Mitt. math. Ges. Hamburg 4 (1901), p. 1—11.

217) *E. Landau*, a. a. O 29) und Über das Nichtverschwinden einer Dirichletschen Reihe, Sitzungsber. Akad Berlin 1906, p. 314—320.

218) Vgl. auch eine Bemerkung von *Remak* bei *E. Landau*, Über imaginärquadratische Zahlkörper mit gleicher Klassenzahl, Gött. Nachr. 1918, p 277—284.

219) Hiermit hängt zusammen, daß die Reihe $\sum\limits_p \frac{\chi(p)}{p^s}$ und das Produkt in (69) auch noch für $\sigma = 1$ konvergieren (beim Hauptcharakter jedoch nur für $t \neq 0$) und daß (69) auch hier richtig bleibt (vgl. Nr. **14**). Ob diese Ausdrücke in der Halbebene $\sigma < 1$ einen einzigen Konvergenzpunkt besitzen, ist noch nicht entschieden. Vgl. *E. Landau*, a) Über die Primzahlen einer arithmetischen Progression, Sitzungsber. Akad. Wien 112, Abt. 2a (1903), p. 493—535; b) Über einen Satz von Tschebyschef, Math. Ann. 61 (1905), p. 527—550, wo eine Reihe früherer Arbeiten über den Gegenstand kritisiert werden; c) a. a. O. 178).

220) *E. Landau*, Handbuch, § 131, wo frühere Resultate desselben Verfassers verschärft werden.

da ferner nach (72) die imaginäre Achse unter Umständen unendlich viele Nullstellen enthalten kann, muß die Vermutung etwa so ausgesprochen werden: „für $\sigma > \frac{1}{2}$ ist $L(s) \neq 0$."[221]) Die Sätze von der Existenz unendlich vieler Nullstellen[222]) auf $\sigma = \frac{1}{2}$ und von der Häufung der Nullstellen in der Nähe dieser Geraden[223]) (vgl. Nr. **19**) gelten auch für die L-Funktionen.

Das Produkt zweier L-Reihen ist, sofern keine von ihnen einem Hauptcharakter entspricht, nach dem Satze von *Stieltjes* (vgl. Nr. **12**) für $\sigma > \frac{1}{2}$ konvergent. *Landau*[224]) beweist den folgenden Satz, der als Spezialfall eine Verschärfung hiervon enthält: Es seien $\chi_1(n)$ und $\chi_2(n)$ zwei beliebige[225]) Charaktere modulo k_1 bzw. k_2. Dann gibt es zwei Konstanten A und B, so daß die *Dirichlet*sche Reihe

$$\sum \frac{a_n}{n^s} = \sum \frac{\chi_1(n)}{n^s} \cdot \sum \frac{\chi_2(n)}{n^s} + A \sum \frac{\log n}{n^s} + B \sum \frac{1}{n^s}$$
$$= L_1(s) L_2(s) - A\zeta'(s) + B\zeta(s)$$

für $\sigma > \frac{1}{3}$ konvergiert. Hierbei ist $A = B = 0$, wenn weder χ_1 noch χ_2 Hauptcharakter ist, und $A = 0$, wenn nur einer von den beiden Hauptcharakter ist. Dieser Satz hat wichtige Anwendungen auf verschiedene zahlentheoretische Probleme (vgl. Nr. **34** und **35**).

30. Die Verteilung der Primzahlen einer arithmetischen Reihe. Nach (69) gilt für $\sigma > 1$

$$(74)\quad \begin{cases} \log L(s) = \sum\limits_{p,\, m} \dfrac{\chi(p^m)}{m\, p^{ms}} = \sum\limits_{n=1}^{\infty} \dfrac{\chi(n) \Lambda(n)}{\log n \cdot n^s} \\ -\dfrac{L'}{L}(s) = \sum\limits_{p,\, m} \dfrac{\chi(p^m) \log p}{p^{ms}} = \sum\limits_{n=1}^{\infty} \dfrac{\chi(n) \Lambda(n)}{n^s}. \end{cases}$$

Hieraus folgt nach den Eigenschaften b) und d) der Charaktere, wenn

221) Eine notwendige und hinreichende Bedingung für die Wahrheit dieser Vermutung gab neuerdings *H. Bohr* mit Hilfe des von ihm eingeführten Begriffes „Quasiperiodizität einer *Dirichlet*schen Reihe": Über eine quasi-periodische Eigenschaft Dirichletscher Reihen mit Anwendung auf die Dirichletschen L-Funktionen, Math. Ann. 85 (1922), p. 115—122. — *J. Großmann* hat die Vermutung durch numerische Untersuchungen gestützt: Über die Nullstellen der Riemannschen ζ-Funktion und der Dirichletschen L-Funktionen, Diss., Göttingen 1913.

222) *E. Landau*, a. a. O. 135).

223) *H. Bohr* und *E. Landau*, a. a. O. 65).

224) *E. Landau*, Über die Anzahl der Gitterpunkte in gewissen Bereichen, Gött. Nachr. 1912, p. 687—771.

225) Hier soll also $\chi_1(n)$ nicht wie oben notwendig den Hauptcharakter bezeichnen.

l eine beliebige zu k teilerfremde ganze Zahl bedeutet,

$$(75)\qquad \begin{cases} \displaystyle\sum_{n\equiv l\,(\mathrm{mod}\,k)} \frac{\Lambda(n)}{\log n\cdot n^s} = \frac{1}{\varphi(k)}\sum_{\nu=1}^{\varphi(k)} \frac{1}{\chi_\nu(l)}\log L_\nu(s)\\ \displaystyle\sum_{n\equiv l\,(\mathrm{mod}\,k)} \frac{\Lambda(n)}{n^s} = -\frac{1}{\varphi(k)}\sum_{\nu=1}^{\varphi(k)} \frac{1}{\chi_\nu(l)}\cdot\frac{L_\nu'}{L_\nu}(s).\end{cases}$$

In beiden Gleichungen (75) wird die rechte Seite bei Annäherung an $s = 1$ unendlich, da dieser Punkt für $L_1(s)$ ein Pol, für die übrigen $L_\nu(s)$ dagegen weder Pol noch Nullstelle ist. Daraus schloß *Dirichlet*[203]), daß in der arithmetischen Reihe $l, l+k, l+2k, \ldots$ unendlich viele Primzahlen vorkommen; sonst würden ja in der Tat die linken Seiten von (75) für alle s endlich bleiben.[226])

Mit Hilfe der tieferen Eigenschaften der L-Funktionen konnten *Hadamard*[162]) und *de la Vallée Poussin*[204]) die dem Primzahlsatz entsprechenden Sätze

$$(76)\qquad \begin{cases} \displaystyle\pi_{k,l}(x) = \sum_{\substack{p\leq x\\ p\equiv l\,(\mathrm{mod}\,k)}} 1 \sim \frac{1}{\varphi(k)} Li(x),\\ \displaystyle\psi_{k,l}(x) = \sum_{\substack{n\leq x\\ n\equiv l\,(\mathrm{mod}\,k)}} \Lambda(n) \sim \frac{1}{\varphi(k)}\, x\end{cases}$$

beweisen. Das Hauptargument beim Beweise bildet der in der vorigen Nummer erwähnte Satz: $L_\nu(1+ti) \neq 0$ für alle ν und alle reellen t. Von *Landau*[227]) wurden die Beweise vereinfacht und die Resultate verschärft, so daß das beste mit Sicherheit bekannte Resultat[228]) so lautet:

$$(77)\qquad \begin{cases} \pi_{k,l}(x) = \dfrac{1}{\varphi(k)} Li(x) + O\left(xe^{-\alpha\sqrt{\log x}}\right)\\ \psi_{k,l}(x) = \dfrac{1}{\varphi(k)}\, x + O\left(xe^{-\alpha\sqrt{\log x}}\right)\end{cases}$$

mit absolut konstantem, d. h. von k und l unabhängigen, α. Die

226) In mehreren speziellen Fällen läßt sich der *Dirichlet*sche Satz elementar beweisen. Vgl. z. B. *L. E. Dickson*, History of the theory of numbers, Bd 1, Washington 1919, p. 419.

227) *E. Landau*, Über die Primzahlen in einer arithmetischen Progression und die Primideale in einer Idealklasse Sitzungsber. Akad. Wien 117, Abt. 2a (1908), p. 1095—1107; a. a. O. 219a); a. a. O. 107); Handbuch, § 119—121, § 131—132.

228) Wäre die verallgemeinerte *Riemann*sche Vermutung für die L-Funktionen bewiesen, so würden natürlich für die Primzahlen einer arithmetischen Reihe zu (57) und (58) analoge Beziehungen gelten. Vgl. *E. Landau*, Handbuch, § 239 und a. a. O. 178).

Hauptglieder rühren natürlich von den Singularitäten von $\log L_1(s)$ bzw. $\frac{L_1'}{L_1}(s)$ in $s = 1$ her. Aus (76) folgt speziell, wenn l_1 und l_2 beide zu k teilerfremd sind,

$$\pi_{k, l_1}(x) \sim \pi_{k, l_2}(x),$$

d. h. die zweite in der vorigen Nummer genannte *Legendre*sche Behauptung. — Die *Riemann-v. Mangoldt*sche Primzahlformel (vgl. Nr. 28) läßt sich auch für die Primzahlen einer arithmetischen Reihe verallgemeinern.[229]) Zunächst gilt für einen beliebigen Charakter $\chi_\nu(n)$ (vgl. (73))[230])

$$(78)\quad \begin{aligned} \sum_{n \leq x} \chi_\nu(n) \Lambda(n) - \frac{1}{2} \chi_\nu(x) \Lambda(x) &= -\frac{1}{2\pi i} \int_{2 - i\infty}^{2 + i\infty} \frac{x^s}{s} \cdot \frac{L_\nu'}{L_\nu}(s)\, ds \\ &= \varepsilon_\nu x - \sum_\varrho \frac{x^\varrho}{\varrho} - \sum_{n=1}^{\infty} \frac{x^{\alpha - 2n}}{\alpha - 2n} + a_0 + a_1 \log x, \end{aligned}$$

wo a_0 und a_1 von x unabhängig sind. ε_ν bedeutet Eins für $\nu = 1$, sonst Null. Aus (75) kann man jetzt, nach dem Eindeutigkeitssatz der *Dirichlet*schen Reihen, (vgl. Nr. 3) eine explizite Formel für die Funktion

$$\overline{\psi}_{k, l}(x) = \tfrac{1}{2}(\psi_{k, l}(x + 0) + \psi_{k, l}(x - 0))$$

erhalten.

Die bisher erwähnten Resultate laufen alle darauf hinaus, daß die Primzahlen auf die $\varphi(k)$ zu k teilerfremden Restklassen *gleichmäßig* verteilt sind. Schon *Tschebyschef*[231]) behauptete — freilich nur für den Fall $k = 4$ — dies könne nur bis zu einer bestimmten Grenze gelten, indem die Reihe $4n + 3$ „viel mehr" Primzahlen als die Reihe $4n + 1$ $(n = 1, 2, \ldots)$ enthalte. Er sprach (ohne Beweis) den Satz aus: es gibt eine Folge $x_1, x_2, \ldots$ mit $x_\nu \to \infty$, derart, daß für wachsendes ν

$$(79)\qquad \pi_{4,3}(x_\nu) - \pi_{4,1}(x_\nu) \sim \frac{\sqrt{x_\nu}}{\log x_\nu}$$

gilt. Dies wurde zuerst von *Phragmén*[181]) und dann einfacher von

229) Vgl. *A. Piltz*, a. a. O. 180) und *G. Torelli*, a. a. O. 159), Nuove formole per calcolare la totalità dei numeri primi etc., Rend. Accad. sc. fis. mat. Napoli (3) 10 (1904), p. 350—362 und (3) 11 (1905), p. 101—109. Vollständig ausgeführt wurde der Beweis erst von *E. Landau*, a. a. O. 194) und Handbuch, § 133—138.

230) Für nicht ganze x bedeuten $\chi(x)$ und $\Lambda(x)$ Null.

231) *P. Tschebyschef*, Lettre à M. Fuss, Bull. cl. phys.-math. Acad. St. Petersburg 11 (1853), p. 208 und Œuvres 1, p. 697—698; Sur une transformation de séries numériques, Nouv. corr. math. 4 (1878), p. 305—308 und Œuvres 2, p. 705—707.

Landau[183]) bewiesen; aus den Untersuchungen von *Littlewood*[184]) folgt aber, daß die obige Differenz, bei zweckmäßiger Wahl von K, für beliebig große Werte von x sowohl $> K\frac{\sqrt{x}}{\log x}\log\log\log x$ als auch $< -K\frac{\sqrt{x}}{\log x}\log\log\log x$ wird. Der Zusammenhang wird gewissermaßen durch die aus (78) folgenden Gleichungen

$$\vartheta_{4,1}(x) = \frac{1}{2}x - x^{\frac{1}{2}} - \frac{1}{2}\left(\sum_{\varrho}\frac{x^{\varrho}}{\varrho} + \sum_{\varrho'}\frac{x^{\varrho'}}{\varrho'}\right) + O\left(x^{\frac{1}{3}}\right),$$

$$\vartheta_{4,3}(x) = \frac{1}{2}x \qquad - \frac{1}{2}\left(\sum_{\varrho}\frac{x^{\varrho}}{\varrho} - \sum_{\varrho'}\frac{x^{\varrho'}}{\varrho'}\right) + O\left(x^{\frac{1}{3}}\right)$$

aufgeklärt. (Es ist $\vartheta_{k,l} = \sum\limits_{\substack{p \leqq x \\ p \equiv l \pmod{k}}} \log p$ gesetzt; ϱ durchläuft die komplexen Nullstellen von $\zeta(s)$ und ϱ' diejenigen von

$$L(s) = \sum_{0}^{\infty}\frac{(-1)^n}{(2n+1)^s} = \sum_{1}^{\infty}\frac{\chi(n)}{n^s},$$

wo $\chi(n)$ den Nicht-Hauptcharakter modulo 4 bezeichnet.) Die oszillierenden Glieder sind hier zwar von höherer Größenordnung als $x^{\frac{1}{2}}$, in der ersten Gleichung tritt aber ein Glied $-x^{\frac{1}{2}}$ *von konstantem Vorzeichen* auf, was wiederum daraus folgt, daß alle *Primzahlquadrate* ($2^2 = 4$ ausgenommen) von der Form $4n+1$ sind.

Tschebyschef[231]) behauptete auch: „wenn c gegen Null abnimmt, so gilt

$$(80) \quad e^{-3c} - e^{-5c} + e^{-7c} + e^{-11c} - \cdots = -\sum_{p}\chi(p)e^{-pc} \rightarrow \infty.\text{“}$$

Von *Hardy-Littlewood*[232]) und *Landau*[233]) wurde gezeigt, daß dieser Satz mit dem folgenden (unbewiesenen) Analogon der *Riemann*schen Vermutung äquivalent ist: „Die zum Nicht-Hauptcharakter modulo 4 gehörige L-Funktion ist für $\sigma > \frac{1}{2}$ von Null verschieden.“

Die allgemeine *Tschebyschef*sche Aussage: „es gibt viel mehr Primzahlen von der Form $4n+3$ als von der Form $4n+1$“ kann also jedenfalls nur in ziemlich beschränktem Maße wahr sein[234]) und

232) *G. H. Hardy* und *J. E. Littlewood*, a. a. O. 31b). (Aus $L(s) \neq 0$ für $\sigma > \frac{1}{2}$ folgt (80)).

233) *E. Landau*, a. a. O. 178). (Aus (80) folgt $L(s) \neq 0$ für $\sigma > \frac{1}{2}$) und Über einige ältere Vermutungen und Behauptungen in der Primzahltheorie, zweite Abhandl., Math. Ztschr. 1 (1918), p. 213—219.

234) *E. Landau*, a. a. O. 178), p. 6, bemerkt, daß aus der Behauptung (80) von *Tschebyschef*, $\pi_{4,3}(x) - \pi_{4,1}(x) = O\left(x^{\frac{1}{2}}\log x\right)$ folgt. Die Differenz $\pi_{4,3} - \pi_{4,1}$

ist z. B. in der Fassung (80), die wenigstens wahr sein *könnte*, noch nicht bewiesen.

Die Resultate von *Phragmén* und *Landau* betreffend die *Tschebyschef*sche Behauptung (79) wurden von *Landau*[235]) für beliebige Moduln k (an der Stelle von 4) verallgemeinert.

31. Andere Primzahlprobleme. *Summen über Primzahlen.* Daß unter den n ersten ganzen Zahlen annäherungsweise $Li(n)$ Primzahlen vorkommen, kann wegen

$$Li(n) = \sum_{2}^{n} \frac{1}{\log n} + O(1)$$

in ungenauer Weise so ausgedrückt werden: „die Wahrscheinlichkeit, daß die beliebig gewählte Zahl n Primzahl ist, ist gleich $\frac{1}{\log n}$." Man wird hiernach vermuten, daß die beiden Reihen

$$(81) \qquad \sum_{p} F(p) \quad \text{und} \quad \sum_{n} \frac{F(n)}{\log n}$$

sich mehr oder weniger ähnlich verhalten müssen. In der Tat besagt ja der Primzahlsatz

$$\sum_{p \leqq x} 1 \quad \sim \sum_{n=2}^{x} \frac{1}{\log n}, \qquad (F(t) = 1)$$

bzw.

$$\sum_{p \leqq x} \log p \sim \sum_{n=2}^{x} 1, \qquad (F(t) = \log t).$$

Nach *Tschebyschef*[236]) sind die Reihen (81) gleichzeitig konvergent oder divergent, sobald $\frac{F(n)}{\log n}$ für hinreichend große n positiv und nie zunehmend ist. *Mertens*[237]) beweist

$$(82) \quad \sum_{p \leqq x} \frac{\log p}{p} = \sum_{n=2}^{x} \frac{1}{n} + O(1) = \log x + O(1) \qquad \text{und}$$

$$(83) \quad \sum_{p \leqq x} \frac{1}{p} \quad = \sum_{n=2}^{x} \frac{1}{n \log n} + A + O\left(\frac{1}{\log x}\right) = \log\log x + B + O\left(\frac{1}{\log x}\right)$$

---

wäre also nach (80) absolut genommen kleiner, als bisher bekannt war — nämlich $O\left(xe^{-\alpha\sqrt{\log x}}\right)$ — eine Folgerung, die ja in der entgegengesetzten Richtung von *Tschebyschefs* Interpretation seiner Behauptung liegt.

235) *E. Landau*, Handbuch, § 200.

236) *P. Tschebyschef*, a. a. O. 155 b).

237) *F. Mertens*, Ein Beitrag zur analytischen Zahlentheorie, Crelles J. 78 (1874), p 46—62.

mit konstantem A und B. Von *de la Vallée Poussin*[173]) wurde (82) zu

$$\sum_{p \leqq x} \frac{\log p}{p} = \log x - C - \sum_{p} \frac{\log p}{p(p-1)} + O\left(e^{-\alpha\sqrt{\log x}}\right)$$

verschärft, wo C die *Euler*sche Konstante bezeichnet. Die Abschätzung des Restgliedes in (83) kann in ähnlicher Weise verschärft werden. Hieraus folgt speziell

$$\lim_{x \to \infty}\left(\log x - \sum_{p \leqq x} \frac{\log p}{p-1}\right) = \lim_{n \to \infty}\left(\sum_{n=1}^{x} \frac{1}{n} - \log x\right) = C$$

und (vgl. (59))
$$\sum_{n=1}^{\infty} \frac{\Lambda(n) - 1}{n} = -2C.$$

Landau[238]) gibt verschiedene Sätze über Summen der Gestalt $\sum_{p \leqq x} F(p)$ und $\sum_{p \leqq x} F(p, x)$ und bespricht insbesondere die Möglichkeit, von einer Formel elementar zu den andern zu gelangen, ohne jedes Mal die Theorie der Zetafunktion zu benutzen (vgl. hierzu Nr. 33). *Mertens*[237]) hat (82) und (83) auch für die Primzahlen einer arithmetischen Reihe verallgemeinert.

Die Konvergenz von $\sum p^{-s}$ und $\sum \chi(p) p^{-s}$ auf der Geraden $\sigma = 1$ wurde schon oben besprochen (vgl. Nr. 14 und Nr. 29, Fußnote 219). Diese Reihen stellen für $\sigma > 1$ die Funktionen dar

$$\sum_{1}^{\infty} \frac{\mu(n)}{n} \log \zeta(ns) \qquad \text{bzw.} \qquad \sum_{1}^{\infty} \frac{\mu(n)}{n} \log L(ns, \chi^n),$$

die über $\sigma = 1$ hinaus bis zu $\sigma = 0$, aber nicht weiter, analytisch fortgesetzt werden können.[239]) — Die Funktion

$$F(z) = \sum_{p} \frac{z^p}{p}$$

wird bei Annäherung an einen „rationalen" Punkt $z = e^{\frac{2m\pi i}{n}}$ des Einheitskreises unendlich groß, sofern n eine quadratfreie Zahl ist.

238) *E. Landau,* Sur quelques problèmes relatifs à la distribution des nombres premiers, Bull. Soc. math. France 28 (1900), p. 25—38; Handbuch § 55—56 (vgl. auch p. 889).

239) *E. Landau* und *A. Walfisz,* Über die Nichtfortsetzbarkeit einiger durch Dirichletsche Reihen definierter Funktionen, Palermo Rend. 44 (1920), p. 82—86.

Vgl. auch *J. C. Kluyver,* Benaderingsformules betreffende de priemgetallen beneden eene gegeven grens, Akad. Wetensk. Amsterdam, Verslagen 8 (1900), p. 672—682 und *E Landau,* a. a. O. 78).

Fatou[240]) schließt hieraus, daß $F(z)$ und

$$zF'(z) = \sum_p z^p$$

nicht über den Einheitskreis fortgesetzt werden können. Nach einer Bemerkung von *Landau*[241]) folgt dies auch direkt aus neueren Sätzen über die *Taylor*sche Reihe.

Die n^{te} Primzahl und die Differenz $p_{n+1} - p_n$. Wenn p_n die n^{te} Primzahl bezeichnet, so folgt aus der Gleichung

$$n = \pi(p_n) = Li(p_n) + O\left(p_n e^{-\alpha\sqrt{\log p_n}}\right)$$

durch Inversion

$$p_n = Li^{-1}(n) + O\left(n \log^2 n\, e^{-\alpha\sqrt{\log n}}\right),$$

wo $Li^{-1}(x)$ die zu $Li(x)$ inverse Funktion bedeutet. Insbesondere ist[242]) also

$$p_n \sim n \log n.$$

Tschebyschef[236]) bewies den früher von *Bertrand*[243]) vermuteten und empirisch bestätigten Satz, daß von einer gewissen Stelle an zwischen x und $2x$ wenigstens eine Primzahl liegt, d. h. daß für große n immer $\frac{p_{n+1}}{p_n} < 2$ ist. Der Primzahlsatz gibt sogar[244])

$$\lim_{n \to \infty} \frac{p_{n+1}}{p_n} = 1$$

oder

$$p_{n+1} - p_n = o(p_n).$$

Aus der genauen Restabschätzung (53) zum Primzahlsatz folgt[245])

$$p_{n+1} - p_n = O\left(p_n e^{-\alpha\sqrt{\log p_n}}\right),$$

240) *P. Fatou*, Sur les séries entières à coefficients entiers, Paris C. R. 138 (1904), p. 342—344.

241) *E. Landau*, a. a. O. 78). Dieselbe Bemerkung hat auch *F. Carlson* gemacht: Über Potenzreihen mit ganzzahligen Koeffizienten, Math. Ztschr. 9 (1921), p. 1—13.

242) Mit der asymptotischen Darstellung von p_n beschäftigten sich u. a. *M. Perwuschin*, Formule pour la détermination approximative des nombres premiers etc., Verhandl. Math.-Kongr. Zürich 1897, Leipzig 1898, p. 166—167; *E. Cesàro*, Sur une formule empirique de M. Pervouchine, Paris C. R. 119 (1894), p. 848—849; *M. Cipolla*, La determinazione assintotica dell' n^{imo} numero primo, Rend. Accad. Sc. Fis. Mat. Napoli (3) 8 (1902), p. 132—166. Vgl. auch *E. Landau*, Handbuch, § 57.

243) *J. Bertrand*, Mémoire sur le nombre de valeurs que peut prendre une fonction quand on y permute les lettres qu' elle renferme, J. Éc. Polyt. 18 (1845), p. 123—140.

244) Ein direkter Beweis dieser Tatsache, der nicht zugleich den Primzahlsatz liefert, scheint nicht bekannt zu sein. Vgl. *E. Landau*, Gelöste und ungelöste Probleme aus der Theorie der Primzahlverteilung und der Riemannschen Zetafunktion, Proc. Fifth. Intern. Congr. Math., Cambridge 1913, 1 p. 93—108.

245) Vgl. *Ch. de la Vallée Poussin*, a. a. O. 173), p. 55.

was die beste mit Sicherheit bekannte Abschätzung darstellt. Wenn die *Riemann*sche Vermutung vorausgesetzt wird, folgt aus (57)

$$p_{n+1} - p_n = O(\sqrt{p_n} \log^2 p_n),$$

wo nach *Cramér*[246]) $\log^2 p_n$ durch $\log p_n$ ersetzt werden kann. Es gibt demnach, wenn die *Riemann*sche Vermutung richtig ist, eine Zahl c, so daß für $n = 2, 3, \ldots$ zwischen n^2 und $(n + c\log n)^2$ immer wenigstens eine Primzahl liegt. *Oppermann*[247]) behauptete, daß dasselbe von dem Intervall $(n^2, (n+1)^2)$ gilt; das ist aber bisher nicht entschieden. *Piltz*[248]) hat sogar die Behauptung

$$p_{n+1} - p_n = O(p_n^{\varepsilon})$$

für jedes $\varepsilon > 0$, ausgesprochen; in dieser Hinsicht ist nur bekannt[249]), daß die Anzahl der $p_n \leqq x$, die der Ungleichung

$$p_{n+1} - p_n > p_n^k, \qquad (0 < k \leqq \tfrac{1}{2})$$

genügen, unter Voraussetzung der *Riemann*schen Vermutung von der Form $O\left(x^{1-\frac{3}{2}k+\varepsilon}\right)$ ist. — Im Mittel muß die Differenz $\delta_n = p_{n+1} - p_n$ von der Ordnung $\log p_n$ sein, denn es gilt

$$\frac{1}{n}(\delta_1 + \delta_2 + \cdots \delta_n) = \frac{1}{n}(p_{n+1} - 2) \sim \log p_n.$$

Nach unten ist keine bessere Abschätzung als die triviale $\delta_n \geqq 2$ für $n > 1$ bekannt; verschiedene Verfasser[250]) vermuten, daß in der Tat

$$p_{n+1} - p_n = 2 \tag{84}$$

---

246) *H. Cramér*, a. a. O. 186).

247) *L. Oppermann*, Om vor Kundskab om Primtallenes Mængde mellem givne Grœnser, Overs. Danske Vidensk. Selsk. Forh. 1882, p. 169—179.

248) *A. Piltz*, a. a. O. 180), p. 46.

249) *H. Cramér*, On the distribution of primes, Proc. Cambr. Phil. Soc. 20 (1921), p. 272—280.

250) Vgl. *J. J. Sylvester*, On the partition of an even number into two primes, Proc. London math. Soc. (1) 4 (1871), p. 4—6 und Collected Math. Pap. 2, p. 709—711; On the Goldbach-Euler Theorem regarding prime numbers, Nature 55 (1896—1897), p. 196—197, 269 und Pap. 4. p. 734—737; *P. Stäckel*, Über Goldbachs empirisches Theorem etc., Gött. Nachr. 1896, p. 292—299; Die Darstellung der geraden Zahlen als Summen von zwei Primzahlen, Sitzungsber. Akad. Heidelberg 1916; Die Lückenzahlen r^{ter} Stufe und die Darstellung der geraden Zahlen als Summen und Differenzen ungerader Primzahlen. I—III, Sitzungsber. Akad. Heidelberg 1917—1918; *J. Merlin*, Un travail sur les nombres premiers, Bull. sc. math. (2) 39 (1915), p. 121—136; *V. Brun*, Über das Goldbachsche Gesetz und die Anzahl der Primzahlpaare, Arch. for. Math. og Naturv., Kristiania 34, Nr. 8 (1915); Sur les nombres premiers de la forme $ap + b$, ebenda 34, Nr. 14 (1917), *G. H. Hardy* und *J. E. Littlewood*, Note on Messrs Shah and Wilson's paper entitled: On an empirical formula connected with Goldbach's

für unendlich viele n gilt, und sogar daß

$$h(x) \sim a\frac{x}{\log^2 x}$$

mit konstantem a ist, wenn $h(x)$ die Anzahl der $p_n \leqq x$ bedeutet, die (84) genügen. *Brun*[251]) beweist

$$h(x) = O\left(\frac{x}{\log^2 x}\right).$$

Der Satz von Goldbach und verwandte Fragen. Goldbach[252]) sprach im Jahre 1742 den bis jetzt unbewiesenen Satz aus: „Jede gerade Zahl kann als Summe von zwei Primzahlen dargestellt werden." Verschiedene Verfasser[250]) vermuteten, daß die Anzahl $G(n)$ solcher Darstellungen einer geraden Zahl n sogar mit n ins Unendliche wächst, und zwar so, daß für alle geraden n

$$G(n) > b\frac{n}{\log^2 n}$$

mit konstantem b gilt.[253]) *Hardy* und *Littlewood*[250]) verallgemeinern das Problem und greifen es zuerst mit analytischen Mitteln an, indem sie in der Potenzreihe

$$f_k(z) = \sum_{n=1}^{\infty} a_n^{(k)} z^n = \left(\sum_p \log p\, z^p\right)^k,$$

(die über den Einheitskreis nicht fortsetzbar ist) ein beliebiges $a_n^{(k)}$ durch das Integral

$$a_n^{(k)} = \frac{1}{2\pi i}\int\limits_{|z|=r<1} \frac{f_k(z)}{z^{n+1}}\,dz$$

ausdrücken, um dann das Verhalten von $a_n^{(k)}$ für große n zu untersuchen. (Auf diese Methode kommen wir in Nr. 38 zurück.) Die Be-

---

theorem, Proc. Cambr. Phil. Soc. 19 (1919), p. 245—254; Some problems of Partitio Numerorum; III: On the expression of a number as a sum of primes, Acta math. 44 (1922), p. 1—70.

251) *V. Brun,* Le crible d'Eratosthène et le théorème de Goldbach, Vidensk selsk. Skrifter, Mat-naturv. Kl. Kristiania 1920, Nr. 3 und Paris C. R. 168 (1919), p. 544—546. Vgl. auch: La série $\frac{1}{5} + \frac{1}{7} + \cdots$ où les dénominateurs sont „nombres premiers jumeaux" est convergente ou finie, Bull. sc. Math. (2) 43 (1919), p. 1—9.

252) Vgl. Briefwechsel zwischen *Euler* und *Goldbach* bei *P. H Fuss,* Correspondance math. phys. 1, St. Petersbourg 1843, p. 127, 135. Vgl. in bezug auf die ältere Geschichte des Satzes *L. E. Dickson,* a. a. O. 226), p. 421—425. Über die numerische Prüfung des Satzes vgl. z. B. *P. Stäckel,* a a. O. 250). — Für $n = 2$ kann der Satz offenbar nur richtig sein, wenn 1 als Primzahl mitgezählt wird.

253) *E. Landau,* Über die zahlentheoretische Funktion $\varphi(n)$ und ihre Beziehung zum Goldbachschen Satz, Gött. Nachr. 1900, p. 177—186, zeigt, daß $G(2) + \cdots G(n) \sim \frac{n^2}{2\log^2 n}$ ist. Hieraus folgt, daß eine früher von *Stäckel,* a. a. O. 250), vorgeschlagene Formel falsch ist.

hauptung von *Goldbach*, $a_n^{(2)} > 0$ für alle geraden $n > 2$, läßt sich zwar nicht beweisen, es wird aber die Formel[254])

$$G(n) \sim c \frac{n}{\log^2 n} \prod_q \frac{q-1}{q-2}, \qquad (n \text{ gerade})$$

als wahrscheinlich hingestellt. Hierin ist c konstant, und q durchläuft die ungeraden Primteiler von n. Wenn die (unbewiesene) Annahme gemacht wird, daß *die obere Grenze der reellen Teile der Nullstellen von* $\zeta(s)$ *und von allen L-Funktionen kleiner als* $\frac{3}{4}$ *ist,* so läßt sich der folgende Satz beweisen: „Jede hinreichend große *ungerade* Zahl kann als Summe von *drei* Primzahlen dargestellt werden." — *Brun*[251]) beweist durch Anwendung einer Modifikation des sog. Siebverfahrens von *Eratosthenes* den Satz: „Jede hinreichend große gerade Zahl kann als Summe von zwei ganzen Zahlen dargestellt werden, die höchstens je neun Primfaktoren enthalten." Die beiden letztgenannten Sätze sind offenbar direkte Folgerungen aus dem *Goldbach*schen.

Das Problem, die Bedingungen für die Lösbarkeit einer unbestimmten Gleichung $ax + by + c = 0$ mittelst zweier Primzahlen x und y zu finden, ist eine Verallgemeinerung des *Goldbach*schen; es wurde auch von den oben erwähnten Verfassern behandelt. Mit der *Hardy-Littlewood*schen Methode lassen sich endlich auch verschiedene Probleme der Art: „Gibt es unendlich viele Primzahlen von der Form $n^2 + 1$, von der Form $n'^3 + n''^3 + n'''^3$," usw., angreifen. Auch hier läßt sich nichts beweisen, die Methode führt aber auf gewisse asymptotische Formeln, die in mehreren Fällen mit gutem Erfolg numerisch geprüft wurden.

IV. Weitere zahlentheoretische Funktionen.[255])

32. Die Funktionen $\mu(n)$, $\lambda(n)$ und $\varphi(n)$. Für $\sigma > 1$ gilt (vgl. Nr. 22)

$$(85) \qquad \sum_1^\infty \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}.$$

Die für $\sigma > 1$ unbedingt konvergente Reihe $\sum \mu(n) n^{-s}$ stellt also eine für $\sigma \geqq 1$ reguläre Funktion dar; daß die Reihe auch noch für

254) Mehr oder weniger ähnliche Formeln waren von den oben erwähnten Verfassern schon früher vorgeschlagen worden. Die *Hardy-Littlewood*sche Formel wurde von *N. M. Shah* und *B. M. Wilson* numerisch geprüft: On an empirical formula connected with Goldbach's theorem, Proc. Cambr. Phil. Soc. 19 (1919), p. 238—244.

255) Betreffs älterer Untersuchungen zu diesem Kapitel sei auf I C 3 verwiesen.

$s=1$ konvergiert, hat schon *Euler*[256]) vermutet. Dies wurde von *v. Mangoldt*[257]) unter Benutzung der *Hadamard*schen Sätze über die Produktzerlegung von $(s-1)\zeta(s)$ (vgl. Nr. 15) bewiesen; nach (85) ist dann

$$\sum_1^\infty \frac{\mu(n)}{n} = 0, \quad \text{d. h.} \quad g(x) = \sum_1^x \frac{\mu(n)}{n} = o(1).$$

Landau[258]) zeigt, daß dieses Resultat auch elementar aus dem Primzahlsatz abgeleitet werden kann (vgl. Nr. 33). Eine unmittelbare Folgerung ist

$$M(x) = \sum_1^x \mu(n) = o(x),$$

und man kann nun nach dem Konvergenzsatz von *M. Riesz* (vgl. Nr. 5) schließen, daß (85) auf der ganzen Geraden $\sigma = 1$ gültig bleibt.[259]) *Landau*[260]) hat sogar die Konvergenz von

$$\sum_1^\infty \frac{\mu(n)(\log n)^q}{n^{1+ti}}$$

für beliebige reelle q und t festgestellt. Er[261]) gab — mit seiner bei dem Primzahlsatz angewandten Methode — die Abschätzungen

$$(86) \quad \begin{cases} M(x) = O\left(x e^{-\alpha\sqrt{\log x}}\right) \\ g(x) = O\left(e^{-\alpha\sqrt{\log x}}\right) \\ \displaystyle\sum_1^x \frac{\mu(n)\log n}{n} = -1 + O\left(e^{-\alpha\sqrt{\log x}}\right) \end{cases}$$

256) *L. Euler*, Introductio in analysin infinitorum, 1, Lausanne 1748, p. 229.

257) *H. v. Mangoldt*, Beweis der Gleichung $\sum_1^\infty \frac{\mu(k)}{k} = 0$, Sitzungsber. Akad. Berlin 1897, p. 835—852.

258) *E. Landau*, Neuer Beweis der Gleichung $\sum_1^\infty \frac{\mu(k)}{k} = 0$, Diss. Berlin 1899.

259) Vgl. *Landau*, a. a. O. 21). Das war natürlich nicht der erste Beweis dieses Satzes.

260) *E. Landau*, Über die zahlentheoretische Funktion $\mu(k)$, Sitzungsber. Akad. Wien 112, Abt 2a (1903), p. 537—570. Die Konvergenz von $\sum \frac{\mu(n)\log n}{n}$ wurde schon von *A F. Möbius* vermutet: Über eine besondere Art von Umkehrung der Reihen, Crelles J. 9 (1832), p. 105—123 und Werke 4 (1887), p 589—612.

261) *E. Landau*, a. a O. 29) und Handbuch, § 163—164. *Ch. de la Vallée Poussin*, a a O 173), hatte eine unschärfere Abschätzung gegeben.

und verallgemeinerte[262]) alle diese Resultate für den Fall, daß n nur die Zahlen einer arithmetischen Reihe durchläuft.

Wie bei dem Primzahlsatz, so ist bei den Gleichungen (86) die Frage nach der möglichen Verschärfung der Abschätzungen eng mit der *Riemann*schen Vermutung verbunden. Von *Stieltjes*[263]) und *Mertens*[264]) wurde

$$M(x) = O(\sqrt{x}) \tag{87}$$

vermutet; *Stieltjes* behauptete in der Tat auf diesem Wege die *Riemann*sche Vermutung bewiesen zu haben, denn aus (87) würde (vgl. Nr. 2) die Konvergenz von $\sum_1^\infty \mu(n) n^{-s}$ für $\sigma > \frac{1}{2}$, und damit die *Riemann*sche Vermutung, folgen. — Daß auch umgekehrt aus der *Riemann*schen Vermutung die Konvergenz von $\sum_1^\infty \mu(n) n^{-s}$ für $\sigma > \frac{1}{2}$ folgt, wurde zuerst von *Littlewood*[138]) im Laufe seiner in Nr. 20 besprochenen Untersuchungen über die Zetafunktion bewiesen. Demnach ist die *Riemann*sche Vermutung mit der Behauptung

$$M(x) = O\left(x^{\frac{1}{2}+\varepsilon}\right) \quad \text{für jedes} \quad \varepsilon > 0 \tag{88}$$

vollständig äquivalent.[265]) Die weitere Vermutung von *Stieltjes*[263]) daß $\sum_1^\infty \mu(n) n^{-s}$ auch noch für $s = \frac{1}{2}$ konvergiert, ist aber nach *Landau*[178]) sicher nicht richtig. A fortiori kann also (88) für kein negatives ε gelten.

Aus (85) folgt $\sum_1^\infty n^{-s} \cdot \sum_1^\infty \mu(n) n^{-s} = 1$ und hieraus für jedes ganze $n > 1$

$$\sum_{d \mid n} \mu(d) = 0,$$

---

262) Vgl. *J. C. Kluyver*, Reeksen, afgeleid uit de reeks $\sum \frac{\mu(m)}{m}$, Akad Wetensk. Amsterdam, Verslagen 12 (1904), p. 432—439, und *E. Landau*, Bemerkungen zu der Abhandlung von Herrn Kluyver etc., ebenda 13 (1905), p. 71—83, Handbuch, § 169—175.

263) *T. J. Stieltjes*, a. a. O. 160), Lettre 79 und Sur une fonction uniforme, Paris C. R. 101 (1885), p. 153—154.

264) *F. Mertens*, Über eine zahlentheoretische Funktion, Sitzungsber. Akad. Wien 106, Abt. 2a (1897), p. 761—830. Über die numerische Prüfung dieser Vermutung vgl. etwa *R. D. v. Sterneck*, Sitzungsber. Akad. Wien 110, Abt. 2a (1901), p. 1053—1102.

265) Aus (87) würde dagegen mehr als die *Riemann*sche Vermutung folgen, z. B. daß alle Wurzeln von $\zeta(s)$ einfach sind. Vgl. auch *H. Cramér* und *E. Landau*, Über die Zetafunktion auf der Mittellinie des kritischen Streifens, Arkiv för Mat., Astr. och Fys. 15 (1921), Nr. 28.

sowie für jedes $x \geqq 1$
$$\sum_{n=1}^{x} \mu(n)\left[\frac{x}{n}\right] = 1.$$

Auf diesen Eigenschaften von $\mu(n)$ beruhen die sog. zahlentheoretischen Umkehrungsformeln. Es läßt sich z. B. die Gleichung (67), Nr. 28, leicht aus ihnen ableiten.

Die Funktion $\lambda(n)$ ist durch
$$\sum_1^\infty \frac{\lambda(n)}{n^s} = \frac{\zeta(2s)}{\zeta(s)} = \sum_1^\infty \frac{1}{n^{2s}} \cdot \sum_1^\infty \frac{\mu(n)}{n^s}, \qquad (\sigma > 1)$$
definiert; es folgt hieraus
$$\sum_1^x \lambda(n) = \sum_1^{\sqrt{x}} M\left(\frac{x}{n^2}\right),$$
und mit Hilfe dieser Identität lassen sich alle obigen Ergebnisse für $\lambda(n)$ verallgemeinern.[266]) Insbesondere zeigt es sich, daß es unter den N ersten ganzen Zahlen asymptotisch ebenso viele gibt, die aus einer geraden, als solche, die aus einer ungeraden Anzahl von Primfaktoren bestehen[267])

Für die *Euler*sche Funktion $\varphi(n)$ gilt offenbar immer $\varphi(n) \leqq n - 1$, und sobald n eine Primzahl ist, muß hier das Gleichheitszeichen benutzt werden Andererseits beweist *Landau*[268])
$$\liminf_{n \to \infty} \frac{\varphi(n) \log\log n}{n} = e^{-C}.$$
Daß die summatorische Funktion $\Phi(x)$ asymptotisch gleich $\frac{3}{\pi^2}x^2$ ist, war schon *Dirichlet*[269]) bekannt; nach *Mertens*[270]) gilt sogar
$$\Phi(x) = \frac{3}{\pi^2}x^2 + O(x \log x),$$
was völlig elementar bewiesen werden kann. Merkwürdigerweise ist es bisher nicht gelungen, aus der Beziehung $\sum_1^\infty \varphi(n) n^{-s} = \frac{\zeta(s-1)}{\zeta(s)}$ mit analytischen Mitteln eine bessere Abschätzung des Restgliedes zu

266) *E. Landau*, Handbuch, § 166—167, 169—172

267) *E. Landau*, Handbuch, p. 571.

268) *E. Landau*, Über den Verlauf der zahlentheoretischen Funktion $\varphi(x)$, Arch. Math. Phys. (3) 5 (1903), p. 86—91.

269) *P. G. Lejeune-Dirichlet*, Über die Bestimmung der mittleren Werte in der Zahlentheorie, Abhandl. Akad. Berlin 1849, math. Abhandl. p. 69—83 (Werke 2, p. 49—66).

270) *F. Mertens*, Über einige asymptotische Gesetze der Zahlentheorie, Crelles J. 77 (1874), p. 289—338.

erhalten.[271]) — Nach *Landau*[272]) gilt

$$\sum_{n=1}^{x} \frac{1}{\varphi(n)} \sim \frac{315\,\zeta(3)}{2\pi^4} \log x.$$

Landau[273]) beweist mit Hilfe der Theorie der Multiplikation *Dirichlet*scher Reihen (vgl. Nr. 12) die Konvergenz verschiedener hierhergehöriger Reihen, z. B.

$$\sum_{1}^{\infty} \frac{\chi(n)\,\mu(n)}{n}, \quad \sum_{1}^{\infty} \frac{\chi(n)\,\lambda(n)}{n}, \quad \sum_{1}^{\infty} \frac{\chi(n)\,\varphi(n)}{n^2},$$

wo $\chi(n)$ ein beliebiger Charakter (bei der letztgenannten Reihe jedoch nicht der Hauptcharakter) nach einem beliebigen Modul ist.

33. Zusammenhangssätze. Die im vorhergehenden erwähnten tieferen Ergebnisse der analytischen Zahlentheorie waren alle mit Hilfe der Theorie der Zetafunktion, bzw. deren Verallgemeinerungen, erreicht. Für die systematische Darstellung der Theorie erscheint es wichtig, die verschiedenen Hauptresultate in bezug auf ihre „Tiefe" zu vergleichen und insbesondere die Möglichkeit zu untersuchen, aus einem von ihnen die anderen *elementar* abzuleiten, ohne nochmals die transzendenten Methoden zu benutzen.

Die wichtigsten in dieser Richtung durch *Landau*[274]) und *Axer*[275]) bekannten Tatsachen lassen sich dahin zusammenfassen, daß die vier Gleichungen

$$\psi(x) = \sum_{1}^{x} \Lambda(n) = x + o(x), \tag{89}$$

$$\sum_{1}^{\infty} \frac{\Lambda(n) - 1}{n} = -2C, \tag{90}$$

$$M(x) = \sum_{1}^{x} \mu(n) = o(x), \tag{91}$$

$$\sum_{1}^{\infty} \frac{\mu(n)}{n} = 0, \tag{92}$$

271) Da $\Phi(x)$ unendlich viele Sprünge von der Größenordnung x macht, kann das Restglied jedenfalls nicht von niedrigerer Größenordnung als $O(x)$ sein.

272) *E. Landau*, a. a. O. 253).

273) *E. Landau*, a. a. O. 78) und Handbuch, § 184—195.

274) *E. Landau*, a. a. O. 258) und 21), sowie Über die Äquivalenz zweier Hauptsätze der analytischen Zahlentheorie, Sitzungsber. Akad. Wien **120**, Abt. **2a** (1911), p. 973—988.

275) *A. Axer*, Beitrag zur Kenntnis der zahlentheoretischen Funktionen $\mu(n)$ und $\lambda(n)$, Prace Mat. Fiz. 21 (1910), p. 65—95

alle in dem Sinne äquivalent sind, daß aus irgendeiner von ihnen die drei übrigen *elementar* folgen. [Nach den Ergebnissen von Nr. **23** können natürlich auch die (89) entsprechenden Formeln mit $\Pi(x)$, $\pi(x)$ und $\vartheta(x)$ hinzugesetzt werden.] In (90) und (92) ist nur die *Konvergenz* der betreffenden Reihe wesentlich, ist diese einmal festgestellt, so folgt die Wertbestimmung aus einfachen Stetigkeitsbetrachtungen. — Etwas tiefer liegt der Satz

$$\sum_1^\infty \frac{\mu(n)\log n}{n} = -1,$$

der mit einer schärferen Form von (92), nämlich mit

$$\sum_1^x \frac{\mu(n)}{n} = o\left(\frac{1}{\log x}\right),$$

äquivalent ist.[276]) — Der Übergang (durch partielle Summation) von (90) zu (89), bzw. von (92) zu (91), ist trivial; die anderen Übergänge folgen aus gewissen allgemeinen Grenzwertsätzen. *Landau*[277]) gibt einen Satz, aus dem alle jene Übergänge durch Spezialisierung folgen. *Hardy* und *Littlewood*[278]) zeigen, daß die Übergänge auch mit Hilfe von „*Tauber*schen" Sätzen (vgl. Nr. **5**) über die „*Lambert*schen Reihen"

$$\sum_1^\infty \frac{a_n}{e^{n s}-1}$$

ausgeführt werden können.

34. Teilerprobleme. Die Funktionen $d(n)$ und $\sigma(n)$, die Anzahl und die Summe der Teiler von n, sind vielfach untersucht worden. Über die Größenordnung dieser Funktionen ist zunächst trivial, daß immer

$$d(n) \geqq 2, \quad \sigma(n) \geqq n+1$$

ist, sowie daß in beiden Beziehungen unendlich oft (nämlich für alle Primzahlen) das Gleichheitszeichen gilt. Andererseits beweisen *Wigert*[279]) und *Gronwall*[280])

276) *A. Axer*, Über einige Grenzwertsätze, Sitzungsber. Akad. Wien 120, Abt. 2a (1911), p. 1253—1298.

277) *E. Landau*, Über einige neuere Grenzwertsätze, Palermo Rend. 34 (1912), p. 121—131.

278) *G. H. Hardy* und *J. E. Littlewood*, On a Tauberian theorem for Lambert's series and some fundamental theorems in the analytic theory of numbers, Proc. London math. Soc. (2) 19 (1919), p. 21—29.

279) *S. Wigert*, a) Sur l'ordre de grandeur du nombre des diviseurs d'un entier, Arkiv för Mat., Astr och Fys. 3 (1906—1907), No. 18; b) Sur quelques fonctions arithmétiques, Acta Math. 37 (1914), p. 113—140.

280) *H. Gronwall*, Some asymptotic expressions in the theory of numbers, Trans. Amer. math. Soc. 14 (1913), p. 113—122.

$$\limsup_{n\to\infty} \frac{\log d(n)\cdot \log\log n}{\log n} = \log 2,$$

$$\limsup_{n\to\infty} \frac{\sigma(n)}{n \log\log n} = e^C.$$

Gronwall gibt auch entsprechende Beziehungen für $\sigma_\alpha(n)$, die Summe der α^{ten} Potenzen der Teiler von n. *Ramanujan*[281]) beweist viele ins einzelne gehende Sätze über den Verlauf der Funktion $d(n)$. Er zeigt insbesondere, daß $d(n)$, wenn die *Riemann*sche Vermutung richtig ist, die „maximale Größenordnung“

$$2^{Li(\log n) + O(\log^\alpha n)} \qquad (\alpha < 1)$$

hat. Er nennt eine Zahl n „highly composite“, wenn $d(n) > d(\nu)$ für $\nu = 1, 2, \ldots n - 1$ ist, und zeigt, wie man mit elementaren Mitteln erstaunend genaue Resultate über die Reihe der Exponenten in der Darstellung einer solchen Zahl als Produkt von Primzahlpotenzen ableiten kann. Er findet auch bemerkenswerte Beziehungen zwischen der Funktion $\sigma_\alpha(n)$ und gewissen trigonometrischen Summen; ein spezieller Fall hiervon lautet

$$\sigma(n) = \frac{\pi^2 n}{6} \sum_{\nu=1}^{\infty} \frac{c_\nu(n)}{\nu^2},$$

wo $c_\nu(n) = \sum_\mu \cos \frac{2\pi\mu n}{\nu}$ ist, und μ die $\varphi(\nu)$ zu ν teilerfremden ganzen positiven Zahlen $\leqq \nu$ durchläuft.

Die summatorische Funktion $D(x)$ gibt offenbar die Anzahl der *Gitterpunkte* (Punkte mit ganzzahligen Koordinaten) an, die in der (u, v)-Ebene dem Gebiet

$$u > 0, \quad v > 0, \quad uv \leqq x \tag{93}$$

angehören; hieraus folgt leicht

$$D(x) = \sum_{n=1}^{x} \left[\frac{x}{n}\right] = 2\sum_{n=1}^{\sqrt{x}} \left[\frac{x}{n}\right] - [\sqrt{x}]^2,$$

woraus die von *Dirichlet*[269]) gegebene Formel

$$D(x) = x(\log x + 2C - 1) + O(\sqrt{x})$$

281) *S. Ramanujan*, Highly composite numbers, Proc. London math. Soc. (2) 14 (1915), p. 347—409; On certain trigonometrical sums and their applications in the theory of numbers, Trans. Cambr. Phil. Soc. 22 (1918), p. 259—276. Vgl. auch: On certain arithmetical functions, Trans. Cambr. Phil. Soc. 22 (1916), p. 159—184, wo gewisse, die Funktion $\sigma_\alpha(n)$ enthaltende Summen untersucht werden.

gefolgert werden kann. Dieses Resultat wurde erst von *Voronoï*[282]) verschärft; er zieht in zweckmäßig gewählten Punkten der Hyperbel $uv = x$ die Tangenten, zerlegt dadurch das Gebiet (93) in mehrere Teilgebiete, schätzt die Anzahl der Gitterpunkte in jedem Teilgebiet ab und erhält

$$(94) \qquad D(x) = x(\log x + 2C - 1) + O\left(x^{\frac{1}{3}}\log x\right).$$

Neuerdings ist es *van der Corput*[306]) gelungen, die Abschätzung des Restgliedes sogar zu $O(x^M)$ zu verbessern, wo $M < \frac{33}{100}$ ist.

Schon vor *Voronoï* hatte *Pfeiffer*[283]) einen vermeintlichen Beweis von (94) — mit $O\left(x^{\frac{1}{3}+\varepsilon}\right)$ anstatt $O\left(x^{\frac{1}{3}}\log x\right)$ — veröffentlicht; seine Methode war freilich nicht einwandfrei, wurde aber von *Landau*[284]) umgearbeitet und u. a. zum Beweis von (94) benutzt. Diese „*Pfeiffer*sche Methode", auf die wir in Nr. **36** zurückkommen, beruht auf „reell-analytischer" Grundlage. Andererseits ist[285]) (vgl. Nr. 4 und 22)

$$(95) \qquad \overline{D}(x) = \frac{1}{2\pi i}\int\limits_{2-i\infty}^{2+i\infty} \frac{x^s}{s}(\zeta(s))^2\,ds;$$

dieser für die Primzahltheorie grundlegende „komplex-analytische" Ansatz schien lange auf das Teilerproblem nicht anwendbar zu sein, es gelang jedoch *Landau*[286]) ihn zum Beweis von (94) zu benutzen. In (95) tritt die Zetafunktion nicht im Nenner auf; die Schwierigkeiten rühren daher nicht wie bei den Primzahlproblemen von den komplexen ζ-Nullstellen her, sie sind hier von ganz anderer Natur und sind hauptsächlich mit dem Aufsuchen einer oberen Grenze für das Integral

$$\int\limits_{-\varepsilon-Ti}^{-\varepsilon+Ti} \frac{x^s}{s}(\zeta(s))^2\,ds \qquad (\varepsilon > 0)$$

verbunden, wobei T eine Funktion von x ist. *Landau*[286]) zeigt, daß

282) *G. Voronoï,* Sur un problème du calcul des fonctions asymptotiques, Crelles J. 126 (1903), p. 241—282.

283) *E. Pfeiffer,* Über die Periodizität in der Teilbarkeit der Zahlen und über die Verteilung der Klassen positiver quadratischer Formen auf ihre Determinanten, Jahresber. d. Pfeifferschen Lehr- und Erzieh.-Anstalt. Jena 1886, p. 1—21.

284) *E. Landau,* Die Bedeutung der Pfeifferschen Methode für die analytische Zahlentheorie, Sitzungsb. Akad. Wien 121, Abt. 2a (1912), p. 2195—2332.

285) Für ganzzahlige x muß wie oben der Hauptwert des Integrals genommen werden.

286) *E. Landau,* a) a. a. O. 224); b) Über die Anzahl der Gitterpunkte in gewissen Bereichen, zweite Abhandl., Gött. Nachr. 1915, p. 209—243; c) Über Dirichlets Teilerproblem, Sitzungsb. Akad. München 1915, p. 317—328.

diese Schwierigkeit bei einer ausgedehnten Klasse von Problemen überwunden werden kann, wo an der Stelle von $(\zeta(s))^2$ eine Funktion steht, die eine Funktionalgleichung vom Typus der *Riemann*schen besitzt und gewissen anderen Bedingungen genügt. Mit dieser Methode wurde z. B. der in Nr. 29 erwähnte Satz über die Multiplikation zweier L-Reihen bewiesen, der übrigens (94) — mit der Fehlerabschätzung $O\left(x^{\frac{1}{3}+\varepsilon}\right)$ — als Spezialfall enthält, da die Konvergenz von

$$(96) \qquad \sum_1^{\infty} \frac{d(n) - \log n - 2C}{n^s} = (\zeta(s))^2 + \zeta'(s) - 2C\zeta(s)$$

für $\sigma > \frac{1}{3}$ daraus folgt.[287])

Das Problem, die untere Grenze γ derjenigen α zu bestimmen, für welche

$$\Delta(x) = D(x) - x(\log x + 2C - 1) = O(x^{\alpha})$$

gilt (γ ist also die Konvergenzabszisse von (96)), wird als „*Dirichlets* Teilerproblem" bezeichnet. Nach dem Obigen ist jedenfalls $\gamma < \frac{33}{100}$. Eine nicht triviale *untere* Abschätzung von γ hat *Hardy*[288]) gegeben, er beweist nämlich $\gamma \geqq \frac{1}{4}$. Er untersucht die Funktion

$$f(s) = \sum_1^{\infty} d(n) e^{-s\sqrt{n}} = \frac{1}{\pi i} \int_{2-i\infty}^{2+i\infty} \Gamma(2z) s^{-2z} \zeta^2(z)\, dz,$$

die in allen Punkten $s = \pm 4\pi i \sqrt{q}$ $(q = 1, 2, \ldots)$ algebraische Unendlichkeitsstellen von der Ordnung $\frac{3}{2}$ aufweist, während

$$f(s) + \frac{4(\log s - 1)}{s^2}$$

für $s = 0$ regulär ist. Hieraus folgt nach *Hardy* $\gamma \geqq \frac{1}{4}$ und sogar der schärfere Satz, daß bei zweckmäßiger Wahl einer positiven Konstanten K die Ungleichungen

$$(97) \qquad \begin{cases} \Delta(x) > Kx^{\frac{1}{4}} \\ \Delta(x) < -Kx^{\frac{1}{4}} \end{cases}$$

beide beliebig große Lösungen besitzen. *Hardy* deutet auch an, wie man durch die Anwendung der von *Littlewood* (vgl. Nr. 27 und 28) für die entsprechenden Probleme der Primzahltheorie geschaffenen

287) *Landau* gibt auch einen Beweis von (94) mit einer arithmetischen Methode, deren Grundgedanke von *Piltz* herrührt: Über Dirichlets Teilerproblem, Gött. Nachr. 1920, p. 13—32. Er hat auch (94) für den Fall verallgemeinert, daß nur solche Teiler, die einer gegebenen arithmetischen Reihe angehören, mitgezählt werden; vgl. a. a. O. 224) und 284).

288) *G. H. Hardy,* On Dirichlets Divisor Problem, Proc. London math. Soc (2) 15 (1916), p. 1—25.

Methode in (97) sogar $x^{\frac{1}{4}}$ durch $(x \log x)^{\frac{1}{4}} \log \log x$ ersetzen kann. *Landau*[289]) beweist mit der vorhin erwähnten komplex-analytischen Methode einen allgemeinen Satz, der insbesondere $\gamma \geqq \frac{1}{4}$ ergibt.

Über γ ist also bis jetzt nur $\frac{1}{4} \leqq \gamma < \frac{33}{100}$ bekannt. *Im Mittel* ist aber $|\Delta(x)|$ von der Ordnung $x^{\frac{1}{4}}$; *Cramér*[290]) beweist nämlich (in Verschärfung früherer Resultate von *Hardy*[291]))

$$(98) \qquad \int_1^x (\Delta(t))^2 dt = \frac{x^{\frac{3}{2}}}{6\pi^2} \sum_1^{\infty} \left(\frac{d(n)}{n^{\frac{3}{4}}}\right)^2 + O\left(x^{\frac{5}{4}+\varepsilon}\right)$$

und folgert hieraus

$$(99) \qquad \frac{1}{x} \int_1^x |\Delta(t)| dt = O\left(x^{\frac{1}{4}}\right).$$

Schließlich kennt man auch eine *explizite Formel* für die Funktion $\overline{D}(x)$; nach *Voronoï*[292]) gilt nämlich[293])

$$(100) \qquad \overline{D}(x) = x(\log x + 2C - 1) + \frac{1}{4} + \sqrt{x} \sum_1^{\infty} \frac{d(n)}{\sqrt{n}} \left(Y_1(4\pi\sqrt{nx}) - H_1(4\pi\sqrt{nx})\right),$$

289) *E. Landau,* a) Über die Anzahl der Gitterpunkte in gewissen Bereichen, dritte Abhandl., Gött. Nachr. 1917, p. 96—101; vgl. auch: b) Über die Heckesche Funktionalgleichung, ebenda 1917, p. 102—111.

290) *H. Cramér,* Über zwei Sätze des Herrn G. H. Hardy, Math. Ztschr. 15 (1922), p. 201—210.

291) *G. H. Hardy,* The average order of the arithmetical functions $P(x)$ and $\Delta(x)$, Proc. London math. Soc. (2) 15 (1916), p. 192—213; Additional note on two problems in the analytic theory of numbers, ebenda (2) 18 (1918), p. 201—204.

292) *G. Voronoï,* Sur une fonction transcendante et ses applications à la sommation de quelques séries, Ann. Éc. Norm. (3) 21 (1904), p. 207—268, 459—534.

293) In der folgenden Nummer machen wir über Formeln dieser Art einige allgemeine Bemerkungen. Eine Formel, die im wesentlichen mit (101) übereinstimmt, wurde schon 1891 — mit ungenügendem Beweis — von *L. Lorenz* gegeben: Analytiske Undersøgelser over Primtalmœngderne, Kgl. Danske Vidensk. Selsk. Skrifter, naturv. og math. Afd. (6) 5 (1889—1891), p. 427—450. Er entwickelt diese und sogar die entsprechenden Formeln für das *Piltz*sche Teilerproblem (s. u.) nach einer Methode, die im Grunde mit der — unstreng angewandten — *Pfeiffer*schen Methode identisch ist. Später wurde (100) von *Hardy* a. a. O. 288) unabhängig wiedergefunden. Einen Beweis von (100) mit der *Pfeiffer*schen Methode gab *W. Rogosinski,* Neue Anwendung der Pfeifferschen Methode bei Dirichlets Teilerproblem, Diss. Göttingen 1922. Vgl. auch *E. Landau,* Über Dirichlets Teilerproblem, zweite Mtlg., Gött. Nachr. 1922, p. 8—16 und *A. Walfisz,* a. a. O. 297).

wo $Y_1(v)$ die gewöhnliche „zweite Lösung" der *Bessel*schen Differentialgleichung bezeichnet und

$$H_1(v) = \frac{2}{\pi}\int_1^\infty \frac{te^{-vt}}{\sqrt{t^2-1}}\,dt \qquad (= O(e^{-v}))$$

eine *Hankel*sche Zylinderfunktion ist. Nach der bekannten asymptotischen Entwicklung von Y_1 hat man

$$(101) \quad \overline{D}(x) = x(\log x + 2C - 1)$$
$$+ \frac{x^{\frac{1}{4}}}{\pi\sqrt{2}} \sum_1^\infty \frac{d(n)}{n^{\frac{3}{4}}} \cos\left(4\pi\sqrt{nx} - \tfrac{1}{4}\pi\right) + \tfrac{1}{4} + O\left(x^{-\frac{1}{4}}\right),$$

wo das Glied $O\left(x^{-\frac{1}{4}}\right)$ eine für $x \geqq 1$ stetige Funktion ist. Die Sprünge der Funktion $\overline{D}(x)$ rühren also von der in (101) auftretenden unendlichen Reihe her, die das „kritische Glied" von $\overline{D}(x)$ darstellt. *Voronoï*[293]) gibt auch analoge Formeln für $\sum_1^x d(n)(x-n)^k$, $k = 1, 2, \ldots$.[294])

Wigert[295]) untersucht die summatorischen Funktionen $S(x)$ und $\sum_1^x \frac{\sigma(n)}{n}$; er beweist

$$S(x) = \frac{\pi^2}{12}x^2 + x\Theta_1(x),$$

$$\sum_1^x \frac{\sigma(n)}{n} = \frac{\pi^2}{6}x - \frac{1}{2}\log x + \Theta_2(x),$$

wo für $\nu = 1, 2$

$$\limsup_{x\to\infty} \frac{|\Theta_\nu(x)|}{\log x} \leqq \frac{1}{4}$$

aber jedenfalls *nicht*

$$\Theta_\nu(x) = o(\log\log x)$$

294) *S. Wigert,* Sur la série de Lambert et son application à la théorie des nombres, Acta Math. 41 (1917), p. 197—218, und *E. Landau,* Gött gel. Anz. 1915, p. 377—414, gaben einfachere Beweise für einen Teil der *Voronoï*schen Resultate. *Wigert* benutzt hierfür eine von ihm gefundene *asymptotische Funktionalgleichung* für die *Lambert*sche Reihe $\sum_1^\infty \frac{1}{e^{ns}-1} = \sum_1^\infty d(n)e^{-ns}$, für welche *Landau* einen vereinfachten Beweis gibt: Über die Wigertsche asymptotische Funktionalgleichung für die Lambertsche Reihe, Arch. Math. Phys. (3) 27 (1918), p. 144—146. Vgl. auch *S. Wigert,* Sur une équation fonctionnelle et ses conséquences arithmétiques, Arkiv för Mat., Astr. och Fys. 13 (1918), Nr. 16.

295) *S. Wigert,* a. a. O. 279 b).

gilt. Für die Funktion

$$\sum_1^x \frac{\sigma(n)}{n}(x-n)^k, \quad k = 1, 2, \ldots,$$

gibt er erstens entsprechende asymptotische Formeln, die zum Teil von *Landau*[296]) verschärft wurden, und zweitens explizite Formeln, welche unendliche Reihen mit *Bessel*schen Funktionen enthalten. *Walfisz*[297]) zeigt, daß eine solche Formel auch für den Fall $k = 0$ aufgestellt werden kann, und gibt für $\overline{S}(x)$ die entsprechende Entwicklung[298])

$$\overline{S}(x) = \frac{\pi^2}{12}x^2 - \frac{1}{2}x + \frac{1}{24} - x\sum_1^\infty \frac{\sigma(n)}{n} J_2(4\pi\sqrt{nx})$$

$$= \frac{\pi^2}{12}x^2 - \frac{1}{2}x + \frac{x^{\frac{3}{4}}}{\pi\sqrt{2}}\sum_1^\infty \frac{\sigma(n)}{n^{\frac{5}{4}}}\cos\left(4\pi\sqrt{nx} - \tfrac{1}{4}\pi\right) + O\left(x^{\frac{1}{4}}\right),$$

wobei jedoch die unendlichen Reihen mit *Cesàro*schen Mitteln von der ersten Ordnung summiert werden müssen, da ihre Konvergenz bisher nicht bewiesen werden konnte.

Ramanujan[299]) findet die Beziehung

$$\sum_1^\infty \frac{(d(n))^2}{n^s} = \frac{(\zeta(s))^4}{\zeta(2s)}$$

und schließt daraus $\displaystyle\sum_1^x (d(n))^2 \sim \frac{1}{\pi^2}x\log^3 x.$

Piltz[300]) verallgemeinert das *Dirichlet*sche Teilerproblem, indem er für $k = 2, 3, \ldots$ die Funktion

$$D_k(x) = \sum_1^x d_k(n)$$

---

296) *E. Landau*, Gött. gel. Anz. 1915, p. 377—414.

297) *A. Walfisz*, Über die summatorischen Funktionen einiger Dirichletscher Reihen, Diss Göttingen 1922

298) Die zweite Zeile der Formel ergibt sich durch Zusammenstellung der Ergebnisse von *Walfisz* mit denjenigen von *Wigert* a. a. O. 279 b) und *Landau*, a. a. O. 296).

299) *S. Ramanujan*, Some formulae in the analytic theory of numbers, Mess. of Math. 45 (1916), p. 81—84. Vgl. auch *B. M. Wilson*, Proofs of some formulae enunciated by Ramanujan, Proc. London math Soc. (2) 21 (1922), p. 235—255.

300) *A. Piltz*, Über das Gesetz, nach welchem die mittlere Darstellbarkeit der natürlichen Zahlen als Produkte einer gegebenen Anzahl Faktoren mit der Größe der Zahlen wächst, Diss. Berlin 1881. Vgl. auch *E. Landau*, Über eine idealtheoretische Funktion, Trans. Amer. math. Soc. 13 (1912), p. 1—21.

wo $Y_1(v)$ die gewöhnliche „zweite Lösung" der *Bessel*schen Differentialgleichung bezeichnet und

$$H_1(v) = \frac{2}{\pi}\int_1^\infty \frac{te^{-vt}}{\sqrt{t^2-1}}\,dt \qquad (= O(e^{-v}))$$

eine *Hankel*sche Zylinderfunktion ist. Nach der bekannten asymptotischen Entwicklung von Y_1 hat man

$$(101) \qquad \overline{D}(x) = x(\log x + 2C - 1)$$
$$+ \frac{x^{\frac{1}{4}}}{\pi\sqrt{2}}\sum_1^\infty \frac{d(n)}{n^{\frac{3}{4}}}\cos\left(4\pi\sqrt{nx} - \tfrac{1}{4}\pi\right) + \tfrac{1}{4} + O\left(x^{-\frac{1}{4}}\right),$$

wo das Glied $O\left(x^{-\frac{1}{4}}\right)$ eine für $x \geqq 1$ stetige Funktion ist. Die Sprünge der Funktion $\overline{D}(x)$ rühren also von der in (101) auftretenden unendlichen Reihe her, die das „kritische Glied" von $\overline{D}(x)$ darstellt. *Voronoï*[293]) gibt auch analoge Formeln für $\sum_1^x d(n)(x-n)^k$, $k = 1, 2, \ldots$.[294])

Wigert[295]) untersucht die summatorischen Funktionen $S(x)$ und $\sum_1^x \frac{\sigma(n)}{n}$; er beweist

$$S(x) = \frac{\pi^2}{12}x^2 + x\Theta_1(x),$$

$$\sum_1^x \frac{\sigma(n)}{n} = \frac{\pi^2}{6}x - \frac{1}{2}\log x + \Theta_2(x),$$

wo für $\nu = 1, 2$

$$\limsup_{x\to\infty} \frac{|\Theta_\nu(x)|}{\log x} \leqq \frac{1}{4}$$

aber jedenfalls *nicht*

$$\Theta_\nu(x) = o(\log\log x)$$

294) *S. Wigert,* Sur la série de Lambert et son application à la théorie des nombres, Acta Math. 41 (1917), p. 197—218, und *E. Landau,* Gött gel. Anz. 1915, p. 377—414, gaben einfachere Beweise für einen Teil der *Voronoï*schen Resultate. *Wigert* benutzt hierfür eine von ihm gefundene *asymptotische Funktionalgleichung* für die *Lambert*sche Reihe $\sum_1^\infty \frac{1}{e^{ns}-1} = \sum_1^\infty d(n)e^{-ns}$, für welche *Landau* einen vereinfachten Beweis gibt: Über die Wigertsche asymptotische Funktionalgleichung für die Lambertsche Reihe, Arch. Math. Phys. (3) 27 (1918), p. 144—146. Vgl. auch *S. Wigert,* Sur une équation fonctionnelle et ses conséquences arithmétiques, Arkiv för Mat., Astr. och Fys. 13 (1918), Nr. 16.

295) *S. Wigert,* a. a. O. 279 b).

gilt. Für die Funktion

$$\sum_{1}^{x} \frac{\sigma(n)}{n}(x-n)^k, \quad k=1, 2, \ldots,$$

gibt er erstens entsprechende asymptotische Formeln, die zum Teil von *Landau*[296]) verschärft wurden, und zweitens explizite Formeln, welche unendliche Reihen mit *Bessel*schen Funktionen enthalten. *Walfisz*[297]) zeigt, daß eine solche Formel auch für den Fall $k=0$ aufgestellt werden kann, und gibt für $\overline{S}(x)$ die entsprechende Entwicklung[298])

$$\overline{S}(x) = \frac{\pi^2}{12}x^2 - \frac{1}{2}x + \frac{1}{24} - x\sum_{1}^{\infty} \frac{\sigma(n)}{n} J_2(4\pi\sqrt{nx})$$

$$= \frac{\pi^2}{12}x^2 - \frac{1}{2}x + \frac{x^{\frac{3}{4}}}{\pi\sqrt{2}}\sum_{1}^{\infty} \frac{\sigma(n)}{n^{\frac{5}{4}}} \cos\left(4\pi\sqrt{nx} - \tfrac{1}{4}\pi\right) + O\left(x^{\frac{2}{3}}\right),$$

wobei jedoch die unendlichen Reihen mit *Cesàro*schen Mitteln von der ersten Ordnung summiert werden müssen, da ihre Konvergenz bisher nicht bewiesen werden konnte.

Ramanujan[299]) findet die Beziehung

$$\sum_{1}^{\infty} \frac{(d(n))^2}{n^s} = \frac{(\zeta(s))^4}{\zeta(2s)}$$

und schließt daraus $\displaystyle\sum_{1}^{x} (d(n))^2 \sim \frac{1}{\pi^2} x \log^3 x.$

Piltz[300]) verallgemeinert das *Dirichlet*sche Teilerproblem, indem er für $k = 2, 3, \ldots$ die Funktion

$$D_k(x) = \sum_{1}^{x} d_k(n)$$

296) *E. Landau*, Gött. gel. Anz. 1915, p. 377—414

297) *A. Walfisz*, Über die summatorischen Funktionen einiger Dirichletscher Reihen, Diss Göttingen 1922

298) Die zweite Zeile der Formel ergibt sich durch Zusammenstellung der Ergebnisse von *Walfisz* mit denjenigen von *Wigert* a. a. O. 279 b) und *Landau*, a. a. O. 296).

299) *S. Ramanujan*, Some formulae in the analytic theory of numbers, Mess. of Math. 45 (1916), p. 81—84. Vgl. auch *B. M. Wilson*, Proofs of some formulae enunciated by Ramanujan, Proc. London math Soc. (2) 21 (1922), p. 235—255.

300) *A. Piltz*, Über das Gesetz, nach welchem die mittlere Darstellbarkeit der natürlichen Zahlen als Produkte einer gegebenen Anzahl Faktoren mit der Größe der Zahlen wächst, Diss. Berlin 1881. Vgl. auch *E. Landau*, Über eine idealtheoretische Funktion, Trans. Amer. math. Soc. 13 (1912), p. 1—21.

betrachtet, wobei
$$\sum_1^\infty \frac{d_k(n)}{n^s} = (\zeta(s))^k$$
gilt und $d_k(n)$ also die Anzahl der Zerlegungen von n in k Faktoren bezeichnet; insbesondere ist $d_2(n) = d(n)$. Er zeigte, daß — analog wie bei $k = 2$ — die Hauptglieder von $D_k(x)$ von dem Pol $s = 1$ der Funktion $\frac{x^s}{s}(\zeta(s))^k$ herrühren. Wird das dortige Residuum durch $xp_{k-1}(\log x)$ bezeichnet, wobei also p_{k-1} ein Polynom $(k-1)^{\text{ten}}$ Grades ist, und wird
$$D_k(x) = xp_{k-1}(\log x) + \Delta_k(x)$$
gesetzt, so weiß man nach *Hardy* und *Littlewood*[301]), daß
$$\Delta_k(x) = O\left(x^{\frac{k-2}{k}+\varepsilon}\right)$$
für jedes $\varepsilon > 0$ und alle $k \geqq 4$ ist. Für $k = 3$ wurde das schärfste Resultat von *Landau*[286]) gegeben, indem er
$$\Delta_k(x) = O\left(x^{\frac{k-1}{k+1}} \log^{k-1} x\right)$$
für alle $k \geqq 2$ beweist.[302]) — *Hardy*[288]) hat die durch (97) ausgedrückte Eigenschaft von $\Delta_2(x)$ für beliebige k verallgemeinert, wobei der Exponent $\frac{1}{4}$ durch $\frac{k-1}{2k}$ zu ersetzen ist. Die expliziten Formeln (100) und (101) wurden von *Walfisz*[297]) und *Cramér*[303]) verallgemeinert; das „kritische Glied" von (101) wird durch
$$\frac{x^{\frac{k-1}{2k}}}{\pi\sqrt{k}} \sum_1^\infty \frac{d_k(n)}{n^{\frac{k+1}{2k}}} \cos\left(2k\pi\sqrt[k]{nx} + \frac{k-3}{4}\pi\right)$$
ersetzt, wo von der unendlichen Reihe nur bekannt ist, daß sie durch *Cesàro*sche Mittel von der Ordnung $\left[\frac{k-1}{2}\right]$ summierbar ist. Der Fall $k = 2$ ist somit der einzige, wo die Konvergenz der auftretenden Reihen festgestellt ist.

---

301) *G. H. Hardy* und *J. E. Littlewood*, a. a. O. 129)

302) *Landau*, a. a. O. 224), bemerkt, daß aus der *Riemann*schen Vermutung $\Delta_k(x) = O\left(x^{\frac{1}{2}+\varepsilon}\right)$ für jedes $\varepsilon > 0$ folgen würde. Die Behauptung $\frac{1}{x}\int_1^x |\Delta_k(t)|\,dt = O\left(x^{\frac{1}{2}+\varepsilon}\right)$ für $k = 2, 3, \ldots$ ist nach *Hardy* und *Littlewood*, a. a. O. 301), der „*Lindelöf*schen Vermutung" $\zeta(\frac{1}{2}+it) = O(t^\varepsilon)$ äquivalent.

303) *H. Cramér*, Über das Teilerproblem von Piltz, Arkiv för Mat., Astr. och Fys. 16 (1922), No. 21.

35. Ellipsoidprobleme. Wenn $r(n)$ für $n \geqq 0$ die Anzahl der additiven Zerlegungen von n in zwei Quadrate bedeutet, gibt die summatorische Funktion $R(x) = \sum_0^x r(n)$ die Anzahl der Gitterpunkte (u, v) an, die der Kreisfläche $u^2 + v^2 \leqq x$ angehören. *Gauß*[304]) bewies durch eine einfache geometrische Überlegung

$$R(x) = \pi x + O(\sqrt{x});$$

der Flächeninhalt des Kreises stellt somit einen Annäherungswert für $R(x)$ dar. Die folgenden Hauptsätze über $R(x)$ entsprechen genau denjenigen über $D(x)$ und werden auch durch analoge Methoden bewiesen:

1. Nach *Sierpiński*[305]), der die *Voronoï*sche[282]) Methode benutzte, gilt

$$R(x) = \pi x + O\left(x^{\frac{1}{3}}\right); \tag{102}$$

diese Abschätzung wurde neuerdings von *van der Corput*[306]) zu $O(x^M)$ mit $M < \frac{1}{3}$ verschärft

2. Nach *Hardy*[307]) und *Landau*[308]) kann das Restglied für kein $h < \frac{1}{4}$ von der Form $O(x^h)$ sein.

3. *Im Mittel* ist das Restglied von der Größenordnung $x^{\frac{1}{4}}$; für die Funktion $R(x) - \pi x$ gelten nämlich Formeln, die zu (98) und (99) analog sind.

4. Die explizite Formel für $\bar{R}(x) = \frac{1}{2}(R(x+0) + R(x-0))$

304) *C. F. Gauss*, De nexu inter multitudinem classium etc., Werke 2 (1863), p. 269—291.

305) *W. Sierpiński*, O pewnem zagadnieniu z rachunku funkcyj asymptotycznych, Prace Mat.-Fiz. 17 (1906), p. 77—118. Vgl. auch *E. Landau*, a. a. O. 224), 286b), 284), Über die Zerlegung der Zahlen in zwei Quadrate, Ann. Mat. pura ed appl. (3) 20 (1913), p. 1—28; Über einen Satz des Herrn Sierpiński, Giorn. di Mat. di Battaglini 51 (1913), p. 73—81; Über die Gitterpunkte in einem Kreise, erste Mtlg., Gött. Nachr. 1915, p. 148—160; Über die Gitterpunkte in einem Kreise, Math. Ztschr. 5 (1919), p. 319—320; *S. Wigert*, Über das Problem der Gitterpunkte in einem Kreise, Math. Ztschr. 5 (1919), p. 310—318.

306) *J. G. van der Corput*, a) Verschärfung der Abschätzung beim Teilerproblem, Math. Ann. 87 (1922), p. 39—65; b) Sur quelques approximations nouvelles, Paris C. R. 175 (1922), p. 856—859.

307) *G. H. Hardy*, On the expression of a number as the sum of two squares, Quart. J. 46 (1915), p. 263—283.

308) *E. Landau*, Über die Gitterpunkte in einem Kreise, zweite Mtlg., Gött. Nachr. 1915, p. 161—171; Neue Untersuchungen über die *Pfeiffer*sche Methode zur Abschätzung von Gitterpunktanzahlen, Sitzungsb. Akad. Wien 124, Abt. 2a (1915), p. 469—505.

lautet[309]) nach *Hardy*[307])

$$(103)\qquad \begin{aligned} \bar{R}(x) &= \pi x + \sqrt{x}\sum_1^\infty \frac{r(n)}{\sqrt{n}} J_1(2\pi\sqrt{nx}) \\ &= \pi x + \frac{x^{\frac14}}{\pi}\sum_1^\infty \frac{r(n)}{n^{\frac34}} \sin\left(2\pi\sqrt{nx} - \tfrac14\pi\right) + O\left(x^{-\frac14}\right). \end{aligned}$$

Für $n \geqq 1$ ist bekanntlich $r(n) = 4(d_1(n) - d_3(n))$, (vgl. I C 2, c), wo $d_\nu(n)$ die Anzahl der Divisoren von n von der Form $4k + \nu$ bedeutet[310]); hieraus folgt für $\sigma > 1$

$$\sum_1^\infty \frac{r(n)}{n^s} = 4\sum_1^\infty \frac{1}{n^s}\cdot\sum_1^\infty \frac{\chi(n)}{n^s} = 4\zeta(s)\,L(s),$$

wenn $\chi(n)$ der Nicht-Hauptcharakter modulo 4 ist. Der Satz von *Landau* (vgl. Nr. 29) ergibt die Konvergenz von

$$\sum_1^\infty \frac{r(n)-\pi}{n^s} = 4\zeta(s)\,L(s) - \pi\zeta(s)$$

für $\sigma > \frac13$; nach *van der Corput*[306]) ist diese Reihe sogar über $\sigma = \frac13$ hinaus konvergent, für $\sigma < \frac14$ ist sie aber jedenfalls divergent.

Das obige „Problem der Gitterpunkte in einem Kreise" ist als Spezialfall in dem Problem enthalten, die Anzahl der Gitterpunkte in dem k-dimensionalen ($k \geqq 2$) Ellipsoid

$$F(u_1, u_2, \ldots u_k) = \sum_{\mu,\nu=1}^{k} a_{\mu\nu}u_\mu u_\nu \leqq x \qquad (a_{\mu\nu} = a_{\nu\mu})$$

abzuschätzen, wenn F eine positiv-definite quadratische Form ist. Diese Anzahl ist nach *Landau*[311]) gleich

---

309) Einen Beweis dieser Formel mit der *Pfeiffer*schen Methode gab *E. Landau*, Über die Gitterpunkte in einem Kreise, dritte Mtlg., Gött. Nachr. 1920, p. 109—134. Vgl. auch *G. Voronoï*, Sur le développement, à l'aide des fonctions cylindriques, des sommes doubles $\sum f(pm^2 + 2qmn + rn^2)$ où $pm^2 + 2qmn + rn^2$ est une forme positive à coefficients entiers, Verhandl. des dritten intern. Math.-Kongr. Heidelberg 1904, p. 241—245.

310) Hieraus folgt insbesondere $r(n) \leqq 4d(n)$ und somit nach der vorigen Nummer eine obere Abschätzung für $r(n)$.

311) *E. Landau*, a. a. O. 224), 286b), Zur analytischen Zahlentheorie der quadratischen Formen, (über die Gitterpunkte in einem mehrdimensionalen Ellipsoid) Sitzungsb. Akad. Berlin 1915, p. 458—476; Über eine Aufgabe aus der Theorie der quadratischen Formen, Sitzungsb. Akad. Wien, 124 Abt. 2a (1915), p. 445—468. Vgl. auch *J. G. van der Corput*, Over definiete kwadratische vormen, Nieuw Arch. voor Wisk. 13 (1919), p. 125—140. — Bei diesen Untersuchungen wird teils die *Pfeiffer*sche Methode benutzt, teils analytische Methoden, wobei die verallgemeinerten Zetafunktionen von *Epstein* (vgl. Nr. 21) zur Anwendung gelangen.

$$\frac{\pi^{\frac{k}{2}}}{\sqrt{\Delta}\,\Gamma\left(\frac{k}{2}+1\right)}x^{\frac{k}{2}}+O\left(x^{\frac{k(k-1)}{2(k+1)}}\right),$$

wo Δ die Determinante $\begin{vmatrix} a_{11} & \dots & a_{1k} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ a_{k1} & \dots & a_{kk} \end{vmatrix}$

bezeichnet. Insbesondere ergibt sich für die dreidimensionale Kugel $u^2+v^2+w^2 \leqq x$ als Anzahl der Gitterpunkte

$$\tfrac{4}{3}\pi x^{\frac{3}{2}}+O\left(x^{\frac{3}{4}}\right),$$

was schon von *Cauer*[312]) gefunden war. *Landau*[311]) verallgemeinert seine Sätze nach verschiedenen Richtungen und gibt auch[313]) Verallgemeinerungen der Eigenschaft 2. von $R(x)$. Die *Hardy*sche Formel (103) wird von *Walfisz*[297]) für ein k-dimensionales Ellipsoid (mit ganzen $a_{\mu\nu}$) verallgemeinert[314]); für $k > 2$ kann hierbei nur Summabilität, nicht Konvergenz der auftretenden Reihen bewiesen werden.

Die „expliziten Formeln", die in dieser und der vorhergehenden Nummer erwähnt sind, besitzen alle Eigenschaften, die denjenigen der *Riemann*schen Primzahlformel (vgl. Nr. 28) entsprechen. Da die auftretenden Reihen unstetige Funktionen darstellen, können sie jedenfalls nicht für alle x gleichmäßig konvergieren (bzw. summierbar sein); in jedem Intervall, das von Unstetigkeitspunkten frei ist, sind sie zwar gleichmäßig konvergent (bzw. summierbar), in keinem Falle jedoch unbedingt konvergent. In einigen Fällen ist es gelungen, derartige Formeln mit der „*Pfeiffer*schen Methode" zu beweisen[315]); im allgemeinen war es jedoch notwendig, die komplexe Funktionentheorie zu benutzen. Durch formale gliedweise Integration[316]) erhält man zunächst Formeln, die unbedingt konvergente Reihen enthalten und deshalb leicht bewiesen werden können. Es gilt z. B.[317])

$$\int_0^x \bar{R}(t)\,dt = \frac{1}{2}\pi x^2 + \frac{x}{\pi}\sum_1^\infty \frac{r(n)}{n} J_2(2\pi\sqrt{nx}); \tag{104}$$

---

312) *D. Cauer,* Neue Anwendungen der Pfeifferschen Methode zur Abschätzung zahlentheoretischer Funktionen, Diss. Göttingen 1914.

313) *E. Landau,* a. a. O. 289) und 308).

314) *Hardy* hatte schon früher die Formel für eine Ellipse aufgestellt (a. a. O. 307)); vgl. auch *G. Voronoi* a. a. O. 309).

315) Vgl. 293) und 309).

316) Die in jedem Falle hinreichend oft auszuführen ist.

317) *E. Landau,* Über die Gitterpunkte in einem Kreise, Math. Ztschr. 5 (1919), p. 319—320.

die Zulässigkeit der gliedweisen Differentiation, die auf (103) führt, folgt nun aus dem Konvergenzsatz von *M. Riesz* (vgl. Nr. 5), der hier auf die *Dirichlet*sche Reihe $\sum_{1}^{\infty} \frac{r(n)}{n^{\frac{3}{4}}} e^{-s\sqrt{n}}$ anzuwenden ist. Die zahlentheoretischen Funktionen erscheinen hierbei gewissermaßen als Randwerte von analytischen Funktionen. *Steffensen*[171]) entwickelt eine ganz verschiedene Auffassungsweise; wenn eine zahlentheoretische Funktion $f(n)$ gegeben ist, interpoliert er nämlich die Folge $f(1)$, $f(2), \ldots$ durch eine ganze Funktion $f(z)$. Es sei z. B. $\varphi(s) = \sum f(n) n^{-s}$ für $\sigma \geqq 2$ unbedingt konvergent; dann definiert

$$f(z) = -\frac{\sin 2\pi z}{2\pi} \sum_{1}^{\infty} \varphi(n+1) z^n$$

für $|z| < 1$ eine ganze Funktion der gewünschten Art. *Steffensen* gibt verschiedene in der ganzen Ebene geltenden Darstellungen der Interpolationsfunktionen und wendet sie zur asymptotischen Untersuchung der zahlentheoretischen Funktionen an (vgl. Nr. 26).

Aus (104) ergibt sich leicht ein Beweis von (102), indem man $\int_x^{x+h} \bar{R}(t)\,dt$ bildet und dabei $h = x^{\frac{1}{3}}$ nimmt.[317]) Diese Differenzenbildung stellt einen sehr allgemein verwendbaren Kunstgriff dar.

36. Allgemeinere Gitterpunktprobleme. In den beiden vorhergehenden Nummern wurden verschiedene Spezialfälle der folgenden Aufgabe behandelt: Ein Gebiet G in der (u, v)-Ebene ist gegeben; man soll die Anzahl der in G oder auf der Begrenzung liegenden Gitterpunkte bestimmen. In allen jenen Spezialfällen konnte eine Annäherung an die gesuchte Anzahl sowie eine grobe Abschätzung des Fehlers durch triviale Mittel erhalten werden, und diese Abschätzung konnte durch neuere Methoden verschärft werden; die Aufgabe, die beste mögliche Abschätzung zu finden, war aber noch nicht gelöst. — Es gelingt nun, entsprechende Resultate auch bei viel allgemeineren Gebieten zu erhalten, und zwar gibt es hierfür mehrere verschiedene Methoden.

Die erste Methode, die auf solche allgemeinere Gebiete angewandt wurde, war die sog. *Pfeiffer*sche, die von *Landau* (vgl. Nr. 34) streng gemacht wurde. Wenn kein Gitterpunkt auf dem Rande von G liegt, und wenn außerdem gewisse Voraussetzungen über die Beschaffenheit des Randes gemacht werden, so kann die gesuchte Gitterpunktanzahl, wie *Landau* zeigt, durch

$$\lim_{m\to\infty}\iint_G \varphi_m(u)\,\varphi_m(v)\,du\,dv$$

ausgedrückt werden, wo

$$\varphi_m(u)=\sum_{-m}^{m}\cos 2n\pi u$$

gesetzt ist. *Landau*[318]), *Cauer*[319]) und *Hammerstein*[320]) benutzten diesen Ansatz, um bei verschiedenen speziellen Gebieten, von denen die wichtigsten in den vorhergehenden Nummern erwähnt wurden, die Gitterpunktanzahl abzuschätzen. *Van der Corput*[321]) faßt alle diese Ergebnisse in einem allgemeinen Satz zusammen, bei dem über den Rand von G nur sehr allgemeine Voraussetzungen gemacht werden. Er beweist diesen Satz auch mit der geometrischen *Voronoï*schen Methode (vgl. Nr. 34). *Landau* und *van der Corput*[322]) geben verschiedene analoge Sätze und Vereinfachungen der Beweise, wobei u. a. die arithmetische „*Piltz*sche Methode"[287]) zum Beweis von allgemeinen Gitterpunktsabschätzungen benutzt wird.

37. Verteilung von Zahlen, deren Primfaktoren vorgeschriebenen Bedingungen genügen. Es sei

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_\nu^{\alpha_\nu} \tag{105}$$

die Darstellung von n als Produkt von Primzahlpotenzen. Die α sollen stets positiv und die p alle verschieden sein; p_μ soll *nicht* notwendig die μ^{te} Primzahl bezeichnen. Es liegt nahe, nach der Verteilung derjenigen Zahlen n zu fragen, deren Exponenten $\alpha_1 \dots \alpha_\nu$ gegebenen Bedingungen genügen. Soll z. B. stets $\nu = 1$, $\alpha_1 = 1$ sein, so deckt sich diese Aufgabe offenbar mit derjenigen, die Verteilung der Primzahlen zu untersuchen. Als Verallgemeinerung hiervon kann das Problem aufgefaßt werden, die Verteilung der h Primfaktoren enthaltenden Zahlen zu bestimmen. Dies kann wiederum auf drei verschiedene Weisen aufgefaßt werden, die zu den folgenden Bedingungen führen:

---

318) *E. Landau*, a. a. O. 284), 293), 305), 308), 309).

319) *D. Cauer*, a. a. O. 312) und Über die Pfeiffersche Methode, Math. Abhandl., H. A. Schwarz zu seinem fünfzigjähr. Doktorjubiläum gewidmet, Berlin 1914, p 432—447.

320) *A. Hammerstein*, a. a. O. 200).

321) *J. G. van der Corput*, Over roosterpunten in het platte vlak (De beteekenis van de methoden van Voronoï en Pfeiffer), Diss. Leiden 1919; Über Gitterpunkte in der Ebene, Math Ann. 81 (1920), p. 1—20.

322) *E. Landau* und *J. G. van der Corput*, Über Gitterpunkte in ebenen Bereichen, Gött. Nachr. 1920, p. 135—171; *J G. van der Corput*, Zahlentheoretische Abschätzungen nach der Piltzschen Methode, Math. Ztschr. 10 (1921), p. 105—120; Zahlentheoretische Abschätzungen, Math. Ann. 84 (1921), p. 53—79.

$$1)\quad \nu = h,\ \alpha_1 = \alpha_2 = \cdots = \alpha_h = 1,$$
$$2)\quad \nu = h,$$
$$3)\quad \alpha_1 + \alpha_2 + \cdots \alpha_\nu = h.$$

Landau[323]) zeigt, daß die Anzahl der diesen Bedingungen genügenden Zahlen unterhalb x in jedem der drei Fälle asymptotisch gleich

$$\frac{1}{(h-1)!}\cdot\frac{x(\log\log x)^{h-1}}{\log x}$$

ist; für den Fall 1. war dies schon von *Gauß*[324]) vermutet worden. *Landau* gibt auch genauere Ausdrücke für jene Anzahlen. *Van der Corput*[325]) untersucht verschiedene allgemeinere Probleme dieser Art.

Läßt man ν unbestimmt, schreibt aber $\alpha_1 = \alpha_2 = \cdots = \alpha_\nu = 1$ vor, so bekommt man die sog. quadratfreien Zahlen. Bedeutet $Q(x)$ die Anzahl der quadratfreien Zahlen $\leqq x$, so beweist man leicht[326])

$$Q(x) \sim \frac{6}{\pi^2}x.$$

Für $\sigma > 1$ gilt offenbar (wenn auch 1 als quadratfreie Zahl mitgerechnet wird)

$$\sum_1^\infty \frac{Q(n)-Q(n-1)}{n^s} = \prod_p \left(1+\frac{1}{p^s}\right) = \frac{\zeta(s)}{\zeta(2s)} = \sum_1^\infty \frac{1}{n^s}\cdot\sum_1^\infty \frac{\mu(n)}{n^{2s}},$$

und die aus der Primzahltheorie geläufigen Methoden geben hier[327])

$$Q(x) = \frac{6}{\pi^2}x + O\left(\sqrt{x}\,e^{-\alpha\sqrt{\log x}}\right)$$

mit konstantem α. Wenn unter den $Q(x)$ quadratfreien Zahlen $\leqq x$ $Q_1(x)$ aus einer ungeraden, $Q_2(x)$ aus einer geraden Anzahl von Primfaktoren besteht, so folgt aus (86)

$$\frac{Q_1(x)}{Q_2(x)} = 1 + O\left(e^{-\alpha\sqrt{\log x}}\right).$$

Hardy und *Ramanujan*[328]) lassen in (105) p_μ die μ^{te} Primzahl be-

323) *E. Landau*, Über die Verteilung der Zahlen, welche aus ν Primfaktoren zusammengesetzt sind, Gött. Nachr. 1911, p. 362—381; vgl. auch a. a. O. 238).

324) Vgl. *F. Klein*, Bericht über den Stand der Herausgabe von Gauß' Werken, neunter Bericht, Gött. Nachr. 1911, Geschäftl. Mitt., p. 26—32.

325) *J. G. van der Corput*, On an arithmetical function connected with the decomposition of the positive integers into prime factors, Proceed. Akad. Amsterdam 19 (1916), p. 826—855

326) *L. Gegenbauer*, Asymptotische Gesetze der Zahlentheorie, Denkschriften Akad. Wien, 49 : 1 (1885), p. 37—80. Es werden hier auch analoge Beziehungen für „h^{te} potenzfreie Zahlen" bewiesen

327) *A. Axer*, a. a. O. 276).

328) *G. H. Hardy* und *S. Ramanujan*, Asymptotic formulae for the distribution of integers of various types, Proc. London math. Soc. (2) 16 (1917),

zeichnen und führen die Bedingung $\alpha_1 \geqq \alpha_2 \geqq \alpha_3 \geqq \cdots$ ein. Für Anzahl $A(x)$ der $n \leqq x$, die dieser Bedingung genügen, wird

$$\log A(x) \sim 2\pi \sqrt{\frac{\log x}{3 \log \log x}}$$

bewiesen.

Werden λ verschiedene zu k teilerfremde Restklassen mod. k gegeben, und wird vorgeschrieben, daß in (105) jedes p_μ einer von diesen Restklassen angehören muß, so ist nach *Landau*[329]) die Anzahl der $n \leqq x$ asymptotisch gleich

$$a x (\log x)^{\frac{\lambda}{\varphi(k)} - 1}, \quad (a > 0).$$

Die Summe $\sum 2^{\nu}$, über dieselben $n \leqq x$ erstreckt, ist dagegen asymptotisch gleich

$$b x (\log x)^{\frac{2\lambda}{\varphi(k)} - 1}, \quad (b > 0),$$

was schon *Lehmer*[330]) in einem speziellen Fall bewiesen hatte. Ein ähnlicher Satz von *Landau*[329]) enthält insbesondere das Resultat[331]): es gibt unterhalb x asymptotisch

$$c \frac{x}{\sqrt{\log x}}, \quad (c > 0),$$

ganze Zahlen, die als Summen von zwei Quadraten darstellbar sind Hieraus folgt, wenn $B_\mu(x)$ die Anzahl der ganzen Zahlen $\leqq x$ bezeichnet, zu deren additiven Darstellung *genau* μ Quadrate erforderlich sind (bekanntlich ist $B_\mu(x) = 0$ für $\mu > 4$):

$$B_1(x) \sim \sqrt{x}, \quad B_2(x) \sim c \frac{x}{\sqrt{\log x}}, \quad B_3(x) \sim \frac{5}{6} x, \quad B_4(x) \sim \frac{1}{6} x.$$

38. Neuere Methoden der additiven Zahlentheorie. Als der Abschnitt über additive Zahlentheorie in I C 3 geschrieben wurde, war vor allem das große „*Waring*sche Problem" noch ungelöst. *Waring*[332]) vermutete 1782, daß jede ganze Zahl $n \geqq 0$ als Summe

p. 112—132. In der Abhandlung: The normal number of prime factors of n, Quart. J. 48 (1917), p. 76—92, beschäftigen sich die beiden Verfasser mit Problemen, die zu den in dieser Nummer behandelten Fragestellungen in einer gewissen Beziehung stehen.

329) *E. Landau*, a. a. O. 78); Bemerkungen zu Herrn D. N. Lehmers Abhandlung in Bd 22 dieses Journals, Amer. J. of math. 26 (1904), p. 209—222; Lösung des Lehmerschen Problems, ebenda 31 (1909), p. 86—102.

330) *D. N. Lehmer*, Asymptotic Evaluation of certain Totient Sums, Amer. J. of math. 22 (1900), p. 293—335.

331) *E. Landau*, Über die Einteilung der positiven ganzen Zahlen in vier Klassen nach der Mindestzahl der zu ihrer additiven Zusammensetzung erforderlichen Quadrate, Arch. Math. Phys. (3) 13 (1908), p. 305—312.

332) *E. Waring*, Meditationes Algebraicae, 3. Aufl. Cambridge 1782, p. 349—350

einer festen (d. h. nur von k, nicht von n, abhängenden) Anzahl von positiven k^{ten} Potenzen dargestellt werden konnte, und zwar für $k = 1, 2, \ldots$. Bis 1909 war dies nur für einige spezielle Werte von k bewiesen; es gelang aber *Hilbert*[333]) einen allgemeinen Beweis zu finden. Dieser Beweis benutzt zwar die Hilfsmittel der Integralrechnung; durch spätere Vereinfachungen[334]) ist aber gezeigt worden, daß dies gänzlich vermieden werden kann, so daß die Methode im Grunde eine rein algebraische ist und deshalb hier nicht eingehender besprochen werden soll.

Rein analytisch ist dagegen die Methode, welche neuerdings von *Hardy* und *Littlewood*[335]) auf das Problem angewandt worden ist. Es sei $k > 2$, und es werde für $|x| < 1$

---

333) *D. Hilbert*, Beweis für die Darstellbarkeit der ganzen Zahlen durch eine feste Anzahl n^{ter} Potenzen (Waringsches Problem), Gött. Nachr. 1909, p. 17—36 und Math. Ann. 67 (1909), p. 281—300.

334) Vgl. z. B. *F. Hausdorff*, Zur Hilbertschen Lösung des Waringschen Problems, Math. Ann. 67 (1909), p. 301—305; *E. Stridsberg*, Sur la démonstration de M. Hilbert du théorème de Waring, Math. Ann. 72 (1912), p. 145—152; Några elementära undersökningar rörande fakulteter och deras aritmetiska egenskaper, Arkiv för Mat., Astr. och Fys. 11 (1917), No. 25; *R. Remak*, Bemerkung zu Herrn Stridsbergs Beweis des Waringschen Theorems, Math. Ann. 72 (1912), p. 153—156. Für die ältere Literatur zum *Waring*schen Problem vgl. die Göttinger Dissertationen von *A. J. Kempner*, Über das Waringsche Problem und einige Verallgemeinerungen, 1912, und *W. S. Baer*, Beiträge zum Waringschen Problem, 1913.

335) *G. H. Hardy* und *J. E. Littlewood*, A new solution of Waring's problem, Quart. J. 48 (1919), p. 272—293; Some problems of Partitio numerorum, I: A new solution of Waring's problem, Gött. Nachr. 1920, p. 33—54, II: Proof that any large number is the sum of at most 21 biquadrates, Math. Ztschr. 9 (1921), p. 14—27, (III: a. a. O. 250)), IV: The singular series in Waring's problem and the value of the number $G(k)$, Math. Ztschr. 12 (1922), p. 161—188; *G. H. Hardy*, Some famous problems of the Theory of Numbers, and in particular Waring's problem, Inaugural lecture, Oxford 1920. Vgl. auch *E. Landau*, a) Zur Hardy-Littlewoodschen Lösung des Waringschen Problems, Gött. Nachr. 1921, p. 88—92; b) Zum Waringschen Problem, Math. Ztschr 12 (1922), p. 219—247; c) Über die Hardy-Littlewoodschen Arbeiten zur additiven Zahlentheorie, Jahresb. d. deutschen Math.-Ver. 30 (1921), p 179—185; *H Weyl*, Bemerkungen über die Hardy-Littlewoodschen Untersuchungen zum Waringschen Problem, Gött. Nachr. 1921, p. 189—192; *A Ostrowski*, Bemerkungen zur Hardy-Littlewoodschen Lösung des Waringschen Problems, Math. Ztschr. 9 (1921), p. 28—34. *E. Landau* (b.) berücksichtigt auch gewisse Verallgemeinerungen, die zuerst von *Kamke* mit der *Hilbert*schen Methode behandelt wurden: Verallgemeinerungen des Waring-Hilbertschen Satzes, Math. Ann. 83 (1921), p. 85—112. — Die im Texte gewählte Bezeichnungsweise weicht etwas von der *Hardy-Littlewood*schen ab und schließt sich an *Landau* (b.) an

$$f(x) = \sum_{n=0}^{\infty} x^{n^k}, \quad f^s(x) = \sum_{n=0}^{\infty} r(n) x^n$$

gesetzt, wo also $r(n)$ von s und k abhängt. Um die *Waring*sche Vermutung, $r(n) > 0$ für $s > s_0 = s_0(k)$, zu beweisen, setzen *Hardy* und *Littlewood*

$$r(n) = \frac{1}{2\pi i} \int\limits_{|x| = 1 - \frac{1}{n}} \frac{f^s(x)}{x^{n+1}}\, dx.$$

Bei dem Versuch, aus dieser Integraldarstellung ein asymptotisches Ergebnis über $r(n)$ zu gewinnen, stößt man auf ungeheure Schwierigkeiten, da die Funktion unter dem Integralzeichen nicht über den Einheitskreis fortgesetzt werden kann. *Hardy* und *Littlewood* bemerken nun, daß die Einheitswurzeln $\varrho = e^{\frac{2p\pi i}{q}}$ gewissermaßen die „schwersten" Singularitäten von $f(x)$ sind; bei radialer Annäherung an den Punkt $x = \varrho$ wird $f^s(x)$ asymptotisch gleich einer Hilfsfunktion, die durch eine Potenzreihe von der einfachen Form[336]) $\sum \nu^{\alpha}\left(\frac{x}{\varrho}\right)^{\nu}$, mit einer Konstanten multipliziert, dargestellt wird. Der Hauptgedanke der Methode ist nun, $f^s(x)$ durch eine Summe solcher Hilfsfunktionen, d. h. $r(n)$ durch die entsprechende Summe der Koeffizienten von x^n, zu approximieren. Die Durchführung dieses Ansatzes gelingt natürlich nur durch ziemlich verwickelte Überlegungen, wobei die Untersuchungen von *Weyl*[114]) über Diophantische Approximationen eine wichtige Rolle spielen. Das folgende Hauptresultat wird erhalten: Für alle $s \geqq s_0(k)$ ist bei unendlich wachsendem n

$$(106) \qquad r(n) \sim \frac{\left(\Gamma\left(1 + \frac{1}{k}\right)\right)^s}{\Gamma\left(\frac{s}{k}\right)}\, n^{\frac{s}{k} - 1} S,$$

wo S die sog. „singuläre Reihe"

$$S = \sum_{q=1}^{\infty} \sum_{\substack{p=0 \\ (p,q)=1}}^{q-1} \left(\frac{S_{pq}}{q}\right)^s e^{-\frac{2\pi i n p}{q}},$$

mit

$$S_{pq} = \sum_{h=1}^{q} e^{\frac{2\pi i h^k p}{q}}$$

bezeichnet. Die Reihe S ist für $s \geqq s_0(k)$ konvergent und $> \sigma$, wo $\sigma = \sigma(k, s)$ nicht von n abhängt. Für s_0 ist insbesondere die Zahl $s_0 = (k - 2)2^{k-1} + 5$ wählbar.

336) *Landau*, a. a. O. 335 b), zeigt, daß man sogar eine noch einfachere, durch eine Binomialreihe dargestellte Hilfsfunktion benutzen kann.

Die *Hardy-Littlewood*sche Methode ergibt also wesentlich mehr als die *Hilbert*sche, welche nur einen Existenzsatz lieferte. Insbesondere folgt, daß es zu jedem k eine *kleinste* Zahl $G(k)$ gibt, so daß alle *hinreichend großen* n als Summen von höchstens je $G(k)$ k^{ten} Potenzen darstellbar sind, und daß $G(k) \leqq (k-2)2^{k-1}+5$ ist. Jede hinreichend große Zahl ist also die Summe von höchstens 21 Biquadraten; für den Fall $k=3$ liefert der Satz aber nur $G(k) \leqq 9$, während schon früher durch *Landau*[337]) bekannt war, daß jede hinreichend große Zahl die Summe von höchstens 8 Kuben ist. Andererseits ist bekannt[338]), daß immer $G(k) \geqq k+1$, und im Falle $k=2^m$ sogar $G(k) \geqq 4k$ ist.

Im Falle $k=2$ wird die erzeugende Funktion $f(x)$ durch eine Thetareihe ausgedrückt:

$$f(x) = \tfrac{1}{2}\left(\sum_{-\infty}^{\infty} x^{n^2} - 1\right),$$

und die Transformationstheorie der Thetafunktionen gestattet nun, viel genauere Resultate als im allgemeinen Falle zu erhalten.[339]) Die asymptotische Gleichung (106) bleibt auch hier für $s \geqq 4$ richtig; es kann sogar für $3 \leqq s \leqq 8$ in (106) das Zeichen $\sim$ durch $=$ ersetzt werden. Im Falle $s=3$ ist $S=0$ für unendlich viele n (nämlich für $n=4^a(8b+7)$). Durch Umformung der so erhaltenen Ausdrücke erhält man neue Beweise der klassischen Formeln für die Anzahl der Darstellungen einer Zahl als Summe von Quadraten. Besonders wichtig für diese Untersuchungen waren einige neuere Arbeiten von *Mordell*[340]), der die Darstellung von Zahlen durch Quadratsummen mit Hilfe der Theorie der Modulfunktionen systematisch untersuchte.

337) *E. Landau,* Über eine Anwendung der Primzahltheorie auf das Waringsche Problem in der elementaren Zahlentheorie, Math. Ann. 66 (1909), p. 102—105. Bei dem Beweis wird ein Satz über Primzahlen in arithmetischen Reihen benutzt. Nach *Wieferich* ist *jede* Zahl die Summe von höchstens 9 Kuben; es gibt auch tatsächlich Zahlen (23, 239), die 9 Kuben erfordern: Math. Ann. 66 (1909), p. 95—101.

338) Außerdem kennt man z. B $G(6) \geqq 9$. Eine Zusammenstellung der bekannten Resultate geben *Hardy* und *Littlewood*, Partitio numerorum IV (a. a. O. 335). Auf die Funktion $g(k)$, die man erhält, wenn man in der Definition von $G(k)$ die Wörter „hinreichend große" ausläßt, gehen wir hier nicht ein; es sei nur bemerkt, daß aus der Existenz von $G(k)$ unmittelbar die Existenz von $g(k)$ folgt.

339) *G. H. Hardy,* On the representation of a number as the sum of any number of squares, and in particular of five, Trans. Amer. math. Soc. 21 (1920), p. 255—284. Vgl. hierzu *S. Ramanujan,* On certain trigonometrical sums and their applications in the theory of numbers, Trans. Cambridge Phil. Soc. 22 (1919), p. 259—276.

340) *L. J. Mordell,* On the representations of numbers as a sum of $2r$

Über die Anwendung der Methode auf den „*Goldbach*schen Satz“ und verwandte Primzahlprobleme wurde schon in Nr. **31** berichtet. — Zum ersten Male wurde die Methode nicht auf *Warings* Satz angewendet, sondern auf das Problem der Abschätzung der Funktion $p(n)$, welche die Anzahl der „unbeschränkten Partitionen“ von n, d. h. die Anzahl der positiven ganzzahligen Lösungen von

$$n = x + 2y + 3z + 4u + \cdots,$$

angibt. Als erzeugende Funktion tritt hier

$$f(x) = 1 + \sum_1^\infty p(n)x^n = \prod_1^\infty \frac{1}{1-x^n}$$

auf. *Hardy* und *Ramanujan*[341]) beweisen über $p(n)$ sehr genaue asymptotische Sätze, aus denen insbesondere

$$(107) \qquad p(n) \sim \frac{1}{4n\sqrt{3}} e^{\frac{1}{3}\pi\sqrt{6n}}$$

folgt. Der Hauptgedanke ist wieder derselbe: die nicht fortsetzbare Funktion $f(x)$ wird durch eine Summe fortsetzbarer Funktionen approximiert, die in je einer Einheitswurzel singulär sind. Die rechte Seite in (107) rührt übrigens von der „schwersten“ Singularität $x = 1$ her.

39. Diophantische Approximationen. Durch die in den Nummern **7** und **17** besprochenen Anwendungen der Theorie der Diophantischen Approximationen wurde ein lebhaftes Interesse für diese Theorie erweckt. Jene Anwendungen gingen von dem grundlegenden *Kronecker*schen[342]) Satze aus, der in moderner Ausdrucksweise so lautet: Es seien $1, \alpha_1, \alpha_2, \ldots \alpha_k$ $(k \geqq 1)$ *linear unabhängige* Zahlen, und es sei

$$(x) = x - [x]$$

gesetzt; dann liegen die Punkte mit den Koordinaten

$$(108) \qquad x_1 = (n\alpha_1),\ x_2 = (n\alpha_2), \ldots x_k = (n\alpha_k), \quad (n = 1, 2, \ldots)$$

im k-dimensionalen Einheitswürfel überall dicht. *Weyl*[114]) gibt eine

squares, Quart. J. 48 (1917), p. 93—104; On the representations of numbers as the sum of an odd number of squares, Trans. Cambridge Phil. Soc. **22** (1919), p. 361—372.

341) *G. H. Hardy* und *S. Ramanujan*, Une formule asymptotique pour le nombre des partitions de n, Paris C. R. 164 (1917), p. 35—38; Asymptotic formulae in combinatory analysis, Proc. London math. Soc. (2) 17 (1918), p. 75—115.

342) *L. Kronecker*, Die Periodensysteme von Funktionen reeller Variabeln, Sitzungsb. Akad. Berlin 1884, p. 31—46, Werke 3 : 1, p. 1071—1080; Näherungsweise ganzzahlige Auflösung linearer Gleichungen, Sitzungsb. Akad. Berlin 1884, p. 1179—1193, Werke 3 : 1, p. 47—110.

für die genannten Anwendungen wesentliche Vertiefung dieses Satzes, indem er zeigt, daß jene Punkte sogar in jedem Teile des Einheitswürfels *asymptotisch gleich dicht* liegen. Die Anzahl derjenigen unter den N ersten Punkten (108), die einem Teilgebiet vom Inhalt δ angehören, ist also asymptotisch gleich δN. *Weyl* beweist dies durch systematische Benutzung der „analytischen Invariante der Zahlklassen mod. 1“, der Funktion $e^{2\pi i t}$.

Hardy-Littlewood[343]) und *Weyl*[114]) geben auch wichtige Verallgemeinerungen auf den Fall, wo in (108) n durch n^q oder durch ein Polynom ersetzt wird; die hierbei von *Weyl* eingeführten, eleganten Methoden zur Transformation und Abschätzung von Summen mit dem allgemeinen Gliede $e^{2\pi i p(n)}$ (p ein Polynom) waren für die in der vorhergehenden Nummer besprochenen Untersuchungen über *Warings* Problem von grundlegender Bedeutung und haben auch zu neuen Resultaten über die Größenordnung von $\zeta(s)$ auf vertikalen Geraden geführt (vgl. Nr. 18). *Hardy* und *Littlewood* haben insbesondere Summen der Gestalt

$$\sum_1^n e^{\left(\nu-\frac{1}{2}\right)^2 \alpha\pi i}, \quad \sum_1^n e^{\nu^2\alpha\pi i}, \quad \sum_1^n (-1)^{\nu-1} e^{\nu^2\alpha\pi i}$$

untersucht, die mit dem Verhalten der Thetareihen bei Annäherung an die Konvergenzgrenze zusammenhängen. Wenn α irrational ist, sind alle drei Summen von der Form $o(n)$. Auch über die Verteilung der Zahlen $(\lambda_n\alpha)$, wo $\lambda_1, \lambda_2, \ldots$ eine unbegrenzt und monoton wachsende Zahlfolge ist, gibt es Sätze, die den vorhergehenden entsprechen.[344]) Für $\lambda_n = a^n$ hängen diese Sätze mit der Verteilung der Ziffern in (verallgemeinerten) Dezimalbrüchen zusammen.[345])

Für die Summe $\sum(\nu\alpha)$ gilt bei irrationalem α immer

$$\sum_1^n (\nu\alpha) = \tfrac{1}{2}n + o(n).$$

Wird in dieser Formel das Restglied durch ein „besseres“ ersetzt, so

343) *G. H. Hardy* und *J. E Littlewood,* Some problems of diophantine approximation, Intern. Congr. of math. Cambridge 1912, p. 223—229; Acta Math. 37 (1914), p. 155—238. Vgl. auch *J. G. van der Corput,* Über Summen, die mit den elliptischen Θ-Funktionen zusammenhängen, Math. Ann. 87 (1922), p. 66—77

344) Vgl. hierzu auch *R. H. Fowler,* On the distribution of the set of points $(\lambda_n\Theta)$, Proc. London math. Soc. (2) 14 (1914), p. 189—206.

345) *E. Borel,* Les probabilités dénombrables et leurs applications arithmétiques, Palermo Rend. 27 (1909), p. 247—271 (vgl. auch Leçons sur la théorie des fonctions, deuxième éd., Paris 1914). Weitergehende Sätze geben *Hardy* und *Littlewood,* a. a. O. 343).

kann die neue Formel nicht für *alle* irrationalen α gelten; beschränkt man sich dagegen auf spezielle Klassen von Irrationalitäten, so kann die Abschätzung erheblich verschärft werden. Es gilt z. B. für ein α, bei dessen Entwicklung in einen gewöhnlichen Kettenbruch die auftretenden Nenner beschränkt sind[346])

$$\sum_1^n (\nu\alpha) = \tfrac{1}{2}n + O(\log n)$$

Ostrowski[346]) zeigt, daß bei *keinem* irrationalen α hier das O gegen o vertauscht werden kann. *Hardy* und *Littlewood*[346]) zeigen, daß das Problem der Abschätzung von $\sum(\nu\alpha)$ mit der Bestimmung der Gitterpunktanzahl in einem gewissen rechtwinkligen Dreieck nahe verbunden ist. *Hecke*[347]) hat jene Summen für quadratisch irrationale α näher untersucht. Ist insbesondere $\alpha = \sqrt{D}$, wo $4D$ eine positive Fundamentaldiskriminante ist (vgl. Nr. 40), so konvergiert die *Dirichlet*sche Reihe

$$\sum_1^\infty \frac{(n\alpha) - \frac{1}{2}}{n^s}$$

für $\sigma > 0$ und stellt eine überall meromorphe Funktion dar, die in der Halbebene $\sigma \leqq 0$ unendlich viele Pole besitzt.

Es sei
$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}$$
die gewöhnliche Kettenbruchentwicklung einer irrationalen Zahl α; hinsichtlich der Größenordnung der a_n sind u. a. folgende Sätze bekannt:[348])

1. Die Menge der α, für die von einer gewissen Stelle an $a_n > 1$ gilt, hat das Maß Null.

2. Es seien $d_1, d_2, \ldots$ und $k_1, k_2, \ldots$ monoton wachsende Folgen positiver Zahlen, und es sei $\sum \frac{1}{d_n}$ divergent, $\sum \frac{1}{k_n}$ konvergent. „Fast überall“ ist dann von einer gewissen Stelle an $a_n < k_n$, und „fast überall“ ist $a_n > d_n$ für unendlich viele n.

346) *M. Lerch*, L'intermèd. des Math. 11 (1904), p. 145; *G. H Hardy* und *J. E. Littlewood*, Some Problems of diophantine approximation: the lattice-points of a right-angled triangle, Proc. London math. Soc. (2) 20 (1921), p. 15—36; *A. Ostrowski*, Bemerkungen zur Theorie der Diophantischen Approximationen, Abh Math. Seminar Hamburg 1 (1921), p 77—98.

347) *E. Hecke*, Über analytische Funktionen und die Verteilung von Zahlen mod. eins, Abh. Math Seminar Hamburg 1 (1921), p. 54—76.

348) *E. Borel*, a. a. O. 345); *F. Bernstein*, Über eine Anwendung der Mengenlehre auf ein aus der Theorie der säkularen Störungen herrührendes Problem, Math. Ann. 71 (1911), p. 417—439.

Hiermit hängen die Fragen nach der Approximation irrationaler Zahlen durch rationale nahe zusammen.[349]) Zu jedem irrationalen α gibt es unendlich viele rationale $\frac{p}{q}$, so daß

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{5}\, q^2}$$

gilt. Wenn k_n die obige Bedeutung hat, so bilden diejenigen α, die sich durch unendlich viele $\frac{p}{q}$ mit der Genauigkeit

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q k_q}$$

approximieren lassen, eine Menge vom Maß Null. Wenn α eine algebraische Zahl vom Grade n ist, so gilt nach einem wichtigen Satze von *Siegel*[350]) für jedes rationale $\frac{p}{q}$

$$\left|\alpha - \frac{p}{q}\right| > \frac{\gamma}{q^2 \sqrt{n}},$$

wo γ nur von α abhängt.

V. Algebraische Zahlen und Formen.

40. Quadratische Formen und Körper.[351]) Die nach *Dirichlet* benannten Reihen wurden von ihm gebraucht, um die Anzahl der verschiedenen Klassen binärer quadratischer Formen einer gegebenen Diskriminante zu berechnen;[352]) die Lösung dieser Aufgabe setzte ihn in den Stand, seinen Satz über die Primzahlen einer arithmetischen Reihe zu beweisen (vgl. Nr. 30). Über die Berechnung jener Klassenanzahl und ihre Beziehung zu den *Gauß*schen Summen ist in I C 3, Nr. 2, über die analogen Probleme bei Formen mit mehr als zwei Veränderlichen in I C 2, d und e, berichtet worden; es seien hier nur einige Ergänzungen für die binären Formen gegeben.

349) Vgl. *A. Hurwitz*, Über die angenäherte Darstellung der Irrationalzahlen durch rationale Brüche, Math. Ann. 39 (1891), p. 279—284; *E. Borel*, Contribution à l'analyse arithmétique du continu, J. math. pures appl. (5) 9 (1903), p. 329—375; *O. Perron*, Irrationalzahlen, Berlin und Leipzig (Ver. wiss. Verleger) 1921.

350) *C. Siegel*, Approximation algebraischer Zahlen, Math. Ztschr. 10 (1921), p. 173—213.

351) Was sich unmittelbar durch Spezialisierung ($n = 2$) aus Formeln der beiden folgenden Nummern ergibt, wird hier nicht erwähnt.

352) *G. Lejeune-Dirichlet*, Recherches sur diverses applications de l'analyse infinitésimale à la théorie des nombres, Crelles J. 19 (1839), p. 324—369 und 21 (1840), p. 1—12, 134—155, Werke 1, p. 411—496.

Die quadratische Form werde in *Kronecker*scher Bezeichnungsweise[353])

$$f(xy) = ax^2 + bxy + cy^2 = (a, b, c)$$

geschrieben, ihre Diskriminante sei

$$b^2 - 4ac = D = Q^2 D_0,$$

wo D_0 eine sog. *Fundamentaldiskriminante*[354]) ist. Es sei

$$(a_1, b_1, c_1), \ldots (a_h, b_h, c_h)$$

ein Repräsentantensystem der *primitiven* — und im Falle $D < 0$ *positiven* — zu D gehörigen Klassen; die Koeffizienten a können dabei immer positiv und die b negativ vorausgesetzt werden. Dann ist für $\sigma > 1$

$$(109) \qquad \sum_{\nu=1}^{h} \sum_{x, y} (a_\nu x^2 + b_\nu xy + c_\nu y^2)^{-s} = \tau \sum_{n=1}^{\infty} \left(\frac{D}{n}\right) \frac{1}{n^s} \cdot \sum_{n}{}' \frac{1}{n^s},$$

wo links x und y alle die zugehörige quadratische Form zu Q teilerfremd machenden Paare ganzer Zahlen exkl. (0, 0) durchlaufen; im Falle $D > 0$ tritt jedoch die Beschränkung

$$0 \leqq y < \frac{2 a_\nu U}{T - b_\nu U} x$$

hinzu, wo (T, U) die „Fundamentallösung" der Gleichung $t^2 - Du^2 = 4$ bezeichnet (vgl. I C 2 c, 2). Rechts durchläuft n in $\sum'$ alle zu Q teilerfremden positiven ganzen Zahlen, und es ist

$$\tau = \begin{cases} 2 & \text{für } D < -4 \\ 4 & \text{„} \ \ D = -4 \\ 6 & \text{„} \ \ D = -3 \\ 1 & \text{„} \ \ D > 0. \end{cases}$$

Das *Kronecker*sche Symbol $\left(\frac{D}{n}\right)$ ist für $n > 0$ ein reeller Charakter mod. $|D|$, und die Reihe $\sum_{1}^{\infty} \left(\frac{D}{n}\right) n^{-s}$ ist deshalb eine von den in Nr. 29 untersuchten L-Reihen. Jede der h Doppelsummen auf der linken Seite von (109) hat für $s = 1$ einen Pol erster Ordnung mit einem von ν unabhängigen Residuum, und man findet durch Ver-

353) *L. Kronecker*, Zur Theorie der elliptischen Funktionen, Sitzungsb Akad. Berlin 1885, p. 770; vgl. auch die ausführliche Darstellung von *J. de Séguier*, Formes quadratiques et multiplication complexe, Berlin 1894.

354) Wenn m eine quadratfreie Zahl bedeutet, so ist *entweder* $D_0 = m$, $m \equiv 1$ (mod. 4), *oder* $D_0 = 4m$, $m \equiv 2$ oder 3 (mod. 4). Es wird immer $D_0 \neq 1$ vorausgesetzt, so daß D keine Quadratzahl ist.

gleichung der Residuen

$$(110)\qquad h = \begin{cases} \tau\dfrac{\sqrt{-D}}{2\pi}L(1) & \text{für } D < 0 \\[2ex] \dfrac{\sqrt{D}}{\log\dfrac{T+U\sqrt{D}}{2}}L(1) & \text{für } D > 0. \end{cases}$$

Die Summierung der unendlichen Reihen gestaltet sich am einfachsten, wenn D eine Fundamentaldiskriminante und daher $Q = 1$ ist; der allgemeine Fall läßt sich hierauf zurückführen. Für diesen Fall gilt

$$(111)\qquad h = \begin{cases} \dfrac{\tau}{2D}\displaystyle\sum_{n=1}^{|D|-1}\left(\frac{D}{n}\right)n & \text{für } D < 0 \\[2ex] \dfrac{1}{\log\dfrac{T+U\sqrt{D}}{2}}\displaystyle\sum_{n=1}^{D-1}\left(\frac{D}{n}\right)\log\sin\frac{n\pi}{D} & \text{für } D > 0. \end{cases}$$

Jede Fundamentaldiskriminante D ist die Grundzahl des durch $\sqrt{D}$ erzeugten quadratischen Zahlkörpers (vgl. I C 4a), und den Klassen quadratischer Formen der Diskriminante D entsprechen umkehrbar eindeutig die Idealklassen des Körpers $k(\sqrt{D})$.[355]) Die Anzahl der Idealklassen wird also auch durch (111) gegeben; diese Anzahl hängt nach I C 4a, Nr. 9 mit dem Residuum im Punkte $s = 1$ der zum Körper gehörigen Zetafunktion $\zeta_k(s)$ zusammen. In der Tat ist $\zeta_k(s)$ gleich der rechten Seite von (109), dividiert durch τ,

$$\zeta_k(s) = \sum_1^{\infty}\left(\frac{D}{n}\right)\frac{1}{n^s}\cdot\zeta(s) = L(s)\,\zeta(s),$$

so daß $\zeta_k(s)$ mit der Theorie von $\zeta(s)$ und den L-Funktionen schon erledigt ist.

Die Ausdrücke (111) können auch nach einer weniger analytischen Methode abgeleitet werden, wobei die *Dirichlet*schen Reihen nicht auftreten.[356]) Verschiedene Transformationen von (111), sowie andere Ausdrücke für Klassenzahlen sind von *Lerch*[357]) gegeben. Er gibt

355) Hierbei sind jedoch zwei Ideale nur dann zur selben Klasse gehörig, wenn ihr Quotient eine Zahl *positiver Norm* ist. Die Anzahl der so definierten Klassen ist entweder gleich der gewöhnlichen Klassenzahl (vgl. I C 4a, Nr. 8) oder doppelt so groß.

356) Vgl. z. B. *Ch. Hermite,* Paris C. R. 55 (1862), p. 684, Oeuvres 2, p. 255.

357) *M. Lerch,* Essais sur le calcul du nombre des classes de formes quadratiques binaires aux coefficients entiers, Mém. présentés Acad. sc. Paris (2) 33 (1906), No. 2; Paris C. R. 135 (1902), p. 1314—1315; Acta Math. 29 (1905), p. 333; 30 (1906), p. 203—293; Sur le nombre des classes de formes quadratiques binaires d'un discriminant positif fondamental, J. math. pures appl. (5) 9 (1903), p. 377—401.

insbesondere Formeln, welche für numerische Berechnung geeignet sind, z. B.

$$h = 2 \operatorname{sgn} D_2 \cdot \sum_{\mu=1}^{\frac{1}{2}|D_1|} \left(\frac{D_1}{\mu}\right) \sum_{\nu=1}^{\mu \left|\frac{D_2}{D_1}\right|} \left(\frac{D_2}{\nu}\right),$$

wo D_1 und D_2 Fundamentaldiskriminanten bedeuten, $D_1 D_2 < 0$ und h die Klassenzahl für die Diskriminante $D_1 D_2$ ist.

Für beliebige Diskriminanten gilt[358])

$$h = O(\sqrt{|D|} \log |D|)$$

und für $D > 0$ sogar $\quad h = O(\sqrt{D})$.

Es gibt unendlich viele positive Diskriminanten mit gleicher Klassenzahl[359]), für negative D ist dies dagegen wahrscheinlich nicht der Fall — in der Tat gilt[360]) für negative Fundamentaldiskriminanten, wenn über die Nullstellen der obigen Funktion $\zeta_k(s)$ eine gewisse unbewiesene Annahme (die insbesondere aus der Richtigkeit der „verallgemeinerten *Riemann*schen Vermutung" für die L-Funktionen folgen würde) gemacht wird,

$$h > c \frac{\sqrt{|D|}}{\log |D|}.$$

Über die „mittlere Anzahl" der Klassen gab schon *Gauß*[361]) (ohne Beweis) einige Sätze; es gilt z. B. nach *Landau*[362]), wenn h_n die Klassenanzahl primitiver positiv-definiter Formen der Diskriminante $-n$ bedeutet,

$$\sum_1^x h_n = \frac{\pi}{18\zeta(3)} x^{\frac{3}{2}} - \frac{3}{2\pi^2} x + O\left(x^{\frac{5}{6}} \log x\right).$$

Die Klassenzahl für negative Diskriminanten, insbesondere die Lehre von den sog. Klassenzahlrelationen, steht zu der Theorie der elliptischen Funktionen und deren komplexer Multiplikation in naher Beziehung[363]), darauf kann hier jedoch nicht eingegangen werden.

358) Vgl. *G. Pólya, J. Schur* und *E. Landau*, a. a. O. 205). *E. Landau* gibt auch analoge Resultate für beliebige algebraische Zahlkörper.

359) *G. Lejeune-Dirichlet*, Über eine Eigenschaft der quadratischen Formen von positiver Determinante, Ber. Verhandl. Akad. Berlin 1855, p. 493—495; Werke 2, p. 185—187.

360) *E. Landau*, Über imaginär-quadratische Zahlkörper mit gleicher Klassenzahl; Über die Klassenzahl imaginär-quadratischer Zahlkörper, Gött. Nachr. 1918, p. 277—295. Vgl. auch *T. Nagel*, Über die Klassenzahl imaginär-quadratischer Zahlkörper, Abhandl. Math. Seminar Hamburg 1 (1922), p. 140—150.

361) *C. F. Gauss*, Disquisitiones arithmeticae, Art. 302—304.

362) *E. Landau*, a. a. O. 284). Vgl. auch *F. Mertens*, a. a. O. 270).

363) Vgl. I C 6, Nr. 12 sowie II B 3, Nr. 75. Eine Darstellung der Lehre

Dirichlet[364]) stellte den Satz auf, daß durch jede primitive — und im Falle $D < 0$ positive — binäre quadratische Form einer nichtquadratischen Diskriminante unendlich viele Primzahlen dargestellt werden können. Er gab auch Andeutungen für den Beweis, der später von *Schering*[365]) und *Weber*[366]) vollständig ausgeführt wurde. *Mertens*[367]), *de la Vallée Poussin*[368]), *Bernays*[369]) und *Landau*[370]) gaben für die Anzahl der darstellbaren Primzahlen $\leqq x$ und für gewisse damit zusammenhängende Summen asymptotische Ausdrücke, die als Spezialfälle in den Sätzen von *Landau*[370]) über Primideale in Idealklassen (vgl. Nr. 42) enthalten sind. Jene Anzahl ergibt sich gleich

$$\frac{1}{h_0} Li(x) + O\left(x e^{-\alpha\sqrt{\log x}}\right)$$

mit konstantem α, wobei $h_0 = 2h$ oder $= h$ ist, je nachdem die betreffende Form einer zweiseitigen Klasse angehört oder nicht. Bei dem Beweis wird die Lehre von der Komposition der Klassen (vgl. I C 2c, 11) benutzt. Diese liefert bekanntlich eine *Abel*sche Gruppe, und indem man die Charaktere dieser Gruppe auf der linken Seite von (109) einführt, gewinnt man neue Funktionen, die den L-Funktionen (vgl. Nr. 29) entsprechen. Der Beweis verläuft dann ähnlich wie bei den Primzahlen in einer arithmetischen Reihe. Die hierbei auftretenden Summen

$$\sum_{x,y} (ax^2 + bxy + cy^2)^{-s},$$

von den Klassenzahlrelationen gab neuerdings *L. J. Mordell,* On class relation formulae, Messenger of Math. 46 (1916), p. 113—135.

364) *G. Lejeune-Dirichlet,* Über eine Eigenschaft der quadratischen Formen, Ber. Verhandl. Akad. Berlin 1840, p. 49—52; Werke 1, p. 497—502.

365) *E. Schering,* Beweis des Dirichletschen Satzes etc., Ges. math. Werke 2, p. 357—365.

366) *H. Weber,* Beweis des Satzes etc., Math. Ann. 20 (1882), p. 301—329; Über Zahlengruppen in algebraischen Körpern, Math. Ann. 48 (1897), p. 433—473; 49 (1897), p. 83—100; 50 (1898), p. 1—26.

367) a. a. O. 270).

368) *Ch. de la Vallée Poussin,* Recherches analytiques sur la théorie des nombres premiers, Troisième partie, Ann. soc. sc. Bruxelles 20 : 2 (1896), p. 363—397; Quatrième partie, ibid. 21 : 2 (1897), p. 251—342.

369) *P. Bernays,* Über die Darstellung von positiven, ganzen Zahlen durch die primitiven, binären quadratischen Formen einer nicht-quadratischen Diskriminante, Diss. Göttingen 1912.

370) *E. Landau,* a) Über die Verteilung der Primideale in den Idealklassen eines algebraischen Zahlkörpers, Math. Ann. 63 (1907), p. 145—204; b) Über die Primzahlen in definiten quadratischen Formen und die Zetafunktion reiner kubischer Körper, Math. Abhandl., H. Schwarz . . . gewidmet, Berlin (Springer) 1914, p. 244—273.

mit verschiedenen Bedingungen für die Summationsvariablen x und y, definieren in der ganzen Ebene eindeutige Funktionen, die nur für $s = 1$ einen Pol haben.[371]) Der Beweis dieses Satzes für positive Diskriminanten, der von *de la Vallée Poussin*[368]) gefunden wurde, war sehr kompliziert und wurde später von *Landau*[372]) vereinfacht. *Dirichlet*[373]) hat auch behauptet, daß unter den durch eine quadratische Form dargestellten Primzahlen unendlich viele vorkommen, die einer gegebenen arithmetischen Reihe angehören, vorausgesetzt, daß die Form überhaupt fähig ist, Zahlen von dieser Reihe darzustellen. *Meyer*[374]) hat diesen Satz bewiesen, *de la Vallee Poussin*[375]) und *Landau*[370]) haben ihn durch Angabe asymptotischer Formeln verschärft.

Aus den neueren Untersuchungen von *Hecke*[376]) geht hervor, daß die Form $ax^2 + bxy + cy^2$, wenn D Fundamentaldiskriminante ist, auch noch dann unendlich viele Primzahlen darstellt, wenn man die ganzzahligen Veränderlichen x und y auf einen beliebigen Winkelraum einschränkt.

Bernays[369]) zeigt, daß die Anzahl der ganzen Zahlen $n \leqq x$, die durch eine quadratische Form darstellbar sind, asymptotisch gleich

$$A \frac{x}{\sqrt{\log x}}$$

mit konstantem A ist (vgl. Nr. 37, am Ende).

371) Vgl. Nr. 21. Vgl. ferner *M. Lerch*, Základové theorie Malmsténovskych řad, Rozpravy české akad., 2. Kl., 1 (1892), Nr. 27; Studie v oboru Malmsténovskych řad a invariantu forem kvadratickych, ibid. 2 (1893), Nr. 4; *G. Herglotz*, Über die analytische Fortsetzung gewisser Dirichletscher Reihen, Math. Ann. 61 (1905), p. 551—560.

372) *E. Landau*, Neuer Beweis eines analytischen Satzes des Herrn de la Vallée Poussin, Jahresb. d. deutschen Math.-Ver. 24 (1915), p. 250—278.

373) *G. Lejeune-Dirichlet*, Extrait d'une lettre etc., Paris C. R. 10 (1840), p. 285—288; J. Math. pures appl. (1) 5 (1840), p. 72—74; Sur la théorie des nombres, Werke 1, p. 619—623.

374) *A. Meyer*, Über einen Satz von Dirichlet, Crelles J. 103 (1888), p. 98 bis 117. Vgl. auch *P. Bachmann*, Die analytische Zahlentheorie, Leipzig (Teubner) 1894, insbes. Abschn. 10.

375) *Ch. de la Vallée Poussin*, Recherches analytiques sur la théorie des nombres premiers; Cinquième partie, Ann. Soc. sc. Bruxelles 21: 2 (1897), p. 343 bis 368.

376) *E. Hecke*, Eine neue Art von Zetafunktionen und ihre Beziehungen zur Verteilung der Primzahlen, Math. Ztschr. 1 (1918), p. 357—376; 6 (1920), p. 11—51.

41. Die Zetafunktionen von Dedekind und Hecke.[377] *Dedekind*[378]) hat die *Riemann*sche Zetafunktion für einen beliebigen algebraischen Zahlkörper k vom Grade n verallgemeinert. Er setzt[379]) (vgl. I C 4a, Nr. 9)

$$\zeta_k(s) = \sum_{\mathfrak{a}} \frac{1}{N\mathfrak{a}^s} = \prod_{\mathfrak{p}} \left(1 - \frac{1}{N\mathfrak{p}^s}\right)^{-1} = \sum_{1}^{\infty} \frac{F(m)}{m^s},$$

wo $\mathfrak{a}$ die Ideale, $\mathfrak{p}$ die Primideale von k durchläuft und $F(m)$ die Anzahl der Darstellungen von m als Norm eines Ideals von k bedeutet. Alle drei Ausdrücke sind für $\sigma > 1$ absolut konvergent; $\zeta_k(s)$ ist demnach in dieser Halbebene regulär und $\neq 0$. Für den Körper der rationalen Zahlen ist $\zeta_k(s)$ mit $\zeta(s)$ identisch. Mit Hilfe einer *Weber*schen[380]) Verschärfung der *Dedekind*schen[378]) Abschätzung der Anzahl aller Ideale von k mit Norm $\leqq x$ zeigt *Landau*[381]), daß $\zeta_k(s)$ auch noch für $\sigma > 1 - \frac{1}{n}$ regulär ist, mit Ausnahme des Punktes $s = 1$, wo sie einen Pol erster Ordnung mit dem schon von *Dedekind*[378]) angegebenen Residuum

$$\frac{2^{r_1 + r_2} \pi^{r_2} R h}{w\sqrt{|d|}}$$

besitzt. Hier bedeutet (vgl. I C 4a) d die Grundzahl, R den Regulator, w die Anzahl der Einheitswurzeln und h die Anzahl der Idealklassen[382]) von k. Ferner bezeichnet r_1 die Anzahl der reellen und $2r_2 = n - r_1$ die Anzahl der nicht-reellen Wurzeln der irreduziblen Gleichung, die von einer den Körper k erzeugenden Zahl befriedigt wird.

Kann nun das Residuum von $\zeta_k(s)$ anderweitig bestimmt werden, so erhält man offenbar einen Ausdruck für die Klassenzahl h. (Im Falle eines quadratischen Körpers ist dieses Verfahren im wesent-

377) Vgl. hierzu II B 7, Nr. 129, wo die Beziehungen zu der allgemeinen Theorie der Thetafunktionen behandelt werden.

378) G *Lejeune-Dirichlet,* Vorlesungen über Zahlentheorie, herausgeg. und m. Zusätzen versehen von *R. Dedekind,* 4. Aufl. 1894, p. 610.

379) Die kleinen deutschen Buchstaben bezeichnen Ideale, $N\mathfrak{a}$ ist die Norm des Ideals $\mathfrak{a}$, $N\mathfrak{a}^s$ bedeutet $(N\mathfrak{a})^s$.

380) *H. Weber,* Über einen in der Zahlentheorie angewandten Satz der Integralrechnung, Gött Nachr. 1896, p. 275—281; Lehrbuch der Algebra 2, 2. Aufl. 1899, p. 672—678, 712.

381) *E. Landau,* a. a. O. 28).

382) Wo nichts anderes gesagt wird, ist der Begriff „Idealklasse" im „weitesten Sinne" genommen, d. h. zwei Ideale sind zur gleichen Klasse gehörig, sobald ihr Quotient eine Körperzahl ist.

lichen mit dem *Dirichlet*schen — vgl. Nr. 40 — identisch.) Dies läßt sich aber nur in wenigen Fällen durchführen[383]), und zwar:

a) Für die Kreiskörper und deren Unterkörper (vgl. I C 4b[384])). Im Körper der ν^{ten} Einheitswurzeln ist

$$\zeta_k(s) = L_1(s)\, L_2(s) \ldots L_{\varphi(\nu)}(s) \cdot \prod_{p \mid \nu} (1 - p^{-fs})^{-q},$$

wo alle $\varphi(\nu)$ L-Funktionen mod. ν (vgl. Nr. 29) multipliziert werden, und f und q gewisse von ν und p abhängige positive ganze Zahlen sind. Die Formel für die Klassenzahl gibt hier also einen neuen Beweis für das Nichtverschwinden sämtlicher Reihen $L(1)$.[385])

b) Für die Klassenkörper der komplexen Multiplikation[386]).

c) Für solche Körper 4. Grades, die durch eine Zahl von der Form $\sqrt{a + b\sqrt{c}}$ mit rationalen a, b, c, sowie $c > 0$, $a \pm b\sqrt{c} < 0$ erzeugt werden. *Hecke*[387]) beweist unter Anwendung der von ihm untersuchten *Gauß*schen Summen in algebraischen Zahlkörpern einen Satz über gewisse Relativklassenzahlen, aus dem insbesondere ein Ausdruck für die Klassenzahl der genannten Körper folgt. Es zeigen sich hierbei eigentümliche Zusammenhänge mit tiefliegenden Fragen aus der Theorie der Thetafunktionen.

Die analytische Fortsetzung von $\zeta_k(s)$ über $\sigma = 1 - \frac{1}{n}$ hinaus konnte lange nur bei speziellen Körpern ausgeführt werden. Ein sehr bedeutender Fortschritt wurde nun von *Hecke*[388]) gemacht, indem es

383) *E. Landau*, Über eine Darstellung der Anzahl der Idealklassen eines algebraischen Körpers durch eine unendliche Reihe, Crelles J. 127 (1904), p. 167 bis 174 (vgl. auch a. a. O. 78)), zeigt, daß die *Dirichlet*sche Reihe für $\zeta_k(s) \cdot \frac{1}{\zeta(s)}$ im Punkte $s = 1$ konvergiert, so daß die Klassenzahl immer durch eine konvergente unendliche Reihe dargestellt werden kann.

384) Vgl. auch die neuere Darstellung von *R. Fueter*, Synthetische Zahlentheorie, Berlin u. Leipzig (Göschen) 1917.

385) *Dirichlet-Dedekind*, a. a. O. 378), p. 625.

386) Jedes Eingehen auf die Theorie der Klassenkörper mußte aus diesem Bericht ausgeschlossen werden. Vgl. hierzu I C 6.

387) *E. Hecke*, Bestimmung der Klassenzahl einer neuen Reihe von algebraischen Zahlkörpern, Gött. Nachr. 1921, p. 1—23; Reziprozitätsgesetz und Gaußsche Summen in quadratischen Zahlkörpern, ibid. 1919, p. 265—278. Vgl. auch *L. J. Mordell*, On the reciprocity formula for the Gauss's sums in the quadratic field, Proc. London math. Soc. (2) 20 (1920), p. 289—296; *K. Reidemeister*. Über die Relativklassenzahl gewisser relativquadratischer Zahlkörper, Abhandl. Math. Seminar Hamburg 1 (1922), p. 27—48.

388) *E. Hecke*, Über die Zetafunktion beliebiger algebraischer Zahlkörper, Gött. Nachr. 1917, p. 77—89; eine Anwendung der Entdeckung auf die Theorie der Klassenkörper gibt die Arbeit: Über eine neue Anwendung der Zetafunktionen auf die Arithmetik der Zahlkörper, Gött. Nachr. 1917, p. 90—95.

ihm gelang, den zweiten *Riemann*schen Beweis für $\zeta(s)$ (vgl. Nr. 14) zu verallgemeinern und dadurch nicht nur die Existenz von $\zeta_k(s)$ in der ganzen Ebene nachzuweisen, sondern auch eine der *Riemann*schen analoge Funktionalgleichung für $\zeta_k(s)$ aufzustellen und mit ihrer Hilfe die *Hadamard*schen Sätze über das Geschlecht und die Produktentwicklung der ganzen Funktion $(s-1)\zeta(s)$ (vgl. Nr. 15) zu verallgemeinern.[389]) Es läßt sich nämlich, wenn das Ideal $\mathfrak{a}$ gegeben ist, die Summe[390])

$$(112)\qquad \zeta(s, \mathfrak{a}) = \sum_{\mathfrak{a}\mathfrak{j} \sim 1} \frac{1}{N\mathfrak{j}^s},$$

die über alle Ideale $\mathfrak{j}$ der Klasse $\mathfrak{a}^{-1}$ erstreckt ist, durch eine Thetareihe

$$(113)\qquad \vartheta(\tau_1, \ldots \tau_n; \mathfrak{a}) = \sum_{\mu \equiv 0 \,(\mathrm{mod}\, \mathfrak{a})} e^{-\frac{\pi}{\sqrt[n]{N\mathfrak{a}^2 |d|}} \sum_{p=1}^{n} \tau_p |\mu^{(p)}|^2}$$

ausdrücken, wobei μ alle Zahlen von $\mathfrak{a}$ durchläuft, $\mu^{(1)}, \ldots \mu^{(n)}$ die konjugierten Zahlen (inkl. μ selbst), in bestimmter Reihenfolge wie in I C 4a, Nr. 7, geordnet, und diejenigen τ_p, welche konjugiert imaginären Körpern entsprechen, einander gleich sind. *Hecke* findet in der Tat durch sinnreiche Überlegungen

$$(114)\qquad \Phi(s, \mathfrak{a}) = A^s \left(\Gamma\left(\frac{s}{2}\right)\right)^{r_1} (\Gamma(s))^{r_2} \zeta(s, \mathfrak{a})$$

$$= \frac{2^{r_1-1} n R}{w} \int_{-\frac{1}{2}}^{\frac{1}{2}} dx_1 \ldots \int_{-\frac{1}{2}}^{\frac{1}{2}} dx_r \int_0^\infty u^{\frac{ns}{2}-1} (\vartheta(\tau_1, \ldots \tau_n; \mathfrak{a}) - 1)\, du,$$

wo

$$A = 2^{-r_2} \pi^{-\frac{n}{2}} \sqrt{|d|},$$

$$\tau_p = u\, e^{2\sum_{q=1}^{r} x_q \log |\varepsilon_q^{(p)}|}$$

gesetzt ist und $\varepsilon_1 \ldots \varepsilon_r$ ein System von Grundeinheiten des Körpers bezeichnet. (Wie üblich ist $r = r_1 + r_2 - 1$.) Die Thetareihe (113) genügt aber der Funktionalgleichung

$$\vartheta(\tau_1, \ldots \tau_n; \mathfrak{a}) = \frac{1}{\sqrt{\tau_1 \tau_2 \ldots \tau_n}}\, \vartheta\left(\frac{1}{\tau_1} \cdots \frac{1}{\tau_n};\ \mathfrak{a}^{-1}\, \mathfrak{d}^{-1}\right),$$

389) Neuerdings wurde der *erste Riemann*sche Beweis von *C. Siegel* für $\zeta_k(s)$ verallgemeinert: Neuer Beweis für die Funktionalgleichung der Dedekindschen Zetafunktion, Math. Ann. 85 (1922), p. 123—128.

390) Die Ausdrücke (112), (113) und (114) sind nicht von dem Ideal $\mathfrak{a}$ selbst, sondern nur von der Klasse von $\mathfrak{a}$ abhängig.

wo $\mathfrak{d}$ das Grundideal von k ist (vgl I C 4a, Nr. 5); hieraus folgt

$$\Phi(s, \mathfrak{a}) - \frac{2^{r_1} R}{w\, s\,(s-1)} =$$

$$= \frac{2^{r_1-1} n R}{w} \int_{-\frac{1}{2}}^{\frac{1}{2}} d x_1 \ldots \int_{-\frac{1}{2}}^{\frac{1}{2}} d x_r \int_1^{\infty} \Big[u^{\frac{n s}{2}} (\vartheta(\tau_1 \ldots \tau_n;\, \mathfrak{a}) - 1) +$$

$$+ u^{\frac{n(1-s)}{2}} (\vartheta(\tau_1 \ldots \tau_n;\, \mathfrak{a}^{-1} \mathfrak{d}^{-1}) - 1) \Big] \frac{d u}{u},$$

was der Gleichung (26) von Nr. **14** entspricht. Da $\mathfrak{a}^{-1} \mathfrak{d}^{-1}$ gleichzeitig mit $\mathfrak{a}$ alle h Idealklassen durchläuft, folgt weiter:

$$Z(s) = s\,(1-s)\, A^s \Big(\Gamma\big(\tfrac{s}{2}\big)\Big)^{r_1} (\Gamma(s))^{r_2}\, \zeta_k(s)$$

ist eine ganze Funktion, die sich bei Vertauschung von s mit $1-s$ nicht ändert. $\zeta_k(s)$ ist also, bis auf den Pol bei $s = 1$, in der ganzen Ebene regulär und besitzt die Funktionalgleichung

$$\zeta_k(1-s) = \Big(\frac{2}{(2\pi)^s}\Big)^n |d|^{s-\frac{1}{2}} \Big(\cos \frac{s\pi}{2}\Big)^{r_1+r_2} \Big(\sin \frac{s\pi}{2}\Big)^{r_1} \Big(\Gamma(s)\Big)^n \zeta_k(s).$$

(Vgl [24] Nr. **14**.) Der Punkt $s = 0$ ist Nullstelle r^{ter} Ordnung, $s = -2, -4, \ldots$ Nullstellen $(r+1)^{\text{ter}}$ Ordnung, $s = -1, -3, \ldots$ Nullstellen r_2^{ter} Ordnung; außerdem gibt es unendlich viele Nullstellen ϱ, die sämtlich dem Streifen $0 \leqq \sigma \leqq 1$ angehören, und es kann wie bei $\zeta(s)$ die Produktentwicklung

$$(s-1)\, \zeta_k(s) = a e^{b s} \frac{1}{s \Big(\Gamma\big(\frac{s}{2}\big)\Big)^{r_1} \Big(\Gamma(s)\Big)^{r_2}} \prod_{\varrho} \Big(1 - \frac{s}{\varrho}\Big) e^{\frac{s}{\varrho}}$$

(vgl. [28], Nr. **15**) abgeleitet werden.

Landau[391]) gibt eine Zusammenfassung der Theorie von $\zeta_k(s)$ und zeigt dabei, daß alle ϱ dem *Innern* des „kritischen Streifens" angehören[392]), und daß sogar das Gebiet (vgl. Nr. **19**)

$$\sigma > 1 - \frac{k}{\log t}, \quad t > t_0$$

391) *E. Landau*, Einführung in die elementare und analytische Theorie der algebraischen Zahlen und der Ideale, Leipzig u. Berlin (Teubner) 1918.

392) *Landau*, a. a. O 28), hat schon vor *Heckes* Entdeckung gezeigt, daß keine Nullstelle auf $\sigma = 1$ liegt, und sogar daß $\left|\frac{\zeta_k'}{\zeta_k}\right| < c \log^9 t$ im Gebiete $\sigma > 1 - \frac{k}{\log^7 t}$, $t > t_0$ gilt, was für die Verallgemeinerung des Primzahlsatzes wichtig war (vgl. Nr. **42**). Vgl. hierzu auch *Landau*, Über die Wurzeln der Zetafunktion eines algebraischen Zahlkörpers, Math. Ann. 79 (1919), p. 388—401.

bei passender Wahl von k und t_0 von Nullstellen frei ist. Die Anzahl der ϱ im Rechteck $0 < \sigma < 1$, $0 < t \leqq T$ ist nach *Landau*[391])

$$N(T) = \frac{n}{2\pi} T \log T + \frac{\log |d| - n - n \log 2\pi}{2\pi} T + O(\log T).^{393)}$$

(Vgl. [30], Nr. **16**.) Ob bei der allgemeinen $\zeta_k(s)$ unendlich viele ϱ auf der Geraden $\sigma = \frac{1}{2}$ liegen (vgl. Nr. **19**), ist bisher nicht entschieden.

Die Sätze über die Werte von $\zeta(s)$ auf einer vertikalen Geraden[394]) und über die Größenordnung von $\zeta(s)$ wurden zum Teil auch für $\zeta_k(s)$ verallgemeinert. Insbesondere ist über die μ-Funktion von $\zeta_k(s)$ (vgl. Nr. **18**) bekannt, daß $\mu(\sigma) = 0$ für $\sigma > 1$ und $\mu(\sigma) = n\left(\frac{1}{2} - \sigma\right)$ für $\sigma < 0$ ist[391]), während für $0 < \sigma < 1$ die μ-Kurve im Dreieck mit den Eckpunkten $\left(0, \frac{n}{2}\right)$, $\left(\frac{1}{2}, 0\right)$, $(1, 0)$ verläuft.

Bei Untersuchungen über die Verteilung der Primideale in den verschiedenen Idealklassen des Körpers, bzw. in den Idealklassen mod. $\mathfrak{f}$ ($\mathfrak{f}$ ein ganzes Ideal), treten gewisse Funktionen auf, die den L-Funktionen des rationalen Körpers entsprechen (vgl. Nr. **29** und für den quadratischen Körper Nr. **40**). Sie werden für $\sigma > 1$ durch die Gleichung

$$\zeta_k(s, \chi) = \sum_{\mathfrak{a}} \frac{\chi(\mathfrak{a})}{N\mathfrak{a}^s} = \prod_{\mathfrak{p}} \left(1 - \frac{\chi(\mathfrak{p})}{N\mathfrak{p}^s}\right)^{-1}$$

definiert, wobei die idealtheoretische Funktion $\chi(\mathfrak{a})$ durch einen Charakter der betreffenden *Abel*schen Gruppe von Idealklassen in analoger Weise wie die zahlentheoretische Funktion $\chi(n)$ bei den L-Funktionen (vgl. Nr. **29**) bestimmt ist. Die analytische Fortsetzung wurde von *Landau*[370a]) bis zu $\sigma = 1 - \frac{1}{n}$, von *Hecke*[395]) über die ganze Ebene ausgeführt und die entsprechenden Funktionalgleichungen wurden von *Hecke*[395]) und *Landau*[396]) aufgestellt. Bei jedem vom Hauptcharakter verschiedenen χ ist $\zeta_k(s, \chi)$, bei dem Hauptcharakter aber $(s - 1) \cdot \zeta_k(s, \chi)$, eine ganze Funktion von Geschlechte Eins, die im Streifen

393) Einen Satz über den Mittelwert des Restgliedes in dieser Formel gibt *H. Cramér*, a. a. O. 186).

394) Vgl. *H. Bohr* und *E. Landau*, a. a. O. 113).

395) *E. Hecke*, Über die L-Funktionen und den Dirichletschen Primzahlsatz für einen beliebigen Zahlkörper, Gött. Nachr. 1917, p. 299—318.

396) *E. Landau*, Über Ideale und Primideale in Idealklassen, Math. Ztschr. 2 (1918), p. 52—154. Es werden hier auch andere Äquivalenzbegriffe, d. h. andere Definitionen des Begriffs „Idealklasse" berücksichtigt.

$0 < \sigma < 1$ unendlich viele Nullstellen besitzt[397]), aber für $\sigma \geqq 1$ durchweg von Null verschieden ist.[398])

Um ein genaueres Studium der Verteilung der Primideale von k zu ermöglichen, führt *Hecke*[376]) eine Klasse *verallgemeinerter Zetafunktionen*

$$\zeta(s, \lambda) = \sum_{\mathfrak{a}} \frac{\lambda(\mathfrak{a})}{\mathfrak{a}^s} = \prod_{\mathfrak{p}} \left(1 - \frac{\lambda(\mathfrak{p})}{N\mathfrak{p}^s}\right)^{-1}$$

ein, wo die $\lambda(\mathfrak{a})$ gewisse der Multiplikationsregel $\lambda(\mathfrak{a})\,\lambda(\mathfrak{b}) = \lambda(\mathfrak{a}\mathfrak{b})$ genügende „Größencharaktere" von $\mathfrak{a}$ sind, die von $n - 1$ „Grundcharakteren" abhängen. Durch die Angabe der Grundcharaktere und der Norm ist das Ideal $\mathfrak{a}$ eindeutig bestimmt. Diese $\zeta(s, \lambda)$ lassen sich wie $\zeta_k(s)$ durch Thetareihen ausdrücken und besitzen auch ähnliche Funktionalgleichungen. Durch Kombination dieser Resultate mit einem Satz von *Weyl* (vgl. Nr. 39) über diophantische Approximationen erhält *Hecke* neue Sätze über die Verteilung der Primideale und der Primzahlen in gewissen zerlegbaren Formen (vgl. Nr. 40). — Endlich sei auch noch erwähnt, daß *Hecke*[399]) neuerdings verschiedene einem Zahlkörper zugeordnete analytische Funktionen *mehrerer* Variablen eingeführt hat, wobei die $\zeta(s, \lambda)$ als Hilfsmittel dienen.

42. Die Verteilung der Ideale und der Primideale. Es war für die Verallgemeinerung des Primzahlsatzes wesentlich, daß die von *Landau* eingeführten Methoden (vgl. Nr. 25) nur das Verhalten von $\zeta(s)$ in der Nähe von $\sigma = 1$ benutzten. Ohne die Existenz der *Dedekind*schen $\zeta_k(s)$ in der ganzen Ebene — die damals nicht bekannt war — vorauszusetzen, gelang es ihm nämlich, den sog. *Primidealsatz*[400]) zu beweisen: Für jeden Körper k vom Grade n ist die Anzahl $\pi_k(x)$ der Primideale mit Norm $\leqq x$ asymptotisch gleich $Li(x)$. Unter Benutzung der Gleichung

$$\log \zeta_k(s) = \sum_{\mathfrak{p}, m} \frac{1}{m\, N\mathfrak{p}^{ms}} \qquad (\sigma > 1)$$

397) Außerdem gibt es nur „triviale" Nullstellen bei $s = 0$ und auf der negativen reellen Achse sowie — im Falle eines uneigentlichen Charakters — auf der imaginären Achse.

398) Vgl. *E. Hecke*, a. a. O. 395); *E. Landau*, a. a. O. 370a) und 396), Zur Theorie der Heckeschen Zetafunktionen, welche komplexen Charakteren entsprechen, Math. Ztschr. 4 (1919), p. 152—162.

399) *E. Hecke*, Analytische Funktionen und algebraische Zahlen, I. Teil Abhandl. Math. Seminar Hamburg 1 (1922), p. 102—126.

400) *E. Landau*, Neuer Beweis des Primzahlsatzes und Beweis des Primidealsatzes, Math. Ann. 56 (1903), p. 645—670.

ergibt sich dies genau wie bei dem Primzahlsatz. Es gilt sogar nach *Landau*[401])

$$\pi_k(x) = Li(x) + O\left(x e^{-\frac{\alpha}{\sqrt{n}}\sqrt{\log x}}\right)$$

$$\vartheta_k(x) = \sum_{N\mathfrak{p} \leqq x} \log N\mathfrak{p} = x + O\left(x e^{-\frac{\alpha}{\sqrt{n}}\sqrt{\log x}}\right)$$

mit absolut konstantem α, was jedoch erst nach der Entdeckung der Fortsetzbarkeit von $\zeta_k(s)$ bewiesen werden konnte. Hierin ist als Spezialfall die Abschätzung von $\pi(x)$ (vgl. Nr. 27) enthalten. Für die Primideale einer Idealklasse (bzw. einer Idealklasse mod. $\mathfrak{f}$, wo $\mathfrak{f}$ ein ganzes Ideal ist) gelten entsprechende Gleichungen[402]), und zwar gilt dies auch bei engerer Auffassung des Klassenbegriffs. Dies enthält wiederum als Spezialfall *Dirichlets* Satz von der arithmetischen Reihe (vgl. Nr. 30). Die *Littlewood*schen Beziehungen (60) von Nr. 27 wurden von *Landau*[396]) für einen beliebigen Körper k und für eine beliebige Idealklasse von k verallgemeinert. Insbesondere ist also

$$\liminf_{x \to \infty} \frac{\pi_k(x) - Li(x)}{\frac{\sqrt{x}}{\log x}\log\log\log x} < 0 < \limsup_{x \to \infty} \frac{\pi_k(x) - Li(x)}{\frac{\sqrt{x}}{\log x}\log\log\log x}.$$

Bei dem Beweis dieses Satzes dient als Hilfsmittel die Verallgemeinerung der *Riemann-v. Mangoldt*schen Primzahlformel[403]); über die Konvergenzeigenschaften von $\sum \frac{x^\varrho}{\varrho}$ gibt es auch hier ähnliche Sätze wie bei $\zeta(s)$ (vgl. Nr. 28).

Wird das Residuum von $\zeta_k(s)$ für $s = 1$ durch $h\lambda$ bezeichnet, so gilt nach *Landau*[404]) für die Anzahl $H(x, K)$ der Ideale mit Norm $\leqq x$, die einer Klasse K von k angehören,

$$H(x, K) = \lambda x + O\left(x^{1-\frac{2}{n+1}}\right)$$

und für die Anzahl $H(x)$ aller Ideale des Körpers mit Norm $\leqq x$

$$H(x) = h\lambda x + O\left(x^{1-\frac{2}{n+1}}\right).$$

---

401) *E. Landau,* a. a. O. 391). Die in der vorigen Fußnote erwähnte Arbeit gibt eine weniger gute Abschätzung.

402) *E. Hecke,* a. a. O. 395); *E. Landau,* a. a. O. 227), 370a und 396)

403) *E. Landau,* a. a. O. 391) und 396). Für einige Anwendungen einer analogen Formel vgl. *H. Cramér,* a. a. O. 186).

404) *E. Landau,* a. a. O. 289b), 391) und 396).

Andererseits zeigt *Walfisz*[397]), der die von *Hardy* bei den Teilerproblemen (vgl. Nr. **34**) angewandte Methode benutzt,

$$\liminf_{x \to \infty} \frac{H(x) - h\lambda x}{x^{\frac{n-1}{2n}}} < 0 < \limsup_{x \to \infty} \frac{H(x) - h\lambda x}{x^{\frac{n-1}{2n}}}.$$

In diesen Beziehungen sind als Spezialfälle verschiedene der in Nr. **35** erwähnten Resultate bei den Kreis- und Ellipsoidproblemen enthalten.[405]) *Walfisz*[397]) hat auch eine explizite Formel für $H(x)$ aufgestellt, die als Spezialfälle die entsprechenden Formeln bei jenen Problemen enthält.

Setzt man, analog wie bei $\zeta(s)$, für $\sigma > 1$

$$\frac{1}{\zeta_k(s)} = \prod_{\mathfrak{p}} \left(1 - \frac{1}{N\mathfrak{p}^s}\right) = \sum_{\mathfrak{a}} \frac{\mu(\mathfrak{a})}{N\mathfrak{a}^s},$$

so läßt sich über die idealtheoretische Funktion $\mu(\mathfrak{a})$ z. B.

$$\sum_{N\mathfrak{a} \leqq x} \mu(\mathfrak{a}) = o(x), \qquad \sum_{\mathfrak{a}} \frac{\mu(\mathfrak{a})}{N\mathfrak{a}} = 0,$$

mit entsprechenden schärferen Abschätzungen, beweisen.[406]) Auch die Zusammenhangssätze der rationalen Zahlentheorie (vgl. Nr. **33**) lassen sich für einen beliebigen Körper k verallgemeinern[407]), sowie auch verschiedene andere der in den Nummern **32**—**37** erwähnten Sätze über zahlentheoretische Funktionen.[408])

405) Die entsprechenden oberen Abschätzungen waren überhaupt für quadratische Körper schon früher bekannt. Vgl. z. B. *E. Landau*, a. a. O. 284); *A. Hammerstein*, a. a. O. 200).

406) *E. Landau*, Über die zahlentheoretische Funktion $\mu(k)$, Sitzungsber. Akad. Wien **112** (1903), Abt. 2a, p. 537—570.

407) *E. Landau*, a. a. O. 21); *A. Axer*, a. a. O. 275).

408) Vgl z. B. *E. Landau*, a. a. O. 300) und 370a); *A. Axer*, Przyczynek do charakterystyki funkcyi idealowej $\varphi(\mathfrak{x})$ [Sur la fonction $\varphi(\mathfrak{x})$ dans la théorie des idéaux], Prace Math.-Fiz. 21 (1910), p. 37—41.

(Abgeschlossen im Mai 1922.)

SUPPLEMENTS

NOTE. — The translation S 1 of the doctoral dissertation is, as far as possible, a faithful image of the original. In particular, the notation has been preserved throughout. The original page-numbers are indicated at the top of the pages at the inner margin.

In the summaries S 2 — S 24 of the other Danish papers reviews of previous results have usually been omitted. The aim has been to indicate briefly the new results contained in the papers and, in most cases, the main trend of the reasoning, following the originals very closely. In the cases where the content of a paper has also been published in another language, the summary has been restricted to a statement of the result of the paper, followed by a reference in square brackets.

All comments by the editors have been confined to the Notes at the end of this volume.

CONTRIBUTIONS TO THE THEORY OF DIRICHLET SERIES

DISSERTATION FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

BY

HARALD BOHR

Translation of BIDRAG TIL DE DIRICHLET'SKE RÆKKERS THEORI.
AFHANDLING FOR DEN FILOSOFISKE DOKTORGRAD.
I Kommission hos G. E. C. Gad, København 1910, i—xii, 1—136.

DEDICATED TO MY PARENTS

Foreword.

By the analytic theory of numbers, we understand, in contrast to the elementary theory of numbers, that branch of number-theory in which, even though the goal is always to seek the laws which hold for integers, still, in order to be able to use the numerous tools of analysis, one introduces and operates with concepts such as continuous variables, infinite series, analytic functions, and so on.

While the first traces of the analytic theory of numbers may be found in Euler's works, Dirichlet, through his celebrated researches on arithmetic progressions and on the class number of binary quadratic forms with a given determinant, must be considered as the actual founder of this branch of mathematics.

One of the most important tools in Dirichlet's investigations of the analytic theory of numbers, as in those of later writers, is the class of infinite series $f(s) = \sum \frac{a_n}{n^s}$, which are now known everywhere as Dirichlet series, after Dedekind's suggestion.

The study of Dirichlet series, apart from their number-theoretic applications, offers considerable interest for purely function-theoretic reasons; for this type of series possesses several characteristic properties, closely connected with the fact that a Dirichlet series (in contrast, e.g., to a power series) possesses a two-dimensional region of the complex plane in which the series is simultaneously conditionally and uniformly convergent.

The previous theory of Dirichlet series, which considers the series exclusively in their regions of convergence, lacks in several respects the stamp of completeness. For example, in contrast to the case of power series, one cannot infer from the

fact that two Dirichlet series converge in a certain region G that the series formed by formal multiplication of these two series will also converge in the region G. In some cases, the product series converges; in other cases, it diverges. It may be further mentioned that the line bounding the region of convergence of a Dirichlet series, again in contrast to the case of power series, does not appear to be a line which is connected in a simple way with analytic properties of the function represented by the series.

It will be shown in this dissertation how, by using the concept of summability introduced by Cesàro, which forms a generalization of the usual concept of convergence, one can extend the theory of Dirichlet series in an essential way. In its extended form, the theory presents in several respects the appearance of a complete whole.

The first basis for a theory of summability of Dirichlet series was given by the author in a note: Sur la série de Dirichlet, *Comptes rendus de l'Académie des Sciences, Paris*, vol. 148, 11 January 1909. It is here stated that there exists a two-dimensional region in the complex plane within which a Dirichlet series is summable without being at the same time convergent.

As touched upon in this note, and as described in more detail in a paper: Über die Summabilität Dirichletscher Reihen, *Nachrichten der Kgl. Gesellschaft der Wissenschaften zu Göttingen, math.-phys. Klasse*, 1909, pp. 247–262, a corresponding theory of summability can also be established for the so-called factorial series $\sum \frac{a_n \cdot n!}{s(s+1)\cdots(s+n)}$ and binomial coefficient series $\sum \frac{a_n \cdot (s-1)\cdots(s-n)}{n!}$, which latter series stand, in point of fact, very close to Dirichlet series.

The above-mentioned note also touches briefly upon the way in which a theory of summability can be similarly established for the so-called generalized Dirichlet series, i.e., series of the form $f(s) = \sum a_n e^{-x_n \cdot s}$ $(x_1 < x_2 \cdots < x_n \cdots, \lim x_n = \infty)$, and for definite integrals of the type $\int_0^\infty a(x) e^{-x \cdot s} dx$.

In the present dissertation, however, I have restricted myself to treating only ordinary Dirichlet series $\sum \frac{a_n}{n^s}$, in order to be able to give the theory of summability a more complete form, as well as to make the exposition as simple and perspicuous as possible.

The dissertation is divided into two parts.

In the first part, we give an exposition of the theory of convergence for Dirichlet series, as it has gradually developed through the investigations of various authors.

Our object in this exposition has not been to give a complete survey of the theory of convergence for these series, but only to set forth the essential features of the theory, in particular those which have natural points of contact with the theory of their summability. This part is concluded with an investigation by the author of the so-called convergence problem for Dirichlet series.

The second part deals with the theory of summability for the series. In an introduction to this part, there will be given a detailed survey of the results contained therein.

Upon finishing this work, I wish to ask my teachers at the University, Professor Zeuthen and Professor Nielsen, to accept my sincere thanks for the valuable guidance which they have given me during my study and for the interest which they have always shown me. I ask Mr. N. E. Nørlund, assistant at the astronomical observatory, to accept my thanks for valuable help in reading the proofs.

Copenhagen, December 1909.

HARALD BOHR

PART ONE

The Theory of Convergence for Dirichlet Series.

By a Dirichlet series, we understand an infinite series of the form

$$\sum_{n=1}^{n=\infty} \frac{a_n}{n^s} = \frac{a_1}{1^s} + \frac{a_2}{2^s} + \frac{a_3}{3^s} + \cdots.$$

The quantities $a_1, a_2, \ldots, a_n, \ldots$ (the coefficients of the series) can be arbitrary real or complex numbers and n^s denotes $e^{s \cdot \log n}$, where $\log n$ represents the real logarithm of the positive integer n while $s = \sigma + it$ denotes the independent complex variable.

For the absolute convergence of a Dirichlet series the following theorem holds:*

Theorem I. *If* $\sum \frac{a_n}{n^s}$ *is absolutely convergent for* $s = s_0 = \sigma_0 + it_0$ (i.e., if $\sum \left|\frac{a_n}{n^{s_0}}\right|$ is convergent), *then* $\sum \frac{a_n}{n^s}$ *is absolutely convergent for every value of* $s = \sigma + it$ *such that* $\sigma \geqq \sigma_0$.

* W. Scheibner, *Über unendliche Reihen und deren Convergenz*, Leipzig 1860, pp. 24–25.

Proof. For $\sigma \geqq \sigma_0$ we have

$$\left|\frac{a_n}{n^s}\right| = \frac{|a_n|}{n^\sigma} \leqq \frac{|a_n|}{n^{\sigma_0}} = \left|\frac{a_n}{n^{s_0}}\right|,$$

and hence the convergence of $\sum \left|\frac{a_n}{n^{s_0}}\right|$ implies the convergence of $\sum \left|\frac{a_n}{n^s}\right|$.

An immediate consequence of Theorem I is that, for any Dirichlet series, there are only the following three possibilities:

1. The series is absolutely convergent for all values of s.
2. The series is not absolutely convergent for any value of s.
3. There exists a straight line $\sigma = l$ orthogonal to the real axis such that $\sum \frac{a_n}{n^s}$ is absolutely convergent in the half-plane to the right of this line (i.e., for $\sigma > l$), while the series is not absolutely convergent in the half-plane to the left of this line (i.e., for $\sigma < l$).

The number l is called *the abscissa of absolute convergence* of the series and the line $\sigma = l$ the *boundary of absolute convergence* of the series.*

It follows also from Theorem I that $\sum \frac{a_n}{n^s}$ is absolutely convergent either at all or at none of the points of the boundary line $\sigma = l$ itself.

As one sees from the foregoing, the question about the type of region in which a Dirichlet series is absolutely convergent answers itself, so to say; however, the situation is essentially different with the analogous question regarding the region of convergence in general (absolute or conditional). This latter question has been solved by Jensen,† who proved in 1884 the fundamental theorem that the region of convergence of a Dirichlet series is also a half-plane bounded by a straight line $\sigma = \lambda_0$ orthogonal to the real axis. This line, of course, must lie to the left of or possibly coincide with the boundary of absolute convergence $\sigma = l$.

The proof of Jensen's theorem is based on the following lemma found by Dedekind:‡

* In order to have uniform notation in the sequel, it will be convenient to include the two extreme cases 1. and 2. under the general case 3. by putting in case 1., $l = -\infty$, and in case 2., $l = +\infty$.

† J. L. W. V. Jensen, Om Rækkers Konvergens, *Tidsskrift for Mathematik*, ser. 5, vol. 2, 1884, p. 70.

‡ Dirichlet, *Vorlesungen über Zahlentheorie.* Herausgegeben und mit Zusätzen versehen von Dedekind, Braunschweig, Second edition 1871, p. 371. For real α and u the theorem is already stated by P. du Bois-Reymond, *Antrittsprogram*, Freiburg 1871, p. 10.

Lemma Ia. *If* $\sum_{n=1}^{n=\infty} u_n$ *is convergent or oscillating between finite bounds* (i.e., if $S_n = \sum_{m=1}^{m=n} u_m$ for all n is numerically smaller than a constant K) *and if the sequence* $\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots$ *satisfies the following two conditions*:

$$\sum_{n=1}^{n=\infty} |\alpha_n - \alpha_{n+1}| \text{ converges} \tag{1}$$

and

$$\lim_{n=\infty} \alpha_n = 0\,, \tag{2}$$

then the series $\sum_{n=1}^{n=\infty} u_n \alpha_n$ *is convergent and its sum is equal to the sum of the absolutely convergent series* $\sum_{n=1}^{n=\infty} S_n(\alpha_n - \alpha_{n+1})$.

Proof. Since

$$S_n = \sum_{m=1}^{m=n} u_m\,,$$

we obtain by partial summation

$$T_n = \sum_{m=1}^{m=n} u_m \alpha_m = \sum_{m=1}^{m=n} (S_m - S_{m-1})\alpha_m = S_n \alpha_n + \sum_{m=1}^{m=n-1} S_m(\alpha_m - \alpha_{m+1})\,. \tag{3}$$

From assumption (2) in conjunction with the assumption

$$|S_n| < K \tag{4}$$

it follows, in the first place, that

$$\lim_{n=\infty} S_n \alpha_n = 0\,,$$

while (1) and (4) imply immediately the convergence (indeed the absolute convergence) of the series $\sum_{n=1}^{n=\infty} S_n(\alpha_n - \alpha_{n+1})$.

Thus from the equation (3) we obtain

$$\lim_{n=\infty} T_n = \sum_{n=1}^{n=\infty} u_n \alpha_n = \lim_{n=\infty} S_n \alpha_n + \lim_{n=\infty} \sum_{m=1}^{m=n-1} S_m(\alpha_m - \alpha_{m+1})$$

$$= \sum_{n=1}^{n=\infty} S_n(\alpha_n - \alpha_{n+1})\,. \qquad \text{q.e.d.}$$

Lemma Ia now puts us in a position to prove the following theorem:*

* Jensen, l.c.—The remark 'or oscillating between finite bounds' is due, however, to E. Cahen, Sur la fonction $\zeta(s)$ de Riemann et sur des fonctions analogues, *Annales scientifiques de l'École Normale supérieure*, ser. 3, vol. 11, 1894, p. 82.

Theorem II. *If the Dirichlet series* $\sum \frac{a_n}{n^s}$ *is convergent or oscillating between finite bounds for* $s = s_0 = \sigma_0 + it_0$, *then* $\sum \frac{a_n}{n^s}$ *is convergent for every* $s = \sigma + it$ *such that* $\sigma > \sigma_0$.

Proof. If we put

$$u_n = \frac{a_n}{n^{s_0}}\ ;\ \alpha_n = \frac{1}{n^{s-s_0}}\ ;\ u_n\alpha_n = \frac{a_n}{n^s},$$

the series $\sum u_n$ is, by assumption, convergent or oscillating between finite bounds; furthermore, since $\sigma - \sigma_0 > 0$, we have

$$\lim_{n=\infty} \alpha_n = \lim_{n=\infty} \frac{1}{n^{s-s_0}} = 0$$

and

$$|\alpha_n - \alpha_{n+1}| = \left|\frac{1}{n^{s-s_0}} - \frac{1}{(n+1)^{s-s_0}}\right| = \left|(s-s_0)\int_n^{n+1} \frac{dx}{x^{1+s-s_0}}\right| < |s-s_0| \frac{1}{n^{1+\sigma-\sigma_0}}\ ;$$

since $\sigma - \sigma_0 > 0$, the last inequality immediately implies the convergence of $\sum |\alpha_n - \alpha_{n+1}|$.

Conditions (1), (2), and (4) being thus fulfilled, it follows from Lemma Ia that $\sum u_n\alpha_n = \sum \frac{a_n}{n^s}$ is convergent. q.e.d.

Furthermore, it follows from Lemma Ia that $\sum \frac{a_n}{n^s}$ for $\sigma > \sigma_0$ can be represented as the sum of the absolutely convergent series

$$\sum_{n=1}^{n=\infty} S_n \left(\frac{1}{n^{s-s_0}} - \frac{1}{(n+1)^{s-s_0}}\right), \tag{5}$$

where

$$S_n = \sum_{m=1}^{m=n} \frac{a_m}{m^{s_0}}.$$

It now follows immediately from Jensen's Theorem II *that an arbitrary Dirichlet series is either everywhere or nowhere convergent, or else there exists a straight line* $\sigma = \lambda_0$ *orthogonal to the real axis such that the series is convergent for* $\sigma > \lambda_0$ *and divergent for* $\sigma < \lambda_0$.

The number λ_0 is called the *abscissa of convergence* of the series.*

* As in the case of absolute convergence, we shall express here also the extreme cases as $\lambda_0 = -\infty$ and $\lambda_0 = +\infty$. The reason for denoting the abscissa of convergence by the indexed letter λ_0 will appear in Part Two.

Concerning the behaviour of the series on the boundary of convergence $\sigma = \lambda_0$ itself we cannot—in contrast to what was true of absolute convergence—conclude that the series is either everywhere or nowhere convergent on this line (in fact, Theorem II is only valid 'for $\sigma > \sigma_0$' and not, like Theorem I, 'for $\sigma \geqq \sigma_0$'). We shall also see later that the three possibilities:

1. the series converges everywhere on the boundary of convergence,
2. the series diverges everywhere on the boundary of convergence, and
3. the series converges at some and diverges at other points of the boundary of convergence,

can all actually occur.

The merit of having first made a closer investigation of the analytic properties of the function represented by a Dirichlet series belongs to Cahen; for use in his investigations, Cahen* proved the following Lemma Ib, which may be considered as an extension of Dedekind's Lemma Ia.

Lemma Ib. *If a series* $\sum_{n=1}^{n=\infty} u_n$ *with constant terms is convergent or oscillating between finite bounds* (i.e., $|S_n| = \left|\sum_{m=1}^{m=n} u_m\right| < K$), *and if* $\alpha_1(s), \alpha_2(s), \ldots, \alpha_n(s), \ldots$ *are single-valued functions of the complex variable s which satisfy the following two conditions, for s belonging to a certain domain G*:

$$\sum_{n=1}^{n=\infty} |\alpha_n(s) - \alpha_{n+1}(s)| \text{ converges uniformly}$$

and

$$\lim_{n=\infty} \alpha_n(s) = 0\,,$$

then the series $\sum_{n=1}^{n=\infty} u_n \alpha_n(s)$ *converges uniformly for s belonging to G.*

Proof. Since

$$|\alpha_n(s) - \alpha_{n+p}(s)| \leqq \sum_{m=n}^{m=n+p-1} |\alpha_m(s) - \alpha_{m+1}(s)|\,,$$

where by assumption the expression on the right-hand side is a section of a uniformly convergent series, we see, in the first place, that $\alpha_n(s)$ converges uniformly for s belonging to the domain G to its limit value (here 0).

* l.c., p. 79.

Thus we have
$$\lim_{n=\infty} S_n\alpha_n(s) = 0$$
uniformly for s belonging to G.

Furthermore, on account of the assumptions made, the series
$$\sum_{n=1}^{n=\infty} S_n\big(\alpha_n(s)-\alpha_{n+1}(s)\big)$$
is evidently uniformly convergent in the domain G.

Therefore the identity
$$T_n(s) = \sum_{m=1}^{m=n} u_m\alpha_m(s) = S_n\alpha_n(s) + \sum_{m=1}^{m=n-1} S_m\big(\alpha_m(s)-\alpha_{m+1}(s)\big)$$
shows that $T_n(s)$ converges uniformly for $n=\infty$ to its limit
$$\sum_{n=1}^{n=\infty} S_n\big(\alpha_n(s)-\alpha_{n+1}(s)\big),$$
i.e., $\sum_{n=1}^{n=\infty} u_n\alpha_n(s)$ is uniformly convergent in the domain G. q.e.d.

By use of Lemma Ib Cahen* has proved the following important theorem:

Theorem III. *If $\sum \frac{a_n}{n^s}$ is a Dirichlet series with abscissa of convergence λ_0, then this series is uniformly convergent in every domain $G = G(\varepsilon, E)$ whose points $s = \sigma + it$ all satisfy the following two conditions:*†
$$\sigma > \lambda_0 + \varepsilon \; (\varepsilon > 0), \textit{ and } |s| < E \; (E < \infty).$$

Proof. The Dirichlet series $\sum \frac{a_n}{n^s}$ is convergent at the point $s_0 = \lambda_0 + \frac{\varepsilon}{2}$. As above, we put
$$u_n = \frac{a_n}{n^{s_0}}, \; \alpha_n = \frac{1}{n^{s-s_0}}, \; u_n\alpha_n = \frac{a_n}{n^s}.$$
Then, for s belonging to the domain G, we have
$$\lim_{n=\infty} \alpha_n(s) = 0$$
and
$$|\alpha_n(s)-\alpha_{n+1}(s)| \leqq |s-s_0| \frac{1}{n^{1+\sigma-\sigma_0}} < (E+|s_0|)\frac{1}{n^{1+\frac{\varepsilon}{2}}}.$$

---

* l.c., p. 83.

† If $\lambda_0 = -\infty$ (i.e., if $\sum \frac{a_n}{n^s}$ is convergent in the whole plane), the first condition should be omitted.

Since the right-hand side of the last inequality is independent of s and the series $\sum \frac{1}{n^{1+\frac{\varepsilon}{2}}}$ is convergent, it follows immediately that $\sum |\alpha_n(s)-\alpha_{n+1}(s)|$ is uniformly convergent.

Applying Lemma Ib we find that $\sum u_n\alpha_n(s) = \sum \frac{a_n}{n^s}$ is uniformly convergent in the domain G. q.e.d.

We now quote a well-known theorem of Weierstrass:*

If $u_1(s), u_2(s), \ldots, u_n(s), \ldots$ are regular analytic functions defined in a certain connected domain G, and if the infinite series $\sum_{n=1}^{n=\infty} u_n(s)$ is uniformly convergent for s belonging to G, then this series represents in the domain G a regular analytic function $U(s)$. Furthermore, the series $\sum_{n=1}^{n=\infty} u_n^{(p)}(s)$ obtained by p term by term differentiations $(p = 1, 2, 3, \ldots)$ *is also convergent in the domain G and represents the function $U^{(p)}(s)$.*†

The single terms $\frac{a_n}{n^s}$ in a Dirichlet series are all functions which are regular analytic in the whole plane (integral functions). Furthermore, when $s = \sigma + it$ is an arbitrary point such that $\sigma > \lambda_0$, one can evidently choose the constants ε and E in Theorem III so that s belongs to the interior of the domain $G = G(\varepsilon, E)$. From these observations, it follows immediately by application of Weierstrass' theorem that we have:

Theorem IV.‡ *A Dirichlet series $\sum \frac{a_n}{n^s}$ represents in its half-plane of convergence* (i.e., for $\sigma > \lambda_0$) *a regular analytic function $f(s)$. Furthermore, in its half-plane of convergence the series may be differentiated term by term an arbitrary number of times; i.e., for $\sigma > \lambda_0$ and an arbitrary p we have the equation*

$$f^{(p)}(s) = \sum_{n=1}^{n=\infty} \frac{a_n \cdot (-\log n)^p}{n^s}.$$

We shall now prove the following theorem which deals with the breadth of the strip of conditional convergence:

---

* *Werke*, vol. 2, 1895, p. 205.

† For the sake of a later application we remark that Weierstrass' theorem can also be stated as follows: *If $S_n(s)$ is regular analytic in the domain G (for all $n = 1, 2, 3, \ldots$) and if $\lim_{n=\infty} S_n(s)$ is uniformly equal to $U(s)$, then $U(s)$ is regular analytic in G, and $U^{(p)}(s) = \lim_{n=\infty} S_n^{(p)}(s)$.*

‡ Cahen, l.c., p. 102.

Theorem V.* *If* $\sum \frac{a_n}{n^s}$ *is convergent for* $s = s_0 = \sigma_0 + it_0$, *or if merely* $\left|\frac{a_n}{n^{s_0}}\right| < K$ (*K independent of n*), *then* $\sum \frac{a_n}{n^s}$ *is absolutely convergent for every s such that* $\sigma > \sigma_0 + 1$.†

Proof. From the inequality

$$\left|\frac{a_n}{n^{s_0}}\right| < K$$

it follows immediately that

$$\left|\frac{a_n}{n^s}\right| = \left|\frac{a_n}{n^{s_0}} \cdot \frac{1}{n^{s-s_0}}\right| = \left|\frac{a_n}{n^{s_0}}\right| \cdot \frac{1}{n^{\sigma-\sigma_0}} < K \frac{1}{n^{\sigma-\sigma_0}},$$

and hence the series $\sum \frac{a_n}{n^s}$ is absolutely convergent for $\sigma > \sigma_0 + 1$. q.e.d.

When l and λ_0 denote as above the abscissa of absolute convergence and the abscissa of convergence, respectively, of an arbitrary Dirichlet series, it follows immediately from Theorem V that:

1. If λ_0 and l are finite, then $\lambda_0 \leqq l \leqq \lambda_0 + 1$.
2. If either l or λ_0 is $-\infty$, then so is the other.
3. If either l or λ_0 is $+\infty$, then so is the other.

Before we continue the systematic development of the theory of Dirichlet series, we shall first show, by a series of examples, that all types of Dirichlet series which are consistent with the above results actually exist. These examples fall naturally into two groups of which the first treats the above-mentioned cases for the convergence numbers l and λ_0 while the second group gives a complete list of examples corresponding to the various cases of absolute convergence, conditional convergence, or divergence, on the boundaries of convergence $\sigma = l$ and $\sigma = \lambda_0$.

Group 1.‡

I. $l = \lambda_0 = -\infty$ when $a_n = \frac{1}{n!}$; for the series $\sum \frac{1}{n!\, n^s}$ is obviously absolutely convergent for every real s.

---

* Scheibner, l.c., pp. 24–25.

† However, from the convergence at a point $s_0 = \sigma_0 + it_0$ it is not possible to infer absolute convergence for $\sigma = \sigma_0 + 1$. In fact, as was pointed out by E. Landau, Über die Grundlagen der Theorie der Fakultätenreihen, *Sitzungsberichte der math.-phys. Klasse der Kgl. Bayerischen Akademie der Wissenschaften*, vol. 36, 1906, p. 173, the series $\sum_{n=2}^{n=\infty} \frac{(-1)^n}{n^s \log n}$ is convergent at the point $s = 0$ but not absolutely convergent for $\sigma = 1$.

‡ The examples in this group, with a single exception (V.), are due to Landau, Über die Grund-

II. $l=\lambda_0=+\infty$ when $a_n=n!$; for the series $\sum \frac{n!}{n^s}$ is obviously not convergent for any real s.

III. l and λ_0 are finite and $l=\lambda_0$ when $a_n=1$; for the series $\sum \frac{1}{n^s}$ (which defines the Riemann function $\zeta(s)$) is absolutely convergent for $s>1$ and divergent for $s \leqq 1$, so that $l=\lambda_0=1$.

IV. l and λ_0 are finite and $l=\lambda_0+1$ when $a_n=(-1)^{n+1}$; for the series $\sum \frac{(-1)^{n+1}}{n^s}$ (which represents the function $\zeta(s)(1-2^{1-s})$) is absolutely convergent for $s>1$, conditionally convergent for $1 \geqq s > 0$, and divergent for $s \leqq 0$, so that $l=1$ and $\lambda_0=0$.

V. l and λ_0 are finite and $\lambda_0 < l < \lambda_0+1$, $l=\lambda_0+\theta$ (θ an arbitrary number between 0 and 1), when $a_n=1+(-1)^{n+1}n^\theta$; in fact, $\sum \frac{a_n}{n^s}$ can be formed by term by term addition of the series $\sum \frac{1}{n^s}$ with the convergence numbers $l=\lambda_0=1$ and the series $\sum \frac{(-1)^{n+1}n^\theta}{n^s}$ with the convergence numbers $l=1+\theta$, $\lambda_0=\theta$, and hence the series $\sum \frac{a_n}{n^s}$ has, as will appear from (6), the convergence numbers $l=1+\theta$, $\lambda_0=1$.

Group 2.*

I. *The lines of convergence coincident* ($l=\lambda_0$).

α) 1. The series absolutely convergent on the line of convergence.

---

lagen ..., l. c., p. 171. As an example of a series falling under case V., Landau (l. c.) considers the series $\sum \frac{a_n}{n^s}$ where $a_n=1$ for odd non-square numbers n, $a_n=-1$ for even non-square numbers n, $a_n=2$ for odd square numbers n, $a_n=0$ for even square numbers n. For this series, we have $l=1$, $\lambda_0=\frac{1}{2}$ and hence $l-\lambda_0=\frac{1}{2}$. For the sake of complete generality, we have exhibited a series with $l-\lambda_0=\theta$ where θ is an arbitrary number between 0 and 1 (both excluded).

* One does not find in earlier papers any treatment of the actual existence of all cases falling under group 2.—The following investigations show not only the existence of such series but also give simple examples (i.e., examples $\sum \frac{a_n}{n^s}$ where a_n is defined by simple expressions) for all the cases in question. In a recent paper, W. Schnee, Über Dirichlet'sche Reihen, *Rendiconti del Circolo Matematico di Palermo*, vol. 27, 1909, pp. 105–113, has considered in detail the question of the existence of Dirichlet series corresponding to the single case which is denoted as case 2. under group 2. Schnee has overlooked, however, the fact that such a series, as shown in the text, can immediately be formed by combining

β) The series absolutely convergent at no points of the line of convergence.

2. The series conditionally convergent at all points of the line of convergence.
3. The series conditionally convergent at some points and divergent at other points of the line of convergence.
4. The series divergent at all points of the line of convergence.

II. *The lines of convergence not coincident* ($l > \lambda_0$).

γ) The series absolutely convergent at all points of the line of absolute convergence $\sigma = l$.

5. The series conditionally convergent at all points of the line of convergence $\sigma = \lambda_0$.
6. The series conditionally convergent at some points and divergent at other points of $\sigma = \lambda_0$.
7. The series divergent at all points of the line $\sigma = \lambda_0$.

δ) The series not absolutely convergent (and hence conditionally convergent) on the line of absolute convergence $\sigma = l$.

8. The series conditionally convergent at all points of the line of convergence $\sigma = \lambda_0$.
9. The series conditionally convergent at some points and divergent at other points of $\sigma = \lambda_0$.
10. The series divergent at all points of the line $\sigma = \lambda_0$.

We shall first prove that we need only give examples for cases 1., 3., and 4., for a single case under γ), and for a single case under δ), since by means of these we can construct series corresponding to all of the remaining cases.

Let us therefore assume for the present that we know Dirichlet series of the types 1., 3., 4., γ), and δ), which series we shall denote respectively by [1], [3], [4], [γ], and [δ].* We now recall that for three series with constant terms $U = \sum u_n$, $V = \sum v_n$, and $W = \sum w_n$ with $w_n = u_n + v_n$ we have that

---

simple series (for instance $\sum \frac{a_n}{n^s}$ where $a_n = (-1)^n + (\log n)^{-1-k}$ (k arbitrary positive)). Thus he had to use very long and complicated considerations in his constructions, considerations analogous to those by which Pringsheim has proved the existence of power series conditionally convergent at all points of the circumference of the circle of convergence.

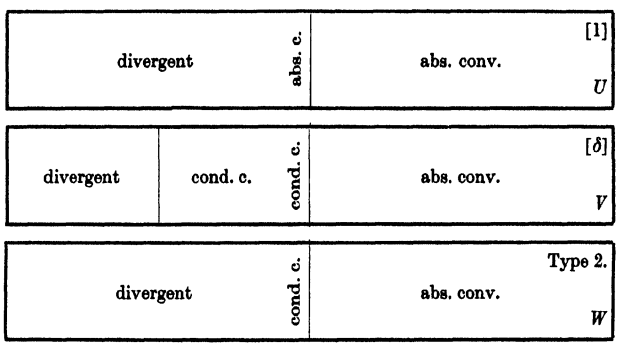
* By a series [γ] of type γ) we mean a series which satisfies the assumption common to all three cases under γ) but about which one need not know to which specific case 5., 6., or 7. it belongs. A similar remark applies to a series [δ] of type δ).

$$\text{from}\quad \left.\begin{array}{llll} U \text{ abs. conv.,} & V \text{ abs. conv.} & & W \text{ abs. conv.} \\ U \text{ abs. conv.,} & V \text{ cond. conv.} & \text{follows} & W \text{ cond. conv.} \\ U \text{ convergent,} & V \text{ divergent} & & W \text{ divergent} \end{array}\right\}. \tag{6}$$

We also recall that the term by term sum of two Dirichlet series $\sum \frac{a_n}{n^s}$ and $\sum \frac{b_n}{n^s}$ is always a new Dirichlet series, namely $\sum \frac{c_n}{n^s}$, where $c_n = a_n + b_n$. The figures below show how these facts allow us to form a series of type 2. and series of types 5., 6., 7. under γ) and 8., 9., 10. under δ) by term by term addition of two suitably chosen series from among the five series [1], ..., [δ] which are supposed known. Before we exhibit these figures we must observe that by a simple transformation of variable $s = s' + k$ we can translate the lines of convergence of a Dirichlet series any distance k, of course with preservation of their mutual distance.

Fig. I shows the construction of a series of type 2.

Fig. I.



$W = U + V.$

Fig. II, III, and IV show the construction of series of types 5., 6., and 7., respectively.

Fig. II.

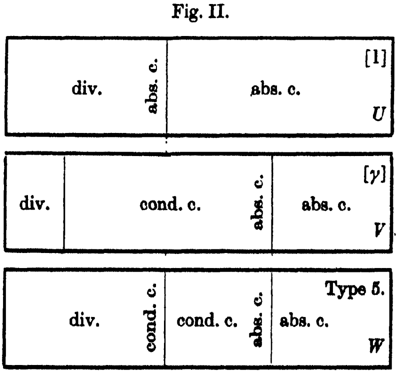


Fig. III.

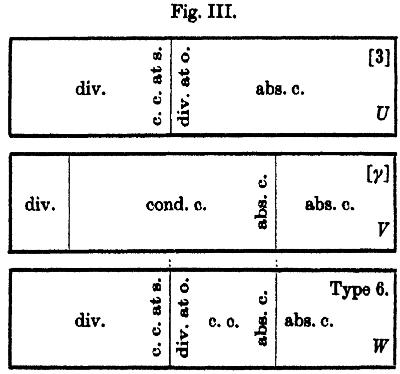
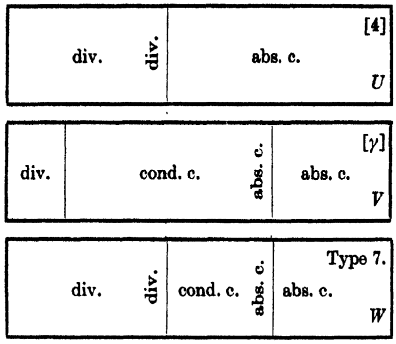


Fig. IV.



If in the three above figures the V-series of γ-type is replaced by a V-series of δ-type, while the U-series are kept unchanged, the corresponding W-series will turn into series of types 8., 9., and 10., respectively.

In order to complete the proof of the existence of Dirichlet series corresponding to all the cases considered under group 2 we need only, in view of the above remarks, give examples of series of the five types 1., 3., 4., γ), and δ). To this end we shall give simple examples whose type can either immediately be decided or whose behaviour on the lines of convergence is already known.

$$\sum_{n=2}^{n=\infty} \frac{a_n}{n^s} \text{ is of } \gamma)\text{-type when } a_n = \frac{(-1)^n}{(\log n)^{1+k}} \; (k > 0);\; (l = 1,\, \lambda_0 = 0)\,.$$

$$\sum_{n=1}^{n=\infty} \frac{a_n}{n^s} \text{ is of } \delta)\text{-type when } a_n = (-1)^{n+1};\; (l = 1,\, \lambda_0 = 0)\,.$$

$$\sum_{n=2}^{n=\infty} \frac{a_n}{n^s} \text{ is of type 1. when } a_n = \frac{1}{(\log n)^{1+k}} \; (k > 0);\; (l = \lambda_0 = 1)\,.$$

$\sum_1^\infty \frac{a_n}{n^s}$ is of type 3. when $a_n = 1$ if n is a prime number p and $a_n = 0$ if n is a composite number; $(l = \lambda_0 = 1)$; for $\sum \frac{1}{p^s}$, as is well known, diverges for $s = 1$, but converges for $s = 1+it$ $(t \neq 0)$.*

---

* Concerning the literature on the convergence of the series $\sum \frac{1}{p^s}$ and $\sum \frac{1}{n^s}$, see for example Landau, Ueber die zu einem algebraischen Zahlkörper gehörige Zetafunktion ..., *Journal für die reine und angewandte Mathematik*, vol. 125, 1903, p. 105.

$\sum\limits_1^\infty \frac{a_n}{n^s}$ is of type 4. when $a_n = 1$; $(l = \lambda_0 = 1)$; for $\sum \frac{1}{n^s}$ is, as one easily shows, divergent at all points of the line $\sigma = 1$.*

This completes the proof of the existence of all the cases under group 2.

From the consideration of these examples we now pass to the proof of the so-called uniqueness theorem of Dirichlet series.

Theorem VI.† *If the series* $f(s) = \sum \frac{a_n}{n^s}$ *and* $g(s) = \sum \frac{b_n}{n^s}$ *are both convergent in a certain half-plane* (from which immediately follows the existence of a number L such that both series are absolutely convergent for $\sigma \geqq L$) *and if there exists an infinite sequence of numbers*

$$s_1 = \sigma_1 + it_1, \ldots, s_p = \sigma_p + it_p, \ldots (\lim_{p=\infty} \sigma_p = +\infty),$$

such that

$$f(s_p) = g(s_p) \quad (\text{for all } p = 1, 2, \ldots),$$

then the series $\sum \frac{a_n}{n^s}$ *and* $\sum \frac{b_n}{n^s}$ *are identical, i.e., we have*

$$a_n = b_n$$

for all $n = 1, 2, 3, \ldots$.

Proof. The series

$$h(s) = f(s) - g(s) = \sum \frac{c_n}{n^s},$$

where $c_n = a_n - b_n$, is absolutely convergent for $\sigma \geqq L$, and assumes the value 0 for $s = s_p$ $(p = 1, 2, 3, \ldots)$. We shall prove that $c_n = 0$ for all n.

Let us assume that this is not the case and denote by c_N the first coefficient in the series $\sum \frac{c_n}{n^s}$ (i.e., the one with the lowest index) which is not 0.

Then, for $\sigma \geqq L$, we have

---

* See the note on the preceding page.

† O. Perron, Zur Theorie der Dirichletschen Reihen, *Journal für die reine und angewandte Mathematik*, vol. 134, 1908, pp. 106–113. In Dirichlet, *Vorlesungen über Zahlentheorie*, l.c., pp. 225–226 we find the following theorem, which, however, deals only with the behaviour on the real axis: 'If $\sum \frac{a_n}{n^s} = \sum \frac{b_n}{n^s}$ for $s > s_0$, then $a_n = b_n$ for all $n = 1, 2, \ldots$.'

$$\left|h(s)-\frac{c_N}{N^s}\right| \leq \sum_{n=N+1}^{n=\infty}\left|\frac{c_n}{n^s}\right|$$

$$=\sum_{n=N+1}^{n=\infty}\left(\left|\frac{c_n}{n^L}\right|\cdot\frac{1}{n^{\sigma-L}}\right) \leq \frac{1}{(N+1)^{\sigma-L}}\sum_{n=N+1}^{n=\infty}\left|\frac{c_n}{n^L}\right| = \frac{K}{(N+1)^\sigma},$$

where K is independent of $s = \sigma+it$.

Consequently, for $\sigma \geqq L$, we have

$$|h(s)| \geqq \frac{|c_N|}{N^\sigma} - \frac{K}{(N+1)^\sigma},$$

from which it immediately follows that for sufficiently large σ (i.e., for $\sigma > E$)

$$|h(s)| > \tfrac{1}{2}\frac{|c_N|}{N^\sigma} > 0, \tag{7}$$

and this contradicts the assumption that $h(s_p) = 0$ for all p. Hence $c_n = 0$ for all n, and Theorem VI is proved.

The determination of the abscissa of convergence of a Dirichlet series from the coefficients of the series is given by Cahen* in the following important theorem:

Theorem VII. *Let* $\sum \frac{a_n}{n^s}$ *be a Dirichlet series with abscissa of convergence* $\lambda_0 \geqq 0$ (*if* $\lambda_0 \neq -\infty$, i.e., if the series is not convergent in the whole plane, *this can always be obtained by a simple transformation of variable* $s = s'+k$).

Then

$$\lambda_0 = \limsup_{n=\infty} \frac{\log\left|\sum_{m=1}^{m=n} a_m\right|}{\log n}.$$

As we shall prove later on (in Part Two) a much more general theorem, which contains Theorem VII as a special case, we shall not dwell here on a proof of this theorem.

Since the abscissa of absolute convergence of the series $\sum\frac{a_n}{n^s}$ is equal to the abscissa of convergence of the series $\sum \frac{|a_n|}{n^s}$ (as is immediately seen by considering the series on the real axis), we obtain from Theorem VII:

Theorem VIII. *If the series* $\sum\frac{a_n}{n^s}$ *has an abscissa of absolute convergence* $l \geqq 0$, *then*

---

* l.c., pp. 89 and 102.

$$l = \limsup_{n=\infty} \frac{\log \sum_{m=1}^{m=n} |a_m|}{\log n}.$$

The following theorem* is of importance for many applications:

Theorem IX. *If*

$$\lim_{n=\infty} a_n = A, \tag{8}$$

then

$$\lim_{s=0} s \cdot \sum_{n=1}^{n=\infty} \frac{a_n}{n^{1+s}} = A,$$

if s, in converging to the point 0, is restricted to an angle with vertex at the origin and extending into the right half-plane so that the boundary lines form angles between $+\frac{\pi}{2}$ (excl.) and $-\frac{\pi}{2}$ (excl.) with the positive real axis.

Proof. From (8) it follows immediately that $\sum \frac{a_n}{n^{1+s}}$ is absolutely convergent for $\sigma > 0$.

1. We consider first the special case $a_n = 1$ (consequently also $A = 1$), in which case we need not assume that s tends to 0 in a limited angle, but merely that $\sigma > 0$ during the limit process. For all $n = 1, 2, \ldots$, we have

$$\frac{1}{n^s} - \frac{1}{(n+1)^s} = \frac{1}{n^s}\left[1 - \left(1 + \frac{1}{n}\right)^{-s}\right] = \frac{1}{n^s}\left[\frac{s}{n} + \frac{F(n, s)}{n^2}\right] = \frac{s}{n^{1+s}} + \frac{F(n, s)}{n^{2+s}}, \tag{9}$$

where $F(n, s)$ is an integral function of s which assumes the value 0 for $s = 0$, and where, as we easily see, $F(n, s)$ is numerically smaller than some constant for all n and $|s| < K$.

We now sum the equation (9) over all $n = 1, 2, \ldots$, under the assumption $\sigma > 0$. We obtain 1 on the left-hand side (since for $\sigma > 0$, $\lim_{n=\infty} \frac{1}{n^s} = 0$), and hence for $\sigma > 0$ we obtain the equation

$$1 = s \sum_{n=1}^{n=\infty} \frac{1}{n^{1+s}} + \sum_{n=1}^{n=\infty} \frac{F(n, s)}{n^{2+s}}.$$

---

* W. Schnee, *Über irreguläre Potenzreihen und Dirichletsche Reihen*, Inauguraldissertation, Berlin 1908, p. 53. For approximation on the real axis, the theorem was given by Dirichlet, Sur un théorème relatif aux séries, *Journal de Mathématiques pures et appliquées*, ser. 2, vol. 1, 1856, pp. 80–81. Numerous similar theorems are to be found particularly in Pringsheim, Zur Theorie der Dirichletschen Reihen, *Mathematische Annalen*, vol. 37, 1890, pp. 38–60.

Since, however, $\sum\limits_{n=1}^{n=\infty} \frac{F(n,s)}{n^{2+s}}$ is uniformly convergent in the neighbourhood of the point 0, and since its terms are continuous functions of s which assume the value 0 for $s = 0$, we have

$$\lim_{s=0} \sum_{n=1}^{n=\infty} \frac{F(n,s)}{n^{2+s}} = 0\,.$$

Consequently, for s tending to 0 under the restriction $\sigma > 0$, we have

$$\lim_{s=0} s \sum_{n=1}^{n=\infty} \frac{1}{n^{1+s}} = 1\,.$$

2. We consider next the general case where the a_n are subjected only to the condition $\lim\limits_{n=\infty} a_n = A$. Since s is here restricted to a limited angle, we can obviously find a constant K such that $\frac{|s|}{\sigma} < K$. Having chosen an arbitrarily small ε, we can next find an integer $N = N(\varepsilon)$ such that for all $n \geqq N$

$$|a_n - A| < \frac{\varepsilon}{4\,K}.$$

From this, it follows that

$$\left| \sum_{n=N}^{n=\infty} \frac{s(a_n - A)}{n^{1+s}} \right| < \frac{\varepsilon}{4\,K} \cdot \frac{|s|}{\sigma} \sum_{n=1}^{n=\infty} \frac{\sigma}{n^{1+\sigma}} < \frac{\varepsilon}{4\,K} \cdot K \cdot 2 = \frac{\varepsilon}{2}$$

provided that s, lying in the angle, is sufficiently near to 0 (i.e. $|s| < \delta_1$). In fact, the sum $\sum\limits_{n=1}^{n=\infty} \frac{\sigma}{n^{1+\sigma}}$ has the limit 1 for $\sigma = 0$ and hence is smaller than 2 for sufficiently small values of σ.

However, for sufficiently small s ($|s| < \delta_2$), the sum of the first $N-1$ terms

$$\sum_{n=1}^{n=N-1} \frac{s(a_n - A)}{n^{1+s}}$$

is also numerically smaller than $\frac{\varepsilon}{2}$.

Consequently we have, under the aforementioned conditions on s (namely the restriction to the angle and $|s| < \delta$ ($\delta < \delta_1$ and $\delta < \delta_2$))

$$\left| \sum_{n=1}^{n=\infty} \frac{s(a_n - A)}{n^{1+s}} \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,;$$

i.e., we have the relation

$$\lim_{\substack{s=0 \\ \text{(in angle)}}} s \sum_{n=1}^{n=\infty} \frac{a_n - A}{n^{1+s}} = 0 . \tag{10}$$

As shown above, however,

$$\lim_{\substack{s=0 \\ (\sigma>0)}} s \sum_{n=1}^{n=\infty} \frac{A}{n^{1+s}} = A , \tag{11}$$

and hence, upon adding (10) and (11), we find that

$$\lim_{\substack{s=0 \\ \text{(in angle)}}} s \sum_{n=1}^{n=\infty} \frac{a_n}{n^{1+s}} = A . \qquad \text{q.e.d.}$$

We mention also the following theorem which treats the behaviour of the function represented by a Dirichlet series when the argument s converges to certain points of the boundary of convergence $\sigma = \lambda_0$.

Theorem X.* *Let* $f(s) = \sum \frac{a_n}{n^s}$ *be a Dirichlet series with the abscissa of convergence* λ_0. *When* s_0 *is a point on the boundary of convergence* $\sigma = \lambda_0$, *let us put*

$$S_n^{(0)} = \sum_{m=1}^{m=n} \frac{a_m}{m^{s_0}} ; S_n^{(1)} = \sum_{m=1}^{m=n} S_m^{(0)} ; \cdots ; S_n^{(r)} = \sum_{m=1}^{m=n} S_m^{(r-1)} ; \cdots .$$

Then, if $\lim S_n^{(0)} = A$ (i.e., if $\sum \frac{a_n}{n^{s_0}}$ is convergent with the sum A), *or if merely for some* r

$$\lim_{n=\infty} \frac{S_n^{(r)} \cdot r!}{n^r} = A ,$$

we have the equation

$$\lim_{s=s_0} f(s) = A ,$$

where s, *in converging to* $s_0 = \sigma_0 + it_0$, *is restricted to an angle with vertex at* s_0 *extending into the half-plane of convergence so that the boundary lines form angles between* $+\frac{\pi}{2}$

---

* Theorem X, for the case where $\sum \frac{a_n}{n^{s_0}}$ converges, was proved by Cahen, l.c., pp. 86–87. (For approximation along the horizontal line $t = t_0$, however, the theorem was proved by Dedekind and appears in his edition of Dirichlet's lectures, *Vorlesungen über Zahlentheorie*, l.c., pp. 374–375.) In its general form, Theorem X is due to W. Schnee, Dissertation, l.c., p. 54. Theorem X forms the complete analogue of a well-known theorem on power series, which is due to Abel, Frobenius, and Hölder, and which will be considered in the introduction to Part Two. We shall not pause here to give a proof of Theorem X (such a proof can be found in Schnee, l.c.) since later on (Part Two, § 6) we shall prove a theorem which contains Theorem X as a very special case.

(excl.) and $-\frac{\pi}{2}$ *(excl.) with the horizontal half-line running to the right from the point* s_0.

We now pass from this to a closer investigation of the regular analytic function defined by a Dirichlet series in its region of convergence and, more specifically, to a consideration of the behaviour of this function when the ordinate t tends to infinity.

Concerning the behaviour of the function in the region of absolute convergence of the series, we have the following evident theorem:

Theorem XI. *If the Dirichlet series* $f(s) = \sum \frac{a_n}{n^s}$ *has the abscissa of absolute convergence* l, *then, for* $\sigma \geqq l+\varepsilon$ (ε an arbitrarily small positive number),

$$|f(s)| \leqq K,$$

where $K = K(\varepsilon)$ *denotes a constant independent of* σ *and* t.

Proof. For $\sigma \geqq l+\varepsilon$ we have

$$|f(s)| \leqq \sum_{n=1}^{n=\infty} \left|\frac{a_n}{n^s}\right| \leqq \sum_{n=1}^{n=\infty} \frac{|a_n|}{n^{l+\varepsilon}} = K. \qquad \text{q.e.d.}$$

Before we pass to the study of what may be called the order of magnitude of the function in the region of conditional convergence, we interpolate the following remark, which defines an abbreviated notation introduced by Bachmann and Landau:

If $g(x)$ *is a positive function defined for all real values of* x *from a certain value on* (i.e., for $x > x_0$), *and if* $f(x)$ *is a real or complex function, also defined for all real values of* x *from a certain value on, then the notation*

$$f(x) = O(g(x)) \quad (\text{read } O \text{ of } g(x))$$

shall mean that

$$\limsup_{x=\infty} \frac{|f(x)|}{g(x)} \quad \textit{is finite},$$

i.e., that there exist two positive numbers ξ *and* K *such that for* $x \geqq \xi$,

$$|f(x)| < K \cdot g(x).^*$$

* In the sequel, when we say that $f(s) = f(\sigma+it)$ is equal to $O(|t|)$ for $\sigma = \sigma_0$, for instance, this relation refers to the numerical value of the complex component of $s = \sigma+it$, i.e., the relation '$f(s) = O(|t|)$ for $\sigma = \sigma_0$' shall indicate the existence of two numbers T and K such that for $\sigma = \sigma_0$ and all $|t| \geqq T$

$$|f(s)| < K \cdot |t|.$$

When we say that $f(s)$ is equal to $O(g(\sigma, |t|))$ for $\sigma_1 \geqq \sigma \geqq \sigma_2$, we mean that this relation holds

Example. By $O(1)$ we mean any function of x which is defined for sufficiently large values of x and which either has a finite limit for $x = +\infty$ or is at least bounded for all sufficiently large x.

Similarly, *when $f(x)$ and $g(x)$ are functions defined from a certain x on and $g(x)$ is positive for sufficiently large values of x, we write*, following Landau,

$$f(x) = o(g(x))$$

if

$$\lim_{x=\infty} \frac{f(x)}{g(x)} = 0\,.$$

Applying this notation we have the following theorem:*

Theorem XII. *Let $\sum \frac{a_n}{n^s}$ be a Dirichlet series with the abscissa of absolute convergence l and the abscissa of convergence $\lambda_0 < l$; then, for $l+\varepsilon \geqq \sigma \geqq \lambda_0+\varepsilon\,(\varepsilon > 0)$ we have*

$$f(s) = O\left(|t|^{\frac{l-\sigma+\varepsilon}{l-\lambda_0}}\right). \tag{12}$$

Proof. If we put $S_n = \sum_{m=1}^{m=n} \frac{a_m}{m^{\lambda_0+\frac{\varepsilon}{2}}}$, we find, since $\sum \frac{a_n}{n^s}$ is convergent at the point $s_0 = \lambda_0+\frac{\varepsilon}{2}$, that S_n is numerically smaller than a constant K for all n. Hence, as shown on page 6, $f(s)$ can, for $\sigma > \lambda_0+\frac{\varepsilon}{2}$, be represented by the absolutely convergent series

$$\sum_{n=1}^{n=\infty} S_n \left(\frac{1}{n^{s-s_0}} - \frac{1}{(n+1)^{s-s_0}}\right).$$

From this and the fact that for $\sigma > \lambda_0+\frac{\varepsilon}{2}$

$$\left|\frac{1}{n^{s-s_0}} - \frac{1}{(n+1)^{s-s_0}}\right| = |s-s_0| \cdot \left|\int_n^{n+1} \frac{dx}{x^{1+s-s_0}}\right| \leqq |s-s_0| \frac{1}{n^{1+\sigma-s_0}}$$

---

uniformly for σ belonging to the interval σ_1, σ_2, i.e., it indicates the existence of two numbers T and K such that

$$|f(s)| < K \cdot g(\sigma, |t|)$$

for all s for which $\sigma_1 \geqq \sigma \geqq \sigma_2$ and $|t| \geqq T$.

* Landau, Über das Konvergenzproblem der Dirichlet'schen Reihen, *Rendiconti del Circolo Matematico di Palermo*, vol. 28, 1909, p. 121. As one sees, the proof of Theorem XII given in the text is essentially based on a general function-theoretic theorem due to Lindelöf. A more direct, but also more complicated, proof, which avoids the use of Lindelöf's theorem, can be found in Landau, l. c.

it follows that, for $\sigma \geqq \lambda_0+\varepsilon = s_0+\frac{\varepsilon|}{2}$, we have

$$|f(s)| \leqq \sum_{n=1}^{n=\infty} |S_n| \left| \frac{1}{n^{s-s_0}} - \frac{1}{(n+1)^{s-s_0}} \right| \leqq K \cdot |s-s_0| \sum_{n=1}^{n=\infty} \frac{1}{n^{1+\frac{\varepsilon}{2}}} = K_1 \cdot |s-s_0| \, .$$

The last inequality shows immediately that for $l+\varepsilon \geqq \sigma \geqq \lambda_0+\varepsilon$, we have the relation

$$f(s) = O(|t|) \, .$$

Furthermore we have $f(s) = O(1)$ for $\sigma = l+\varepsilon$.

Now, Lindelöf* has proved the following interesting function-theoretic theorem: *If $\sigma_1 < \sigma_2$ and if the analytic function $f(s)$ has the following properties*:

1. *$f(s)$ is regular for $\sigma_1 \leqq \sigma \leqq \sigma_2$*;

2. *$f(s) = O(|t|^k)$ for $\sigma = \sigma_1$, where $k \gtreqless 0$ is constant*;

3. *$f(s) = O(1) = O(|t|^0)$ for $\sigma = \sigma_2$*, and

4. *there exists a positive constant c such that $f(s) = O(|t|^c)$ for $\sigma_1 \leqq \sigma \leqq \sigma_2$*;

then for $\sigma_1 \leqq \sigma \leqq \sigma_2$ we have

$$f(s) = O\left(|t|^{k\frac{\sigma_2-\sigma}{\sigma_2-\sigma_1}}\right) \text{ (the exponent varies linearly from } k \text{ to } 0).$$

This theorem of Lindelöf is applied to our function $f(s) = \sum \frac{a_n}{n^s}$, which obviously satisfies all four conditions when $\sigma_1 = \lambda_0+\varepsilon$, $\sigma_2 = l+\varepsilon$, and $k = c = 1$. We find immediately that, for $l+\varepsilon \geqq \sigma \geqq \lambda_0+\varepsilon$, we have

$$f(s) = O\left(|t|^{\frac{l+\varepsilon-\sigma}{l-\lambda_0}}\right). \qquad \text{q.e.d.} \quad (12)$$

From Theorem XII, we infer in particular that $\frac{f(s)}{s}$ tends uniformly to 0 when the variable s tends to infinity in such a way that the condition $\sigma \geqq \lambda_0+\varepsilon_1$ is fulfilled. In fact, (12) shows (when ε is chosen smaller than the given number ε_1) that this is the case for $l+\varepsilon \geqq \sigma \geqq \lambda_0+\varepsilon_1$, and $|f(s)|$ is bounded for $\sigma \geqq l+\varepsilon$.

We shall now pass to a theorem which, although in its nature essentially different from the two preceding theorems, may nevertheless be said to treat the behaviour of a function represented by a Dirichlet series for infinitely large values of the ordinate t.

---

* Quelques remarques sur la croissance de la fonction $\zeta(s)$, *Bulletin des Sciences mathématiques*, ser. 2, vol. 32, 1908, pp. 346–348. See also Landau, Über das Konvergenzproblem der Dirichlet'schen Reihen, l.c., p. 146.

Theorem XIII.* *If* $f(s) = \sum_{n=1}^{n=\infty} \frac{a_n}{n^s}$ *is a Dirichlet series with abscissa of convergence* λ_0, *then for every* $\sigma_0 > \lambda_0$, *the limit*

$$\lim_{T=+\infty} \frac{1}{T}\int_0^T f(\sigma_0+it)dt$$

exists and is equal to a_1 (the first coefficient of the series).

Proof. Since the series $f(\sigma_0+it) = \sum \frac{a_n}{n^{\sigma_0+it}}$ is uniformly convergent for $0 \leqq t \leqq T$, we get

$$\int_0^T f(\sigma_0+it)dt = \sum_{n=1}^{n=\infty} \frac{a_n}{n^{\sigma_0}} \int_0^T \frac{dt}{n^{it}} = a_1 T + i \sum_{n=2}^{n=\infty} \frac{a_n}{n^{\sigma_0}\cdot \log n}\left(\frac{1}{n^{iT}}-1\right),$$

and hence (since, by a well-known theorem, the convergence of $\sum_{n=2}^{n=\infty} \frac{a_n}{n^s}$ for $\sigma > \lambda_0$ implies the convergence of $\sum_{n=2}^{n=\infty} \frac{a_n}{n^s\cdot \log n}$ for $\sigma > \lambda_0$):

$$\int_0^T f(\sigma_0+it)dt = a_1 T + i \sum_{n=2}^{n=\infty} \frac{b_n}{n^{\sigma_0+iT}} + c, \tag{13}$$

where $b_n = \frac{a_n}{\log n}$ and c is independent of T.

The convergence of $g(s) = \sum_{n=2}^{n=\infty} \frac{b_n}{n^s}$ for $\sigma > \lambda_0$ implies however, by Theorem XII, that

$$g(\sigma_0+iT) = \sum_{n=2}^{n=\infty} \frac{b_n}{n^{\sigma_0+iT}} = o(T);$$

hence we infer from formula (13), after division by T and a subsequent passage to the limit, that

$$\lim_{T=+\infty} \frac{1}{T}\int_0^T f(\sigma_0+it)dt = a_1 . \qquad \text{q.e.d.}$$

We shall conclude this first part of the dissertation with a detailed discussion of the so-called *convergence problem* for Dirichlet series, a problem which on account

* Theorem XIII is due to Landau, Beiträge zur analytischen Zahlentheorie, *Rendiconti del Circolo Matematico di Palermo*, vol. 26, 1908, p. 264; (for $\sigma > l$, however, l being the abscissa of absolute convergence, the theorem was proved in Hadamard, Théorème sur les séries entières, *Acta Mathematica*, vol. 22, 1899, pp. 60–63). For the generalization of this theorem to combinations of more than one Dirichlet series, see Landau, l.c.—In Part Two it will be shown how Theorem XIII may be generalized to include certain domains outside the region of convergence of the series.

of its importance as well as on account of the special, and hitherto unresolved, difficulties which its treatment presents, has in recent times more and more attracted the attention of mathematicians working in this field.

Let $f(s) = \sum \frac{a_n}{n^s}$ be a given Dirichlet series. One may then naturally ask: *how far will this series converge?*

Such a question may be understood in various ways. Since the series $\sum \frac{a_n}{n^s}$ is completely and uniquely determined by its coefficients $a_1, a_2, \ldots, a_n, \ldots$ one may ask: how is the boundary of convergence determined, or, what comes to the same thing, how is the abscissa of convergence λ_0 determined, by the coefficients of the series? This question, which is analogous to the question for a power series $\sum a_n x^n$ about the radius of convergence as a function of its coefficients, is solved by Cahen's formula for λ_0, mentioned on page 16. The solution is analogous to the corresponding solution for power series as expressed in the formula for the radius of convergence due to Cauchy and Hadamard.

The question: how far does a given Dirichlet series converge, may, however, also be understood in quite another way, namely: how is the boundary of convergence $\sigma = \lambda_0$ determined, not from the Dirichlet series itself, (i.e., from its coefficients), but on the contrary from the mere knowledge of the analytic function represented by the series? The problem in this formulation has been called the convergence problem for Dirichlet series.

For a power series, the analogous question has the well-known answer that the circle of convergence contains just that singular point of the function which lies nearest to the origin. In analogy with this, one might expect that a Dirichlet series would converge just as far as the function represented by it is regular, or, in other words, that there should be singular points of the function on, or infinitely close to the left of the line of convergence $\sigma = \lambda_0$.*

* For a power series, it is impossible for the function (or rather function element) represented by the series to be regular at all points of the circle of convergence while infinitely close to and outside the circle of convergence there are singular points of the function; that such a phenomenon would involve a contradiction follows immediately from the fact that the singular points outside the circle would have at least one cluster point lying on the circumference, and that this point as a cluster point of singular points would itself be a singular point.—For a Dirichlet series, however, the situation is different; here, as suggested in the text, the possibility can very well be imagined that the analytic function represented by the series is regular not only for $\sigma > \lambda_0$, but also at all points of the line of convergence $\sigma = \lambda_0$. For, the line of convergence not being a bounded curve, it might be that no finite point of the line $\sigma = \lambda_0$ is a cluster point of singular points, but that these so to say cluster around the

As one immediately sees, however, this is not always the case. Thus the series $\sum \frac{(-1)^{n+1}}{n^s}$ has the abscissa of convergence $\lambda_0 = 0$, while the function $\zeta(s)(1-2^{1-s})$ represented by the series is regular beyond the boundary of convergence $\sigma = 0$. Indeed, it is an integral function.

'infinite point' of the line of convergence.—Since the question whether there really exist Dirichlet series of the type mentioned is not treated in the literature, and since it seems to be of some interest that this point should be clarified, I shall give below a brief proof of the following theorem from which it immediately follows that the above existence question must be answered affirmatively:

Theorem. *Let $s_1 = \sigma_1 + it_1, \ldots, s_p = \sigma_p + it_p, \ldots$ be an infinite sequence of complex numbers for which $0 < \sigma_1 < \sigma_2 \ldots < \sigma_p \ldots (\lim\limits_{p=\infty} \sigma_p = 1)$ and $0 < t_1 < t_2 \ldots < t_p \ldots$ $(\lim\limits_{p=\infty} t_p = +\infty)$. Then there exists a Dirichlet series with abscissa of convergence $\lambda_0 = 1$ such that the function represented by it is a meromorphic function in the whole finite plane with the points s_p $(p = 1, 2, 3, \ldots)$ as its only singularities (poles of the first order).*

Proof. For all $p = 1, 2, \ldots$, let

$$f_p(s) = \sum_{n=1}^{n=\infty} \frac{a_{p,n}}{n^s} = \sum_{n=1}^{n=\infty} \frac{n^{s_p-1}}{n^s} = \sum_{n=1}^{n=\infty} \frac{1}{n^{s-s_p+1}} = \zeta(s-s_p+1).$$

The series $f_p(s) = \sum \frac{a_{p,n}}{n^s}$ is absolutely convergent for $\sigma > \sigma_p$ (< 1), from which follows the existence of a constant K_p such that $\sum \left|\frac{a_{p,n}}{n^s}\right| < K_p$ for $\sigma \geqq 1$; furthermore, $f_p(s) = \zeta(s-s_p+1)$ is, as is well known, a function meromorphic in the whole plane with the single pole (of order 1) $s = s_p$.

Finally, let k_p be determined so that the absolute value of $f_p(s)$ is smaller than k_p for $|s| < \frac{1}{2} \cdot |s_p + s_{p-1}|$, throughout which domain $f_p(s)$ is regular.

Then we consider the function

$$F(s) = \sum_{p=1}^{p=\infty} \varepsilon_p \cdot f_p(s),$$

where the numbers $\varepsilon_p \neq 0$ are subjected to the conditions

$$|\varepsilon_p| < \frac{c_p}{K_p} \text{ and } |\varepsilon_p| < \frac{c_p}{k_p},$$

the c_p being positive numbers such that $\sum c_p$ is convergent.

1) We can then prove that $F(s)$ represents a function meromorphic in the whole finite plane with the points $s_1, s_2, \ldots, s_p, \ldots$ as its only poles (of order 1).

This follows immediately from the fact that inside each of the circles C_P $(P = 1, 2, \ldots)$ with center at the origin and radius $R_P = \frac{1}{2}|s_P + s_{P-1}|$ we have the equation

$$F(s) = \sum_{p=1}^{p=P-1} \varepsilon_p f_p(s) + \sum_{p=P}^{p=\infty} \varepsilon_p f_p(s).$$

Here the first series on the right-hand side of the equality sign is seen to represent a meromorphic function with poles at $s_1, s_2, \ldots, s_{P-1}$ only. The terms $\varepsilon_p f_p(s)$ of the second series are all regular for $|s| \leqq R_P$

(If, however, we restrict ourselves to considering only series with *positive* coefficients, as for instance $\zeta(s) = \sum \frac{1}{n^s}$, the situation is as simple as for power series. For such series, Landau* has proved that the point where the real axis and the line of convergence $\sigma = \lambda_0$ intersect is always a singular point for the function represented by the series, whether or not the series converges at the point $s = \lambda_0$. That the Dirichlet series with positive coefficients form a very special sub-class of the class of all Dirichlet series appears immediately from the fact that for series with positive coefficients, the existence of a two-dimensional region—a strip—inside which the series is conditionally convergent is excluded.)

It has been shown above that it is not possible, by using only the fact that the function $f(s)$ defined by a Dirichlet series $\sum \frac{a_n}{n^s}$ is regular for $\sigma > \sigma_0$, to infer the convergence of $\sum \frac{a_n}{n^s}$ for $\sigma > \sigma_0$. In other words: it has been shown that this condition, which is necessary in consequence of Theorem IV, is by no means sufficient for the convergence of the series in the half-plane $\sigma > \sigma_0$. This shows that if we want to determine the line of convergence only from the knowledge of the analytic behaviour of the function $f(s)$, it is necessary to take into consideration characteristic properties of the function other than those which are given through the mere knowledge of the regularity or singularity of the function. The results obtained above, in parti-

---

and in consequence of the inequality

$$|\varepsilon_p f_p(s)| \leqq |\varepsilon_p|\, k_p < \frac{c_p}{k_p} \cdot k_p = c_p,$$

valid for $p \geqq P$ and $|s| \leqq R_P$, the second series is uniformly convergent inside the circle C_P; it will thus inside this circle represent a regular analytic function.

2) Furthermore, $F(s)$ can be represented for $\sigma \geqq 1$ by a convergent (and even absolutely convergent) Dirichlet series $\sum \frac{a_n}{n^s}$, namely by the series

$$F(s) = \sum_{p=1}^{p=\infty} \varepsilon_p \cdot f_p(s) = \sum_{p=1}^{p=\infty} \varepsilon_p \sum_{n=1}^{n=\infty} \frac{a_{p,n}}{n^s} = \sum_{n=1}^{n=\infty} \frac{1}{n^s} \sum_{p=1}^{p=\infty} \varepsilon_p \cdot a_{p,n} = \sum_{n=1}^{n=\infty} \frac{a_n}{n^s}.$$

In fact, the change of order of summation (with respect to p and n) is permissible for $\sigma \geqq 1$, since for $\sigma \geqq 1$

$$\sum_{p=1}^{p=\infty} |\varepsilon_p| \sum_{n=1}^{n=\infty} \left|\frac{a_{p,n}}{n^s}\right| < \sum_{p=1}^{p=\infty} |\varepsilon_p| \cdot K_p < \sum_{p=1}^{p=\infty} \frac{c_p}{K_p} \cdot K_p = \sum_{p=1}^{p=\infty} c_p \text{ (which is convergent).}$$

* Über einen Satz von Tschebyschef, *Mathematische Annalen*, vol. 61, 1905, p. 536.

cular Theorem XII, lead naturally, as Franel* has emphasized, to an attempt to solve the convergence problem by taking into account the behaviour of the function for infinitely large values of the ordinate t.

The first successful attempt in this direction was made by Landau,† whose investigations were later carried on by W. Schnee,‡ and most recently taken up again by Landau§ himself.

As their main result, these mathematicians have found the following interesting theorem:||

Theorem XIV. *Let* $\sum \frac{a_n}{n^s}$ *be a Dirichlet series whose coefficients* a_n *satisfy the condition* $\lim\limits_{n=\infty} \frac{a_n}{n^\delta} = 0$ *for every* $\delta > 0$, *from which it follows that the series is absolutely convergent at least for* $\sigma > 1$; *furthermore, let the function* $f(s)$ *defined by the series be regular for* $\sigma > \eta$ $(\eta < 1)$ *and for* $\sigma > \eta$ *satisfy the relation*

$$f(s) = O(|t|^k) \quad (k \geqq 0)\,.$$

Then $\sum \frac{a_n}{n^s}$ *is convergent for* $\sigma > \frac{\eta+k}{1+k}$ (< 1).

(The special role of the point $s = 1$ in Theorem XIV is of course due only to the formulation of the theorem and may be eliminated by a simple transformation of variable $s = s'+c$.)

Theorem XIV contains as a special case (corresponding to $k = 0$) the following noteworthy theorem: ¶

---

* J. Franel, *L'intermédiaire des Mathématiciens*, vol. 3, 1896, p. 103, adds to a formulation of the convergence problem the following remark: 'When the straight boundary line (the line of convergence) does not pass through any singular point of the function $f(s)$ it will be reasonable to investigate how this function behaves when the point s tends to infinity along a line parallel to the imaginary axis and which lies in that part of the plane where the series $\sum \frac{a_n}{n^s}$ does not converge absolutely.'

† Beiträge zur analytischen Zahlentheorie, l.c., pp. 252–255.

‡ Zum Konvergenzproblem der Dirichletschen Reihen, *Mathematische Annalen*, vol. 66, 1909, pp. 337–349.

§ Über das Konvergenzproblem der Dirichlet'schen Reihen, l.c., pp. 113–151.

|| For the proof of this deep-lying theorem, we refer to Landau, Über das Konvergenzproblem, l.c.

¶ Schnee, l.c. Theorem XV may also be stated in the following more general form, as noted by Schnee: *If* $f(s)$ *for* $\sigma > \sigma_0$ *is regular and equal to* $O(|t|^\varepsilon)$ *for every* $\varepsilon > 0$, *however small, then* $\sum \frac{a_n}{n^s}$ *is convergent for* $\sigma > \sigma_0$.

Theorem XV. *If $f(s)$, which is assumed to be represented in a certain half-plane by a Dirichlet series $\sum \frac{a_n}{n^s}$, is regular and equal to $O(1)$ for $\sigma > \sigma_0$, then $\sum \frac{a_n}{n^s}$ is convergent for $\sigma > \sigma_0$.*

As will be seen, Theorem XIV (and its corollary Theorem XV) give *sufficient* conditions for the convergence of the series in a certain half-plane in terms of simple analytic properties of the function represented by the series; this theorem, however, gives no method for the *exact* determination of the line of convergence $\sigma = \lambda_0$; in other words, Theorem XIV can by no means be considered to be a solution of the convergence problem for Dirichlet series, but is only a step towards this goal.

Before we pass to the discussion of certain investigations undertaken by the author in order to clarify what kind of results might possibly be obtained through a further advance along the road first taken by Landau, we shall interpolate the following remark, through which a convenient abbreviated notation will be introduced.

Let $f(s)$ be a function which we assume to be defined in a certain half-plane (for $\sigma > \sigma'$) by a convergent Dirichlet series $\sum \frac{a_n}{n^s}$; and for $\sigma > \sigma_0$ ($\sigma_0 < \sigma'$) let $f(s)$ be both regular and of finite order of magnitude with respect to the ordinate t. The last expression means that corresponding to an arbitrarily small ε, there shall exist a constant $c = c(\varepsilon)$ such that $f(s) = O(|t|^c)$ for $\sigma \geqq \sigma_0+\varepsilon$.

Then $f(s) = f(\sigma+it)$ cannot possibly be equal to $O(|t|^{-\delta})$ for any value of $\sigma > \sigma_0$ and any positive number δ, however small; indeed, since $f(s) = O(1)$ for $\sigma > \sigma'+1+\varepsilon$, it would follow at once from Lindelöf's theorem (page 22) that for all sufficiently large values of σ,

$$\lim_{t=\infty} f(\sigma+it) = 0.$$

This relation contradicts the inequality, found on page 16 and valid for sufficiently large values of σ and all t:

$$|f(s)| > \tfrac{1}{2}\frac{|a_N|}{N^\sigma},$$

in which a_N denotes the first non-vanishing coefficient in the series $\sum \frac{a_n}{n^s}$.

By $\mu = \mu(\sigma)$ (defined for every $\sigma > \sigma_0$), we shall understand the finite and, in view of the preceding remark, non-negative real number (determined by a so-called Dedekind cut) *which satisfies the following two conditions:*

1. *$f(\sigma+it) = O(|t|^{\mu+\varepsilon})$ for all $\varepsilon > 0$,*

2. *$f(\sigma+it)$ is not equal to $O(|t|^{\mu-\varepsilon})$ for any $\varepsilon > 0$, however small.*

Since, in consequence of Theorem XI, $f(s)$ is equal to $O(1)$ for $\sigma > l+\varepsilon$ (l being the abscissa of absolute convergence of the series) and $\mu(\sigma)$ is non-negative, we see that $\mu(\sigma) = 0$ for $\sigma > l$.

Furthermore, it can immediately be inferred from Lindelöf's theorem:

1. *that $\mu(\sigma)$ is a continuous function of $\sigma > \sigma_0$,*

2. *that $\mu(\sigma)$* (which for sufficiently large σ is equal to 0) *as soon as it has taken on positive values will be strictly increasing when σ decreases, i.e., that if $\mu(\sigma_1) > 0$ and $\sigma_1 > \sigma_2 > \sigma_0$ then $\mu(\sigma_2) > \mu(\sigma_1)$,* and

3. *that $f(s)$ for $\sigma \geqq \sigma_1$ ($> \sigma_0$) is equal to $O(|t|^{\mu(\sigma_1)+\varepsilon})$ for every $\varepsilon > 0$, but not equal to $O(|t|^{\mu(\sigma_1)-\varepsilon})$ for any $\varepsilon > 0$.*

Using the notation introduced above, we can now formulate Theorem XV (in the more general form indicated in note ¶ on page 27) as follows:

If $f(s)$ is regular and of finite order of magnitude for $\sigma > \sigma_0$, and if $\mu(\sigma) = 0$ for $\sigma > \sigma_0$, then $\sum \frac{a_n}{n^s}$ is convergent for $\sigma > \sigma_0$.

On the other hand, it follows immediately from Theorem XII *that $\sum \frac{a_n}{n^s}$ cannot converge for $\sigma > \sigma_0$ unless $f(s)$ is both regular and of finite order of magnitude and also $\mu(\sigma) \leqq 1$ for $\sigma > \sigma_0$.* Hence the following question arises naturally:*

Does there exist a number g ($0 \leqq g \leqq 1$) such that a Dirichlet series $\sum \frac{a_n}{n^s}$ is convergent for $\sigma > \sigma_0$ when and only when the following two conditions are fulfilled:

1. *$f(s)$ is regular and of finite order of magnitude for $\sigma > \sigma_0$,*

2. *$\mu(\sigma) \leqq g$ for $\sigma > \sigma_0$.*

This question must, contrary to what might be expected, *be answered negatively.* In order to avoid unnecessary repetitions, however, we shall, before proving our statement, recast the question in a much more general, indeed in a certain sense, its most general form. It then divides naturally into the following two separate questions 1 and 2.

---

* For the solution of this question, we find in the literature only the contribution that *if* there exists such a constant g, it must necessarily be equal to $\frac{1}{2}$. This follows from investigations of the special series $\sum \frac{(-1)^{n+1}}{n^s}$ for which series the functional equation of the ζ-function gives a means of discussing the function $\zeta(s)(1-2^{1-s})$ represented by the series.

Question 1. *Does there exist a number $\gamma > 0$ such that the assumptions:*

1. $f(s)$ *is regular and of finite order of magnitude for* $\sigma > \sigma_0$, and
2. $\mu(\sigma) \leqq \gamma$ *for* $\sigma > \sigma_0$,

imply the convergence of the series for $\sigma > \sigma_0$.

Question 2. *Does there exist a number $\Gamma < 1$ such that the assumption:* $\sum \frac{a_n}{n^s}$ *is convergent for $\sigma > \sigma_0$, implies that $\mu(\sigma) \leqq \Gamma$ for $\sigma > \sigma_0$.*

Like the more special question above, *these questions* as we shall now prove *must be both answered negatively.*

The non-existence of a number $\gamma > 0$ satisfying the conditions in Question 1 results immediately from the following theorem:

Theorem XVI. *There exists a Dirichlet series* $f(s) = \sum \frac{a_n}{n^s}$ *with abscissa of convergence* 0 *representing a function $f(s)$ which is regular and of finite order of magnitude for $\sigma > -1$, and for which*

$$\mu(\sigma) = \begin{cases} 0 \text{ for } \sigma \geqq 0 \\ -\sigma \text{ for } 0 \geqq \sigma > -1 . \end{cases}$$

Proof. Let $p_1 < p_2 \ldots < p_m < p_{m+1} \ldots$ be a strictly increasing sequence of positive numbers satisfying the following conditions:

1. $p_{m+1} - p_m > 1$ for all $m = 1, 2, 3, \ldots$,
2. $\sum_{m=1}^{m=n-1} p_m^\sigma = o(p_n^\sigma)$ for every $\sigma > 0$,* and
3. $\sum_{m=1}^{m=\infty} \frac{1}{p_m^\sigma}$ is convergent for every $\sigma > 0$, and the relation $\sum_{m=n+1}^{m=\infty} \frac{1}{p_m^\sigma} = o\left(\frac{1}{p_n^\sigma}\right)$

holds for every $\sigma > 0$.

Then we consider the Dirichlet series

$$\sum_{n=1}^{n=\infty} \frac{a_n}{n^s} = \frac{1}{p_1^s} - \frac{1}{(p_1+1)^s} + \frac{1}{p_2^s} - \frac{1}{(p_2+1)^s} + \ldots + \frac{1}{p_m^s} - \frac{1}{(p_m+1)^s} + \ldots,$$

and we shall prove that it satisfies all the conditions of Theorem XVI.

---

* By $\alpha_n = O(\beta_n)$ and $\alpha_n = o(\beta_n)$, where β_n is positive, we understand that

$$\frac{|\alpha_n|}{\beta_n} < \text{const., and } \lim_{n=\infty} \frac{\alpha_n}{\beta_n} = 0, \text{ respectively.}$$

From the assumption that $\sum \frac{1}{p_m^\sigma}$ converges for $\sigma > 0$, it follows immediately that $\sum \frac{a_n}{n^s}$ converges absolutely for $\sigma > 0$. On the other hand, $\sum \frac{a_n}{n^s}$ cannot converge for $\sigma \leqq 0$, since for such values of σ the terms of the series do not have the limit 0 as the index n tends to infinity. Consequently, we have $l = \lambda_0 = 0$.

The equation

$$f(s) = \sum_{m=1}^{m=\infty} \left[\frac{1}{p_m^s} - \frac{1}{(p_m+1)^s}\right]$$

has so far been proved only for $\sigma > 0$. However, for $\sigma > -1$

$$\left|\frac{1}{p_m^s} - \frac{1}{(p_m+1)^s}\right| = \left|s\int_{p_m}^{p_m+1} \frac{dx}{x^{1+s}}\right| \leqq |s|\,\frac{1}{p_m{}^{1+\sigma}};$$

the series

$$\sum_{m=1}^{m=\infty} \left[\frac{1}{p_m^s} - \frac{1}{(p_m+1)^s}\right],$$

the terms of which are integral functions, is therefore uniformly convergent for $\sigma > -1+\varepsilon$ and $|s| < k$. Hence the function $f(s)$ is everywhere regular for $\sigma > -1$, and we have, for $\sigma > -1+\varepsilon$,

$$|f(s)| < K \cdot |s|,$$

whence, for $\varepsilon \geqq \sigma \geqq -1+\varepsilon$,

$$f(s) = O(|t|).$$

From this last relation (in conjunction with the relation $f(s) = O(1)$, valid for $\sigma \geqq l+\varepsilon = \varepsilon$) we infer on the one hand that $f(s)$ is of finite order of magnitude for $\sigma > -1$, and on the other hand that for $\varepsilon \geqq \sigma \geqq -1+\varepsilon$

$$f(s) = O(|t|^{-\sigma+\varepsilon}), \tag{14}$$

in consequence of Lindelöf's theorem.

We now pass to the determination of $\mu(\sigma)$ for all $\sigma > -1$.

For $\sigma > l = 0$, we have $\mu(\sigma) = 0$, and hence from the continuity of $\mu(\sigma)$ (for $\sigma > -1$) it follows that $\mu(0) = 0$; and from formula (14) it follows that $\mu(\sigma) \leqq -\sigma$ for $0 > \sigma > -1$. In order to prove that $\mu(\sigma)$ cannot be smaller than $-\sigma$ for $0 > \sigma > -1$ (and consequently that $\mu(\sigma) = -\sigma$) it is evidently sufficient to show that for $0 > \sigma_0 > -1$

$$f(s_n) = f(\sigma_0+ip_n) = K_n \cdot p_n^{-\sigma_0} + o(p_n^{-\sigma_0}), \tag{15}$$

where the numerical value of K_n for all n is larger than a certain constant. Indeed from this it immediately follows that $f(\sigma_0+it)$ cannot be equal to $O(|t|^{-\sigma_0-\varepsilon})$ for any $\varepsilon > 0$ however small and hence that $\mu(\sigma_0) \geqq -\sigma_0$.

The validity of equation (15) is proved in the following way:

$$f(s_n) = f(\sigma_0+ip_n) = \sum_{m=1}^{m=\infty}\left[\frac{1}{p_m^{s_n}} - \frac{1}{(p_m+1)^{s_n}}\right]$$

$$= \sum_{m=1}^{m=n-1}\left[\frac{1}{p_m^{s_n}} - \frac{1}{(p_m+1)^{s_n}}\right] + \left[\frac{1}{p_n^{s_n}} - \frac{1}{(p_n+1)^{s_n}}\right] + \sum_{m=n+1}^{m=\infty}\left[\frac{1}{p_m^{s_n}} - \frac{1}{(p_m+1)^{s_n}}\right]$$

$$= O\left(\sum_{m=1}^{m=n-1}\frac{1}{p_m^{\sigma_0}}\right) + p_n^{-s_n}\left[1-\left(1+\frac{1}{p_n}\right)^{-s_n}\right] + O\left(|s_n|\sum_{m=n+1}^{m=\infty}\frac{1}{p_m^{1+\sigma_0}}\right)$$

$$= o(p_n^{-\sigma_0}) + p_n^{-\sigma_0}\cdot p_n^{-ip_n}\left[1-\left(1+\frac{1}{p_n}\right)^{-\sigma_0}\cdot\left(1+\frac{1}{p_n}\right)^{-ip_n}\right] + o\left(p_n\cdot\frac{1}{p_n^{1+\sigma_0}}\right)$$

$$= p_n^{-\sigma_0}\cdot p_n^{-ip_n}\cdot[1-e^{-i}] + o(p_n^{-\sigma_0}) = K_n\cdot p_n^{-\sigma_0} + o(p_n^{-\sigma_0}). \qquad \text{q.e.d.}$$

The non-existence of a number $\Gamma < 1$ satisfying the conditions in Question 2 appears from the following theorem.

Theorem XVII. *There exists a Dirichlet series $f(s) = \sum \frac{a_n}{n^s}$ with the abscissa of convergence $\lambda_0 = 0$ such that*

$$\mu(\sigma) = \begin{cases} 0 & \textit{for } \sigma \geqq 1 \\ 1-\sigma & \textit{for } 1 \geqq \sigma > 0. \end{cases}$$

Proof. Let

$$\alpha_1 < t_1 < \beta_1 < \gamma_1 \quad < \alpha_2 < t_2 < \beta_2 < \gamma_2 \dots < \alpha_n < t_n < \beta_n < \gamma_n \dots$$

be an infinite sequence of positive integers which satisfy the following conditions:

$$\alpha_n < \sqrt{t_n};$$

$$\beta_n = t_n^{1+\delta_n}, \text{ where } \lim_{n=\infty}\delta_n = 0 \text{ and } \lim_{n=\infty} t_n^{-\delta_n} = 0;$$

$$\gamma_n > t_n^2.$$

Let the coefficients a_m in the series $\sum_{m=1}^{m=\infty}\frac{a_m}{m^s}$ be determined so that, when we put $S_m = a_1+a_2+\dots+a_m$,

$$\begin{aligned} S_m &= 0 && \text{for } \alpha_n \leqq m < \beta_n; \\ S_m &= m^{it_n} && \text{for } \beta_n \leqq m \leqq \gamma_n; \\ S_m &= 1 && \text{for } \gamma_n < m < \alpha_{n+1}. \end{aligned}$$

Since for all $m = 1, 2, \dots$ we have $|S_m| \leqq 1$ and since S_m has no limit for $m = \infty$, the series $\sum a_m = \sum \frac{a_m}{m^0}$ oscillates between finite bounds, and hence $\lambda_0 = 0$.*

---

* If a Dirichlet series oscillates between finite bounds at a certain point, this point must, in view of Theorem II, lie on the line of convergence $\sigma = \lambda_0$.

Since $\lambda_0 = 0$ implies that $l \leqq 1$, we see that $\mu(\sigma) = 0$ at least for $\sigma \geqq 1$. We shall now prove that $\mu(\sigma_0) = 1-\sigma_0$ for $0 < \sigma_0 < 1$. That $\mu(\sigma_0)$ cannot be larger than $1-\sigma_0$ follows at once from Lindelöf's theorem when we recall that $f(s)$ is of finite order of magnitude for $\sigma > \lambda_0 = 0$, that $f(\varepsilon+it) = O(|t|)$, and that $f(1+\varepsilon+it) = O(1)$. Hence, in order to prove that $\mu(\sigma_0)$ is exactly equal to $1-\sigma_0$, it is sufficient to show that

$$f(\sigma_0+it_n) = K \cdot t_n^{1-\sigma_0(1+\delta_n)} + o(t_n^{1-\sigma_0(1+\delta_n)}), \tag{16}$$

where K is a constant different from 0. For this will imply that $f(\sigma_0+it)$ cannot be equal to $O(|t|^{1-\sigma_0-\varepsilon})$ for any $\varepsilon > 0$, however small, and hence that $\mu(\sigma_0) \geqq 1-\sigma_0$.

The validity of equation (16) is proved in the following way: Putting $s_n = \sigma_0+it_n$, we have (as shown on page 6), since $\sum \frac{a_m}{m^s}$ oscillates between finite bounds for $s = 0$,

$$f(s_n) = f(\sigma_0+it_n) = \sum_{m=1}^{m=\infty} S_m \left[\frac{1}{m^{s_n}} - \frac{1}{(m+1)^{s_n}}\right] = (\text{since } S_m = 0 \text{ for } \alpha_n \leqq m < \beta_n)$$

$$\sum_{m=1}^{m=\alpha_n-1} S_m \left[\frac{1}{m^{s_n}} - \frac{1}{(m+1)^{s_n}}\right] + \sum_{m=\beta_n}^{m=\infty} S_m \left[\frac{1}{m^{s_n}} - \frac{1}{(m+1)^{s_n}}\right]. \tag{17}$$

The following equations hold for $\left|\frac{s_n}{m}\right| \leqq 1$ and eo ipso for $m \geqq \beta_n$:

$$\frac{1}{m^{s_n}} - \frac{1}{(m+1)^{s_n}} = \frac{1}{m^{s_n}}\left[1-\left(1+\frac{1}{m}\right)^{-s_n}\right] = \frac{1}{m^{s_n}}\left[1-e^{-s_n \cdot \log\left(1+\frac{1}{m}\right)}\right]$$

$$= \frac{s_n}{m^{1+s_n}} + \frac{s_n^2 \cdot F(n, m)}{m^{2+s_n}},$$

where $F(n, m)$ is numerically smaller than an absolute constant. We thus obtain from (17)

$$f(\sigma_0+it_n) = \sum_{m=1}^{m=\alpha_n-1} S_m \left[\frac{1}{m^{s_n}} - \frac{1}{(m+1)^{s_n}}\right] + \sum_{m=\beta_n}^{m=\gamma_n} S_m \frac{s_n}{m^{1+s_n}}$$

$$+ \sum_{m=\gamma_n+1}^{m=\infty} S_m \frac{s_n}{m^{1+s_n}} + \sum_{m=\beta_n}^{m=\infty} S_m \frac{s_n^2 \cdot F(n, m)}{m^{2+s_n}}$$

$$= O\left(\sum_{m=1}^{m=\alpha_n-1} \frac{1}{m^{\sigma_0}}\right) + s_n \cdot \sum_{m=\beta_n}^{m=\gamma_n} m^{it_n} \frac{1}{m^{1+\sigma_0+it_n}} + O\left(t_n \cdot \sum_{m=\gamma_n+1}^{m=\infty} \frac{1}{m^{1+\sigma_0}}\right) + O\left(t_n^2 \cdot \sum_{m=\beta_n}^{m=\infty} \frac{1}{m^{2+\sigma_0}}\right)$$

$$= O(\alpha_n^{1-\sigma_0}) + s_n \cdot \sum_{m=\beta_n}^{m=\gamma_n} \frac{1}{m^{1+\sigma_0}} + O(t_n \cdot \gamma_n^{-\sigma_0}) + O(t_n^2 \cdot \beta_n^{-1-\sigma_0})$$

$$= O(t_n^{\frac{1}{2}(1-\sigma_0)}) + s_n\left(\sum_{m=\beta_n}^{m=\infty} \frac{1}{m^{1+\sigma_0}} - \sum_{m=\gamma_n+1}^{m=\infty} \frac{1}{m^{1+\sigma_0}}\right) + O(t_n \cdot t_n^{-2\sigma_0}) + O(t_n^2 \cdot t_n^{-(1+\sigma_0)(1+\delta_n)})$$

$$= o(t_n^{1-\sigma_0(1+\delta_n)}) + \frac{\sigma_0+it_n}{\sigma_0}\cdot[\beta_n^{-\sigma_0}+o(\beta_n^{-\sigma_0})+O(\gamma_n^{-\sigma_0})]+o(t_n^{1-\sigma_0(1+\delta_n)})+O(t_n^{1-\sigma_0(1+\delta_n)}\cdot t_n^{-\delta_n})$$

$$= \frac{\sigma_0+it_n}{\sigma_0}\cdot[t_n^{-\sigma_0(1+\delta_n)}+o(t_n^{-\sigma_0(1+\delta_n)})+O(t_n^{-2\sigma_0})]+o(t_n^{1-\sigma_0(1+\delta_n)})$$

$$= \frac{i}{\sigma_0}\cdot t_n^{1-\sigma_0(1+\delta_n)}+o(t_n^{1-\sigma_0(1+\delta_n)})\,. \qquad \text{q.e.d.}$$

Having thus given a complete answer to Questions 1 and 2, we shall now use Theorems XVI and XVII to prove the following theorem, which is of importance in understanding the convergence problem for Dirichlet series:

Theorem XVIII. *Let $f(s)$ be an analytic function of which we know that it is defined in a certain half-plane by a convergent Dirichlet series; moreover, let the following properties of the function $f(s)$, and no others, be known:*

1. *The knowledge of the number $\sigma_0 \geqq -\infty$, determined by a Dedekind cut, which is the smallest of all real numbers σ' for which $f(s)$ for $\sigma > \sigma'$ is regular and of finite order of magnitude with respect to the ordinate t,* and

2. *The knowledge for all $\sigma > \sigma_0$ of the value of the μ-function corresponding to $f(s)$ and defined for $\sigma > \sigma_0$.*

In general, the function $f(s)$ is not characterized sufficiently by these properties to determine the boundary of convergence of the Dirichlet series defining the function.

The proof of Theorem XVIII will be completed when we have proved the existence of two Dirichlet series $f(s) = \sum \frac{a_n}{n^s}$ and $g(s) = \sum \frac{b_n}{n^s}$ representing functions $f(s)$ and $g(s)$ which are both regular and of finite order of magnitude for σ greater than a certain number σ_0, while neither function satisfies both of these conditions for $\sigma > \sigma_0-\varepsilon$ (ε arbitrarily small positive), and for which furthermore the two μ-functions corresponding to $f(s)$ and $g(s)$ for $\sigma > \sigma_0$ are identical; but nevertheless such that the two series $\sum \frac{a_n}{n^s}$ and $\sum \frac{b_n}{n^s}$ do not possess the same boundary of convergence.

The existence of two such series is shown immediately from the previous investigation as follows:

Let $f_1(s) = \sum \frac{a_n'}{n^s}$ be a Dirichlet series of the type mentioned in Theorem XVI, and let $g_1(s) = \sum \frac{b_n'}{n^s}$ be a Dirichlet series of the type mentioned in Theorem XVII,

where the abscissa of convergence of the second series has been moved by a transformation of variable ($s = s'+1$) from the point 0 to the point -1.

Both of the functions $f_1(s)$ and $g_1(s)$ are then regular and of finite order of magnitude at least for $\sigma > -1$, and they both have the same μ-function for $\sigma > -1$, namely

$$\mu(\sigma) = \begin{cases} 0 \text{ for } \sigma \geqq 0 \\ -\sigma \text{ for } 0 \geqq \sigma > -1 \,. \end{cases}$$

In order to avoid an investigation of the functions $f_1(s)$ and $g_1(s)$ for $\sigma \leqq -1$, which is unnecessary for our purpose, we add term by term to both of the series $\sum \frac{a'_n}{n^s}$ and $\sum \frac{b'_n}{n^s}$ the series $\sum \frac{n^{-\frac{3}{2}}}{n^s}$ which is absolutely convergent for $\sigma > -\frac{1}{2}$ and represents the function $\zeta(s+\frac{3}{2})$ possessing a pole at the point $s = -\frac{1}{2}$.

The series thus obtained

$$f(s) = \sum \frac{a_n}{n^s} = \sum \frac{a'_n + n^{-\frac{3}{2}}}{n^s} = f_1(s) + \zeta(s+\tfrac{3}{2})$$

and

$$g(s) = \sum \frac{b_n}{n^s} = \sum \frac{b'_n + n^{-\frac{3}{2}}}{n^s} = g_1(s) + \zeta(s+\tfrac{3}{2})$$

then obviously satisfy all conditions mentioned in Theorem XVIII. Indeed, first, both of the functions $f(s)$ and $g(s)$ are regular and of finite order of magnitude for $\sigma > -\frac{1}{2}$, but not for $\sigma > -\frac{1}{2}-\varepsilon$; secondly, for $\sigma > -\frac{1}{2}$, $\mu(\sigma)$ is the same function whether it is computed for the function $f(s)$ or for the function $g(s)$, namely

$$\mu(\sigma) = \begin{cases} 0 \text{ for } \sigma \geqq 0 \\ -\sigma \text{ for } 0 \geqq \sigma > -\frac{1}{2}\,; \end{cases}$$

and finally, the series $\sum \frac{a_n}{n^s}$ and $\sum \frac{b_n}{n^s}$ have different boundaries of convergence, namely the lines $\sigma = 0$ and $\sigma = -\frac{1}{2}$ respectively.

While, as mentioned above, the mere knowledge of the μ-function corresponding to a function $f(s)$ defined by a Dirichlet series allows us to establish necessary conditions (for instance $\mu(\sigma) \leqq 1$) and also sufficient conditions (for instance $\mu(\sigma) = 0$) for the convergence of the series in a certain half-plane, it appears immediately from Theorem XVIII that, generally, *it is not possible from the mere knowledge of the μ-function to establish conditions which are both necessary and sufficient for such convergence.* This shows that in general it is not possible to obtain an exact determination of the boundary of convergence $\sigma = \lambda_0$ by considering only those properties of the function $f(s)$ used in previous attempts to solve the convergence problem.

Theorem XVIII, as well as certain results of a quite different nature found by Perron,* seem to indicate that perhaps it is not possible at all to determine the boundary of convergence from simple analytic properties of the function represented by the series. In other words, the boundary of convergence is perhaps not a line which, in comparison with other lines $\sigma =$ constant, plays any particularly prominent role from an analytical point of view for the function in question.

The theory of summability set forth by the author in Part Two also seems to point in that direction. It will be shown how, by the simplest possible generalizations of the notion of convergence, the series may be used to represent a function in regions outside the half-plane of convergence. And we shall see, what is of particular importance in this connection, how it is possible in this way to pursue the series to a line $\sigma = \Lambda$ orthogonal to the real axis determined from just those simple analytic properties of the function $f(s)$ which were shown above not to suffice for the exact determination of the boundary of convergence $\sigma = \lambda_0$.

Before concluding this part, we shall show as a last application of Theorem XVII how this theorem allows us to solve a hitherto open question in the *multiplication theory* of Dirichlet series.

Let $f(s) = \sum \frac{a_n}{n^s}$ and $g(s) = \sum \frac{b_n}{n^s}$ be two Dirichlet series which are both convergent for $\sigma > 0$; and let $h(s) = \sum \frac{c_n}{n^s}$ be the Dirichlet series obtained by formal multiplication of these two Dirichlet series, i.e., let the coefficients c_n be determined by the equation $c_n = \sum a_m \cdot b_l$ where the sum is taken over all combinations m, l for which $m \cdot l = n$.

Since both of the given series are absolutely convergent at least for $\sigma > 1$, we see immediately that $h(s) = \sum \frac{c_n}{n^s}$ is convergent (indeed absolutely convergent) for $\sigma > 1$ and represents the function $f(s)\,g(s)$.

Now, Stieltjes† stated the interesting theorem, *that* $\sum \frac{c_n}{n^s}$ *is always convergent*, not only for $\sigma > 1$, *but for* $\sigma > \frac{1}{2}$.

* Zur Theorie der Dirichlet'schen Reihen, l. c.

† Sur une loi asymptotique dans la théorie des nombres, *Comptes rendus de l'Académie des Sciences, Paris*, vol. 101, 1885, p. 369. A proof of this theorem was first given by Landau, Über die Multiplikation Dirichlet'scher Reihen, *Rendiconti del Circolo Matematico di Palermo*, vol. 24, 1907, p. 112.

Next, Cahen* thought that he had proved the convergence of $\sum \frac{c_n}{n^s}$ for $\sigma > 0$.

Landau† pointed out, however, that the proof given by Cahen was incorrect, but it turned out to be very difficult to decide whether the theorem indicated by Cahen was false and only after various vain attempts‡ did Landau§ succeed in answering this question. In fact, by using properties of the function $\zeta(s)(1-2^{1-s})$ deduced by using Riemann's functional equation for $\zeta(s)$ and by applying Theorem XIII, he proved that the series $\sum \frac{c_{q,n}}{n^s} = \left(\sum \frac{(-1)^{n+1}}{n^s}\right)^q = (\zeta(s)(1-2^{1-s}))^q$ must, for sufficiently large values of the integer q, have its abscissa of convergence greater than 0; since $\sum \frac{(-1)^{n+1}}{n^s}$ is convergent for $\sigma > 0$, the falsity of Cahen's hypothesis was thus established.

The following question, however, has not been answered until now.

What is the exact value of the number α (which certainly exists and which on account of Stieltjes is less than or equal to $\frac{1}{2}$ and on account of Landau greater than 0) *for which the following two conditions are fulfilled:*

1. *If* $\sum \frac{a_n}{n^s}$ *and* $\sum \frac{b_n}{n^s}$ *are convergent for* $\sigma > 0$, *then* $\sum \frac{c_n}{n^s}$ *is always convergent for* $\sigma > \alpha$, and

2. *if* ε *denotes an arbitrarily small positive number, there exist two Dirichlet series* $\sum \frac{a_n}{n^s}$ *and* $\sum \frac{b_n}{n^s}$, *both convergent for* $\sigma > 0$, *but for which the product series* $\sum \frac{c_n}{n^s}$ *has its abscissa of convergence greater than* $\alpha - \varepsilon$.

Through Theorem XVII, the author is enabled to prove the following theorem, from which it immediately appears that the answer to the question is *that the number* α *is precisely the number* $\frac{1}{2}$ *which occurs in Stieltjes' theorem.*

Theorem XIX. *There exist two Dirichlet series* $f(s) = \sum \frac{a_n}{n^s}$ *and* $g(s) = \sum \frac{b_n}{n^s}$,

* Cahen, l.c., p. 100.

† Über die Multiplikation, l.c., p. 120.

‡ l.c., pp. 123–125. Also Hadamard, Sur les séries de Dirichlet, *Rendiconti del Circolo Matematico di Palermo*, vol. 25, 1908, pp. 326–330, and Perron, l.c., who have both made important contributions to the clarification of other theorems given with incorrect proofs by Cahen (l.c.), considered the Cahen multiplication hypothesis without success.

§ Beiträge zur analytischen Zahlentheorie, l.c., p. 265.

both convergent for $\sigma > 0$, such that the product series $h(s) = \sum \frac{c_n}{n^s}$ has as abscissa of convergence precisely the number $\frac{1}{2}$.

Proof. If we put $f(s) = g(s) = k(s) = \sum \frac{d_n}{n^s}$, where $k(s) = \sum \frac{d_n}{n^s}$ is a Dirichlet series of the type mentioned in Theorem XVII with abscissa of convergence 0, the series $h(s) = \sum \frac{c_n}{n^s} = \left(\sum \frac{d_n}{n^s}\right)^2 = (k(s))^2$ cannot possibly have an abscissa of convergence smaller than $\frac{1}{2}$; in fact, for the function $k(s)$ we have

$$\mu(\sigma) = 1-\sigma \text{ for } 0 < \sigma \leqq 1;$$

hence for the function $h(s) = (k(s))^2$ we have

$$\mu(\sigma) = 2-2\sigma \text{ for } 0 < \sigma \leqq 1,$$

and consequently

$$\mu(\sigma) > 1 \text{ for } \sigma < \tfrac{1}{2};$$

but this implies that $h(s) = \sum \frac{c_n}{n^s}$ cannot converge for $\sigma < \frac{1}{2}$, since no Dirichlet series can converge beyond the region for which $\mu(\sigma) \leqq 1$. q.e.d.

PART TWO

The Theory of Summability for Dirichlet Series.

Introduction.

In the classical theory of infinite series one distinguishes between, on the one hand, *convergent* series, which are capable of representing a number (the sum of the series), and, on the other hand, *divergent* series, to which this property is not ascribed.

In the last few decades, however, one has tried by various means and with greater or smaller success to move the boundary between these two classes in order to bring certain types of divergent series into the category of series 'capable of representing a number'.

We shall begin this part by outlining briefly the circumstances which have led to, or one may say compelled, such attempts; and we shall also, through a few examples, elucidate what advantages one may expect to gain and has already gained along this road.

One of the most frequently applied operations in analysis is to combine infinite

series by the usual rules of computation. We shall consider here the circumstances which occur when applying the simplest of such operations, namely addition, subtraction, and multiplication.

Let $\sum u_n$ and $\sum v_n$ be two infinite series; then, as is well known, we have the theorem: If $\sum u_n$ and $\sum v_n$ are convergent with the respective sums U and V, then the series $\sum (u_n+v_n)$ and $\sum (u_n-v_n)$ are also convergent with the respective sums $U+V$ and $U-V$. In other words: The property of convergence is invariant under the operations of term by term addition and subtraction.

If we consider multiplication,* however, the situation is different; here it is well known that the following theorem is by no means true: If $\sum u_n$ and $\sum v_n$ are convergent, then the product series $\sum w_n$ is also convergent. In some cases, the series $\sum w_n$ will be convergent, and in other cases not; just as it may occur, when $\sum u_n$, $\sum v_n$, and $\sum t_n$ are three convergent series, that the product series $\sum w_n$ formed from $\sum u_n$ and $\sum v_n$ will be, let us say, divergent, while the product series, constructed formally from $\sum w_n$ and $\sum t_n$ will again be convergent. Thus convergence is by no means an invariant property under multiplication; starting from convergent series the series obtained as the final result of the multiplication will sometimes be convergent and sometimes divergent.

The following question therefore naturally arises: Is it not possible through a generalization of the notion of convergence to arrive at a wider class of infinite series considered capable of representing a number, which is characterized by a property invariant not only under the operations of addition and subtraction, but also under the operation of multiplication.

This question has been solved in an extremely elegant and complete way by Cesàro in his famous paper 'Sur la multiplication des séries'† which may be said to have been the starting point for the subsequent investigations on divergent series.

The extensions of the notion of convergence introduced by Cesàro in this paper may be defined, in somewhat different notation, as follows:

Let $\sum\limits_{n=1}^{n=\infty} u_n$ *be an infinite series with constant terms, and let us put*

---

* By multiplication of two infinite series $\sum\limits_{1}^{\infty} u_n$ and $\sum\limits_{1}^{\infty} v_n$, we understand here and in the sequel multiplication by Cauchy's rule (i.e., the general term in the product series $\sum\limits_{1}^{\infty} w_n$ is determined by the equation $w_n = u_1v_n+u_2v_{n-1}+\ldots+u_{n-1}v_2+u_nv_1$).

† *Bulletin des Sciences Mathématiques*, ser. 2, vol. 14, 1890, pp. 114–120.

$$S_n^{(0)} = u_1+u_2+\cdots+u_n;$$

$$S_n^{(1)} = S_1^{(0)}+\cdots+S_n^{(0)};\ldots; S_n^{(r)} = S_1^{(r-1)}+\cdots+S_n^{(r-1)}.$$

Then the series $\sum u_n$ *is said to be summable of the* 0^{th} *order* (or convergent) *with the value* (sum) U *if* $\lim\limits_{n=\infty} S_n^{(0)} = U$; *summable of the* 1^{st} *order with the value* U *if* $\lim\limits_{n=\infty} \frac{1}{n} S_n^{(1)} = U$; *in general, summable of the* r^{th} *order with the summability value* U *if*

$$\lim_{n=\infty} \frac{S_n^{(r)}\cdot r!}{n^r} = U.$$

Since, as is well known, $\lim\limits_{n=\infty} S_n^{(0)} = U$ implies that

$$\lim_{n=\infty} \frac{S_n^{(1)}}{n} = \lim_{n=\infty} \frac{S_1^{(0)}+\cdots+S_n^{(0)}}{n} = U,$$

it is seen that a series convergent with the sum U is also summable of the 1^{st} order with the value U, while naturally the converse of this theorem is not true. In a similar way, one can show that generally (i.e., for an arbitrary r), *if*

$$\lim_{n=\infty} \frac{S_n^{(r)}\cdot r!}{n^r} = U,$$

then also

$$\lim_{n=\infty} \frac{S_n^{(r+1)}\cdot (r+1)!}{n^{r+1}} = U;$$

i.e., *that a series summable of the* r^{th} *order is also summable of the* $(r+1)^{\text{th}}$ *order and with the same summability value.*

Starting from these definitions Cesàro proves the following very interesting theorem:

If $\sum u_n$ *is summable of the* p^{th} *order with the value* U *and* $\sum v_n$ *is summable of the* q^{th} *order with the value* V, *then the product series* $\sum w_n$ *is always summable of the* $(p+q+1)^{\text{th}}$ *order with the value* UV.*

In general, we call an infinite series *summable with the value* U if there exists an integer $r \geqq 0$ such that the series is summable of the r^{th} order (and hence of course of the $(r+1)^{\text{th}}$, $(r+2)^{\text{th}}, \ldots$ orders) with the value U. It follows immediately from Cesàro's theorem *that if* $\sum u_n$ *is summable with the value* U *and* $\sum v_n$ *is summable with*

* For $p = q = 0$ Cesàro's theorem states that if $\sum u_n$ and $\sum v_n$ are convergent with the respective sums U and V, then the product series $\sum w_n$ is always summable of the 1^{st} order with the summability value UV. This latter theorem contains as a special case the Abel multiplication theorem (*Œuvres*, vol. 1, p. 225) which, with the same assumptions, states that if $\sum w_n$ is convergent then its sum is always UV.

the value V, then the product series is always summable with the value UV. In other words, the property of summability is invariant under the operation of multiplication just as it is obviously invariant under the operations of addition and subtraction.

Cesàro's generalizations of the notion of convergence depend on the fact that instead of passing at once to the limit with the partial sums $S_n^{(0)}$, as we do in the case of convergence, he passes to the limit with certain more and more smoothed mean values of these partial sums. Incidentally, it must be emphasized that Cesàro is not the first who has noticed the important advantages which may be obtained from the consideration of such mean values in the theory of infinite series. Thus, as early as 1880, Frobenius* proved an interesting theorem which, in the above notation, may be stated as follows:

Let $f(x) = \sum_{n=1}^{n=\infty} a_n x^n$ be a power series with radius of convergence 1, and let the series $\sum_{n=1}^{n=\infty} a_n$, i.e., the power series at the point $x = 1$, be summable of the 1st order with the value A. Then

$$\lim_{x=1} f(x) = A$$

when x converges along the radius vector to the point 1 on the circle of convergence.†

Frobenius' theorem was later extended by Hölder‡ to the following more general theorem:

If the series $\sum a_n$ is summable of the r^{th} order with the value A, then

$$\lim_{x=1} f(x) = A$$

when x converges along the radius vector to the point 1.§

Quite recently, important results have been obtained in various parts of analysis

---

* Über die Leibnitzsche Reihe, *Journal für die reine und angewandte Mathematik*, vol. 89, 1880, pp. 262–264.

†This theorem of Frobenius is a generalization of the following theorem of Abel: *If $\sum a_n$ is convergent with the sum A, then* $\lim f(x) = A$. A particularly interesting example of the application of Frobenius' theorem comes from the power series

$$\frac{1}{1+x} = 1-x+x^2-x^3\ldots;$$

here the series $1-1+1-1\ldots$ is summable of the first order with the value $\frac{1}{2}$, in accordance with the equation $\lim_{x=1} \frac{1}{1+x} = \frac{1}{2}$.

‡ Grenzwerthe von Reihen an der Convergenzgrenze, *Mathematische Annalen*, vol. 20, 1882, p. 535.

§ Hölder, however, considers a different, and for applications less convenient, type of mean value. Schnee, Die Identität des Cesàroschen und Hölderschen Grenzwertes, *Mathematische Annalen*, vol. 67, 1909, pp. 110–125, has however proved that in reality Cesàro's and Hölder's definitions are equivalent.

by application of Cesàro summable series. As perhaps the most important of these results, we shall mention only Fejér's theorems on Fourier series. The type and importance of these theorems, in which, incidentally, only summability of the 1st order occurs, appear from the following theorem, which is contained as a special case in Fejér's results:

If $f(x)$ is a continuous function in the interval 0 to 2π, then the corresponding Fourier series

$$\frac{1}{2\pi}\int_0^{2\pi} f(\alpha)d\alpha + \sum_{n=1}^{n=\infty} \frac{1}{\pi}\int_0^{2\pi} f(\alpha)\cos\,[n(\alpha-x)]d\alpha$$

(which series, as shown by du Bois-Reymond, need by no means be convergent for all values of x between 0 and 2π) *will be summable of the 1st order at every point of the interval* 0 *to* 2π *with the summability value* $f(x)$.

We now pass to the consideration of another situation which also naturally leads to the question of representing numbers by divergent series.

Let $F(x) = \sum_{n=1}^{n=\infty} f_n(x)$ be an infinite series whose terms $f_n(x)$ are functions of a complex variable x, and let us assume that this series is convergent in a certain connected region G where it represents an analytic function, while the series is divergent outside the region G. We assume that the boundary of G is not a natural boundary of the function $F(x)$ (a natural boundary of an analytic function is, as is well known, a curve whose points are all singular points of the function). Then the question naturally arises whether or not it is possible through a generalization of the notion of convergence to make the series usable in regions outside the region of convergence G and in such a way that the function represented by the series in these new regions by means of the generalized definitions is just the analytic continuation of the function $F(x)$ defined by the convergent series inside G.

For power series, this question has been treated by various mathematicians, who have taken Cesàro's work as their starting point; among these we must mention Borel particularly.

As is well known, the region of convergence of a power series is bounded by a circle, the so-called circle of convergence. One might therefore naturally ask: Do there exist regions outside the circle of convergence where the power series is summable by Cesàro's method? This question, however, must be answered *negatively*, since, as is very easily seen, the points at which a power series is summable without being convergent must always—like the points at which the series is conditionally convergent—lie on the circle of convergence itself. On account of this fact, which

is due to the very 'strong' divergence of a power series outside its circle of convergence, Borel, in order to make the power series 'applicable' as the generator of a function in regions outside the circle of convergence, has had to use extensions of the notion of convergence quite different from those of Cesàro, and to resort to much stronger devices than those used by Cesàro in order to force a divergent power series to represent a definite number. Thus, in his definitions, Borel has had to use a double passage to the limit, while in Cesàro's definitions, as in the usual definition of convergence, there is only a single passage to the limit. As we shall see in several cases in the sequel, it is just this latter fact that allows us to operate almost as well with a series summable of the r^{th} order by Cesàro's method as with a convergent series.

We now pass from this to the consideration of our proper subject, the Dirichlet series $\sum \frac{a_n}{n^s}$.

This type of series generally possesses a two-dimensional region inside which the series is conditionally convergent—and it is just this property which, apart from number-theoretic applications, has lent the study of Dirichlet series its special interest. Also the boundary line of the half-plane of convergence of a Dirichlet series, contrary to what is the case for the circle of convergence of a power series, is not a line which seems to play any special role for the analytic properties of the function represented by the series. These observations led the author to the view that perhaps in the case of Dirichlet series, the function-representing properties of the series had been far from fully exploited by the mere consideration of its sum in the region of convergence. In other words, in the case of Dirichlet series it might perhaps be possible in a far simpler way and with much greater advantage than in the case of power series to apply generalizations of the notion of convergence. It turned out that this was the case and even that the very simplest generalizations of the notion of convergence, namely those which are given by the Cesàro summation definitions described above, could be used with advantage and in fact were even, so to say, made for application to Dirichlet series, since these series turned out in general to possess two-dimensional regions inside which the series are summable without being convergent:

Like the region of convergence, the regions of summability of the $1^{\text{st}}, 2^{\text{nd}}, \ldots, r^{\text{th}}, \ldots$ *orders are all half-planes bounded by straight lines orthogonal to the real axis; moreover*— and this gives the whole theory its significance—*inside these regions, the series represents by its summability value precisely the analytic continuation of the function defined by the series inside its region of convergence.*

Furthermore, it turned out that, in a manner of speaking, all theorems valid for Dirichlet series in their region of convergence could be generalized by using the right points of view to the series in their summability regions. This analogy may be carried through in such a way that in reality one need not consider the region of convergence first, but may as well at once establish the theory for the series in their regions of summability of arbitrary r^{th} order. Then, if one wants to obtain the theorems valid in the region of convergence, one need only specialize to the value $r = 0$ in the general theorems already obtained. For instance, we shall see that a Dirichlet series in its r^{th} region of summability represents an everywhere regular analytic function and that this function to the right of the line of summability is of order of magnitude with respect to t smaller than $O(|t|^{r+1})$, etc.

Through the introduction of summability, we thus obtain infinitely many regions (corresponding to $r = 1, 2, \dots$) every one of which plays a role similar to the role of the region of conditional convergence and which together form a region in which the analytic function represented by the series may be studied by consideration of the series alone. Moreover—and this must be considered a main point in the theory of summability for Dirichlet series— *through the introduction of summability it is possible to pursue the function represented by the series to a line which*, in contrast to the line of convergence, *plays a decisive role for the analytic behaviour of the function*, viz. a line which can be determined without any reference to the series itself by the mere knowledge of certain decisive properties of the function.

Having set forth these introductory remarks, we shall now give a brief survey of the content of Part Two of this dissertation, in which an attempt is made to build up a systematic theory of summability for Dirichlet series. In order to make this part as clear as possible, it is divided into smaller sections, each of which, though based upon results of the preceding sections, nevertheless as far as possible possesses a certain unity.

In § 1, we state some general theorems regarding Cesàro summable series; for some of these theorems—in particular Theorem I, which is the generalization to summable series of Weierstrass' fundamental theorem on convergent series—it should be noted that the reason why these theorems have hitherto not been stated in complete generality is perhaps mainly to be found in the fact that the theory of Dirichlet series set forth by the author seems to be the first occasion on which series Cesàro summable of the r^{th} order have been used to represent analytic functions in two-dimensional regions.

In § 2, we determine the type of regions inside which a Dirichlet series is summable of the 1st, 2nd, ..., rth, ... orders; it is shown that these regions are all half-planes bounded by lines $\sigma = \lambda_1$, $\sigma = \lambda_2, \ldots,$ $\sigma = \lambda_r, \ldots$ orthogonal to the real axis. It is also shown that a series summable of the rth order to the right of the line $\sigma = \lambda_r$ represents by its summability value an everywhere regular analytic function which for $\lambda_0 > \sigma > \lambda_r$ is the analytic continuation of the function $f(s)$ defined by the sum of the series inside the region of convergence (i.e. for $\sigma > \lambda_0$).

Next, in order to give immediately an application of the general results found in § 2, we study in § 3 the summability behaviour of a special type of Dirichlet series including, in particular, certain series which play an important role in the analytic theory of numbers. § 3 concludes with a comparison between the summability behaviour of the series $\sum \frac{a_n}{n^s}$ and the series $\sum \frac{a_n \cdot (\log n)^\alpha}{n^s}$ (α arbitrary complex).

In § 4, we determine the rth 'abscissa of summability' λ_r as a function of the coefficients of the series. For later applications, the general results thus found are used to establish the existence of certain Dirichlet series with particularly simple summability behaviour.

§ 5 deals with the problem of the distribution of the abscissae of summability, and a complete solution of this problem is given.

In § 6, we treat the behaviour of the sum-function on approaching points on the boundary of summability $\sigma = \lambda_r$. We prove a general theorem which includes as a very special case the Dedekind-Cahen-Schnee theorem, mentioned in Part One, concerning approximation to points on the boundary of convergence.

In § 7, we give a closer investigation of the analytic function represented by a Dirichlet series in its regions of summability. In particular, we study its behaviour for infinitely large values of the ordinate t. One theorem is given concerning the order of magnitude of the function with respect to the ordinate t, and another theorem states the existence of a mean value of the function on an arbitrary straight line orthogonal to the real axis lying within the region of summability of the series, and gives a formula for it.

Finally, in the last section of the dissertation (§ 8), we solve a problem which properly could be called the *summability problem* for Dirichlet series in analogy to what was termed in Part One the convergence problem for Dirichlet series. This is the problem of determining exactly—from the mere knowledge of the analytic properties of the represented function—the boundary line $\sigma = \Lambda$ of the half-plane

inside which the series is summable of some sufficiently high order. It is shown that such a determination is possible and is dependent upon two essential factors, namely, on the one hand, the regularity or singularity of the function and, on the other hand, the order of magnitude of the function when the ordinate t tends to infinity. For this determination, it turns out that three different cases can be imagined, and it is shown that these three cases really all occur. The section concludes with a general theorem concerning multiplication of Dirichlet series.

§ 1.
Some general theorems on summable series.

Let $\sum\limits_{n=1}^{n=\infty} u_n$ be an infinite series with constant terms, and let us put

$$S_n^{(0)} = \sum_{m=1}^{m=n} u_m;\ S_n^{(1)} = \sum_{m=1}^{m=n} S_m^{(0)};\ldots;\ S_n^{(r)} = \sum_{m=1}^{m=n} S_m^{(r-1)}. \tag{1}$$

Then, as mentioned in the introduction, the series is said to be *summable of the r^{th} order with the value U* if

$$\lim_{n=\infty} \frac{S_n^{(r)} \cdot r!}{n^r} = U\,.$$

Furthermore, $\sum u_n$ is said to be '*summably oscillating of the r^{th} order between finite bounds*' if $\dfrac{S_n^{(r)} r!}{n^r}$ is bounded for all n without converging to a limit for $n = \infty$.

We consider now an infinite series $\sum\limits_{n=1}^{n=\infty} u_n(s)$ the terms of which are functions of a complex variable s. This series is said to be *uniformly summable of the r^{th} order in a domain G with the summability value $U(s)$* if $\dfrac{S_n^{(r)} r!}{n^r}$ (where $S_n^{(r)}$ is formed from the u's through the relations (1) and consequently is now a function of s) for s belonging to the domain G converges uniformly to the limit $U(s)$ as n tends to ∞.

Then we have the following fundamental theorem:

Theorem I. *If $\sum u_n(s)$ is an infinite series which is uniformly summable of the r^{th} order in a certain connected domain G and whose terms $u_n(s)$ are regular analytic functions in G, then the series represents by its summability value $U(s)$ a regular analytic function in G. Furthermore, the series $\sum u_n(s)$ may be differentiated term by term an arbitrary number of times; i.e., the series $\sum u_n^{(p)}(s)$ obtained by differentiating term by*

term p times is again summable of the r^{th} order in G and represents by its summability value precisely the function $U^{(p)}(s)$.

Proof. From the defining relations (1), we obtain at once by induction the identity

$$S_n^{(r)} = u_1\binom{n+r-1}{r} + u_2\binom{n+r-2}{r} + \cdots + u_n\binom{r}{r}. \tag{2}$$

Since $S_n^{(r)}$ is thus a linear function of the first n terms of the series and since by assumption all the terms $u_n(s)$ are regular analytic functions in G, we see at once that the quantities $S_n^{(r)}$ and consequently also the quantities $\frac{S_n^{(r)} r!}{n^r}$ are regular analytic functions in G.

By this last remark, together with the assumption that

$$\lim_{n=\infty} \frac{S_n^{(r)} r!}{n^r} \text{ is uniformly equal to } U(s)$$

for s belonging to G, it follows from Weierstrass' theorem, mentioned on page 9 (note †), that the summability value $U(s)$ is a regular analytic function in G. This proves our first statement.

In order to prove that we may differentiate $\sum u_n(s)$ term by term an arbitrary number of times, say p, we infer first from Weierstrass' theorem that in the domain G

$$\lim_{n=\infty} \left(\frac{S_n^{(r)} r!}{n^r}\right)^{(p)} = U^{(p)}(s). \tag{3}$$

Since $S_n^{(r)}$ is, as mentioned above, a linear function of a finite number of terms of the series $\sum u_n(s)$, it is plain that

$$(S_n^{(r)})^{(p)} = \overset{*}{S}{}_n^{(r)},$$

where $\overset{*}{S}{}_n^{(r)}$ is formed in just the same way from the p times term by term differentiated series $\sum u_n^{(p)}(s)$ as $S_n^{(r)}$ is formed from the original series $\sum u_n(s)$. Hence equation (3) may be written

$$\lim_{n=\infty} \frac{\overset{*}{S}{}_n^{(r)} \cdot r!}{n^r} = U^{(p)}(s);$$

but this last equation states precisely that the series $\sum u_n^{(p)}(s)$ is summable of the r^{th} order and has the quantity $U^{(p)}(s)$ as its summability value. q.e.d.

Closely related to the proof of Theorem I is the proof of the following theorem:

Theorem II. *Let $\sum u_n(s)$ be an infinite series whose terms $u_n(s)$ are functions of a complex variable s and which are all integrable functions of s when the path of integration*

is chosen as a certain curve L of finite length in the complex plane. Furthermore, let the series $\sum\limits_{n=1}^{n=\infty} u_n(s)$ *be uniformly summable of the* r^{th} *order with the summability value* $U(s)$ *for all values of s lying on the curve L. Then* $U(s)$ *itself is integrable on the curve L, and we have the equation*

$$\int_L U(s)ds = \sum_{n=1}^{n=\infty} \int_L u_n(s)ds\,, \tag{4}$$

where the infinite series on the right is summable of the r^{th} *order.*

Proof. Since $S_n^{(r)}$ is a linear function of the quantities $u_1(s), \ldots, u_n(s)$, we see immediately that $\frac{S_n^{(r)} \cdot r!}{n^r}$ is integrable on L for every n and also that

$$\int_L \frac{S_n^{(r)} \cdot r!}{n^r}\, ds = \frac{\overset{*}{S}{}_n^{(r)} \cdot r!}{n^r},$$

where $\overset{*}{S}{}_n^{(r)}$ is formed from the series $\sum\limits_{n=1}^{n=\infty} \int_L u_n(s)ds$ in just the same way as $S_n^{(r)}$ is formed from the original series

$$\sum_{n=1}^{n=\infty} u_n(s).$$

This observation, together with the assumption that $\lim \frac{S_n^{(r)} \cdot r!}{n^r}$ is uniformly equal to $U(s)$, implies, in view of a well-known theorem concerning integration of uniformly convergent sequences, that

$$\lim_{n=\infty} \frac{\overset{*}{S}{}_n^{(r)} \cdot r!}{n^r} = \lim_{n=\infty} \int_L \frac{S_n^{(r)} \cdot r!}{n^r}\, ds = \int_L U(s)ds\,,$$

and this last equation is the same as the equation (4).

We shall now prove a theorem of quite a different type from the foregoing, a theorem which will also have important applications in the sequel.

Theorem III. *Let all infinite series of the infinite sequence*

$$\sum_{n=1}^{n=\infty} a_{1,n};\ \sum_{n=1}^{n=\infty} a_{2,n}; \ldots;\ \sum_{n=1}^{n=\infty} a_{m,n}; \ldots (m = 1, 2, 3, \ldots)$$

be summable of the r^{th} *order with the respective summability values* $\alpha_1, \alpha_2, \ldots, \alpha_m, \ldots$.

Then there exists a sequence of positive numbers $e_1, e_2, \ldots, e_m, \ldots$ *such that when* $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m, \ldots$ *is an arbitrary sequence of numbers subjected only to the conditions*

$$|\varepsilon_1| < e_1, \ldots, |\varepsilon_m| < e_m, \ldots,$$

the series $\sum_{n=1}^{n=\infty} a_n$, *where* a_n *denotes the sum of the convergent series* $\sum_{m=1}^{m=\infty} \varepsilon_m a_{m,n}$, *is summable of the* r^{th} *order and has as summability value the sum of the convergent series* $\sum_{m=1}^{m=\infty} \varepsilon_m \alpha_m$.

Proof. Since, as is easily seen,* the general n^{th} term of a series which is summable of the r^{th} order is numerically smaller than a constant (i.e., a quantity independent of n) multiplied by n^r, we see that there exists an infinite sequence of positive constants $k_1, k_2, \ldots, k_m, \ldots$ such that for all m and n,

$$|a_{m,n}| < k_m \cdot n^r .$$

Furthermore, putting

$$S_{m,n}^{(0)} = a_{m,1} + a_{m,2} + \cdots + a_{m,n}; \ldots ; S_{m,n}^{(r)} = S_{m,1}^{(r-1)} + \cdots + S_{m,n}^{(r-1)}$$

for all $m = 1, 2, 3, \ldots$, we have by assumption

$$\lim_{n=\infty} \frac{S_{m,n}^{(r)} \cdot r!}{n^r} = \alpha_m .$$

Hence there exists a sequence of positive constants $K_1, K_2, \ldots, K_m, \ldots$ such that for all m and n,

$$\left| \frac{S_{m,n}^{(r)} \cdot r!}{n^r} \right| < K_m .$$

We now put for all m,

$$e_m = \text{minimum of } \frac{c_m}{k_m} \text{ and } \frac{c_m}{K_m},$$

where the quantities $c_1, c_2, \ldots, c_m, \ldots$ are positive and chosen such that $\sum_{m=1}^{m=\infty} c_m$ is convergent $\left(\text{e.g. } c_m = \frac{1}{m^2}\right)$. Then we shall show that the quantities $e_1, e_2, \ldots, e_m, \ldots$ will satisfy all of the conditions of Theorem III.

First, the series

$$\sum_{m=1}^{m=\infty} \varepsilon_m a_{m,n} \tag{5}$$

are all (for $n = 1, 2, 3, \ldots$) convergent (and even absolutely convergent) since

$$|\varepsilon_m \cdot a_{m,n}| \leqq e_m \cdot |a_{m,n}| \leqq \frac{c_m}{k_m} \cdot k_m \cdot n^r = c_m \cdot n^r.$$

Denoting now by a_n the sum of the convergent series (5) and putting

* See for instance Bromwich, *Theory of infinite series*, London 1908, pp. 317–318.

$$S_n^{(0)} = a_1 + \cdots + a_n;\ \cdots;\ S_n^{(r)} = S_1^{(r-1)} + \cdots + S_n^{(r-1)},$$

we have to show that $\sum_{n=1}^{n=\infty} a_n$ is summable of the r^{th} order with summability value equal to the sum of the series $\sum_{m=1}^{m=\infty} \varepsilon_m \alpha_m$, which we shall prove to be convergent. That is, we have to show that

$$\lim_{n=\infty} \frac{S_n^{(r)} \cdot r!}{n^r} = \sum_{m=1}^{m=\infty} \varepsilon_m \alpha_m .$$

This is done in the following way: Since the series (5) is convergent for all $n = 1, 2, \ldots$, it follows immediately from the identity (2) that

$$\begin{aligned} S_n^{(r)} &= \sum_{q=1}^{q=n} a_q \binom{n+r-q}{r} = \sum_{q=1}^{q=n} \left[\binom{n+r-q}{r} \sum_{m=1}^{m=\infty} \varepsilon_m a_{m,q} \right] \\ &= \sum_{m=1}^{m=\infty} \left[\varepsilon_m \sum_{q=1}^{q=n} a_{m,q} \binom{n+r-q}{r} \right] = \sum_{m=1}^{m=\infty} \varepsilon_m S_{m,n}^{(r)} \end{aligned}$$

and hence

$$\frac{S_n^{(r)} \cdot r!}{n^r} = \sum_{m=1}^{m=\infty} \varepsilon_m \frac{S_{m,n}^{(r)} \cdot r!}{n^r}.$$

Since

$$\left| \varepsilon_m \frac{S_{m,n}^{(r)} \cdot r!}{n^r} \right| \leq \frac{c_m}{K_m} \cdot K_m = c_m$$

for all m and n, the infinite series $\sum_{m=1}^{m=\infty} \varepsilon_m \frac{S_{m,n}^{(r)} \cdot r!}{n^r}$ is uniformly convergent for all $n = 1, 2, 3, \ldots$. But this, together with the assumption

$$\lim_{n=\infty} \frac{S_{m,n}^{(r)} \cdot r!}{n^r} = \alpha_m$$

implies, by known theorems, that the series

$$\sum_{m=1}^{m=\infty} \left[\varepsilon_m \cdot \lim_{n=\infty} \frac{S_{m,n}^{(r)} \cdot r!}{n^r} \right] = \sum_{m=1}^{m=\infty} \varepsilon_m \alpha_m$$

is convergent, and also that

$$\lim_{n=\infty} \frac{S_n^{(r)} \cdot r!}{n^r} = \lim_{n=\infty} \sum_{m=1}^{m=\infty} \varepsilon_m \frac{S_{m,n}^{(r)} \cdot r!}{n^r} = \sum_{m=1}^{m=\infty} \varepsilon_m \alpha_m . \qquad \text{q.e.d.}$$

By the expression: *a series summable of exactly the r^{th} order* we shall understand a series which is summable of the r^{th} order (and consequently also of the $(r+1)^{\text{th}}$, $(r+2)^{\text{th}}, \ldots$ orders) but is not summable of the $(r-1)^{\text{th}}$ order. We then have the

following theorem, which immediately follows from the fact that $S_n^{(r)}$—and consequently also $\frac{S_n^{(r)} \cdot r!}{n^r}$—is a linear function of the first n terms of the series.

Theorem IV. *If $\sum u_n$ is summable of exactly the r_1th order, and $\sum v_n$ is summable of exactly the r_2th order, and if $w_n = u_n + v_n$, then, for $r_1 > r_2$, the series $\sum w_n$ is summable of exactly the r_1th order; if $r_1 = r_2$, however, we can infer only that $\sum w_n$ is summable of the r_1th order, but not that $\sum w_n$ is summable of exactly the r_1th order.*

We conclude this section by recalling the following (well-known) theorem, which gives an explicit statement of a principle very frequently applied in the theory of infinite series, and which will be of use in the sequel.

Theorem Va. *If*

$$F_n = \sum_{m=1}^{m=p} v_m \beta_{n,m} = v_1\beta_{n,1} + v_2\beta_{n,2} + \cdots + v_p\beta_{n,p},$$

where $p = p(n)$ denotes an integer which increases strictly to infinity with n, and if the following three conditions are fulfilled:

1. $\sum_{m=1}^{m=\infty} |v_m|$ *is convergent,*
2. $|\beta_{n,m}| < K$ *(K independent of n and m),*
3. $\lim\limits_{\substack{n=\infty \\ (m \text{ const.})}} \beta_{n,m} = \beta_m$,

then F_n has a finite limit for $n = \infty$, which is equal to the sum of the absolutely convergent series $\sum_{m=1}^{m=\infty} v_m\beta_m$.

Theorem Va is included in the following theorem, which is somewhat more general, and is particularly convenient for our applications:

Theorem Vb. *Let*

$$F_n(s) = \sum_{m=1}^{m=p} v_m(s)\beta_{n,m} = v_1(s)\beta_{n,1} + \cdots + v_p(s)\beta_{n,p},$$

where $p = p(n)$ is an integer which increases strictly to infinity with n. Let the quantities $v_m(s)$, for s belonging to a certain domain G, satisfy the following two conditions:

$$|v_m(s)| < v_m \quad (v_m \text{ denotes a quantity depending only on } m)$$

and

$$\sum_{m=1}^{m=\infty} |v_m(s)| \text{ converges uniformly in the domain } G.$$

S 1. Translation of A 3.

Let furthermore the quantities $\beta_{n,m}$, which are independent of s, satisfy the conditions

$$|\beta_{n,m}| < K \text{ } (K \textit{ independent of } n \textit{ and } m)$$

and

$$\lim_{\substack{n=\infty \\ (m \text{ const.})}} \beta_{n,m} = \beta_m .$$

Then $F_n(s)$, for s belonging to G, will converge uniformly to the limit

$$\sum_{m=1}^{m=\infty} v_m(s)\beta_m ,$$

where the last series is a uniformly absolutely convergent series, i.e.,

$$\sum_{m=1}^{m=\infty} |v_m(s)\beta_m|$$

is uniformly convergent for s belonging to the domain G.

Proof. First, it is immediately seen that the series $\sum_{m=1}^{m=\infty} |v_m(s)\beta_m|$ is uniformly convergent in the domain G; for $\sum |v_m(s)|$ is uniformly convergent and from $|\beta_{n,m}| < K$ it follows that

$$|\beta_m| = \lim_{n=\infty} |\beta_{n,m}| \leqq K .$$

Denoting by ε an arbitrarily small positive number, we now determine $M = M(\varepsilon)$ such that

$$K \sum_{m=M}^{m=\infty} |v_m(s)| < \frac{\varepsilon}{3}.$$

Then, for all n with $p = p(n) \geqq M$, we have

$$\left| \sum_{m=M}^{m=p} v_m(s) \cdot \beta_{n,m} \right| < \frac{\varepsilon}{3}.$$

After determining M we next determine N so that for $n > N$ and s belonging to the domain G

$$\left| \sum_{m=1}^{m=M-1} v_m(s)\beta_{n,m} - \sum_{m=1}^{m=M-1} v_m(s)\beta_m \right|$$

$$= \left| \sum_{m=1}^{m=M-1} v_m(s)(\beta_{n,m} - \beta_m) \right| \leqq \sum_{m=1}^{m=M-1} v_m \, |\beta_{n,m} - \beta_m| < \frac{\varepsilon}{3}.$$

Then for all $n > N$ and all s belonging to G, we have the inequality

$$\left| F_n(s) - \sum_{m=1}^{m=\infty} v_m(s)\beta_m \right| \leqq \left| \sum_{m=M}^{m=p} v_m(s)\beta_{n,m} \right|$$

$$+ \left| \sum_{m=M}^{m=\infty} v_m(s)\beta_m \right| + \left| \sum_{m=1}^{m=M-1} v_m(s)\beta_{n,m} - \sum_{m=1}^{m=M-1} v_m(s)\beta_m \right| < 3 \cdot \frac{\varepsilon}{3} = \varepsilon .$$

This last inequality states exactly that $F_n(s)$ converges uniformly to the limit $\sum_{m=1}^{m=\infty} v_m(s)\beta_m$ for $n = \infty$, and thus Theorem Vb (and consequently also Theorem Va) is proved.

§ 2.

Determination of the region of summability of the r^{th} order for a Dirichlet series and investigation of the analytic behaviour of the function represented by the series.

The following main theorem concerning summability of arbitrary order* is quite analogous to Jensen's theorem (Theorem II, page 6) and contains this theorem as a special case (corresponding to $r = 0$):

Theorem I. *If the Dirichlet series* $\sum \frac{a_n}{n^s}$ *is summable of the* r^{th} *order or summably oscillating of the* r^{th} *order between finite bounds at the point* $s_0 = \sigma_0 + it_0$, *then* $\sum \frac{a_n}{n^s}$ *is summable of the* r^{th} *order for every* $s = \sigma + it$ *such that* $\sigma > \sigma_0$.

Before passing to the proof of Theorem I, we shall prove the following lemma, which is the analogue for summability of the r^{th} order of Dedekind's convergence lemma on page 5, and which includes this theorem as a special case ($r = 0$).

Lemma Ia. *If* $\sum_{n=1}^{n=\infty} u_n$ *is summable of the* r^{th} *order or summably oscillating of the* r^{th} *order between finite bounds, and if the sequence* $\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots$ *satisfies the following conditions*:

$$\sum_{n=1}^{n=\infty} |\Delta^1\alpha_n|\,,\ \sum_{n=1}^{n=\infty} n\,|\Delta^2\alpha_n|\,, \ldots, \sum_{n=1}^{n=\infty} n^r\,|\Delta^{r+1}\alpha_n| \text{ are all convergent}\dagger \tag{1}$$

and

$$\lim_{n=\infty} \alpha_n = 0\,, \tag{2}$$

then the series $\sum_{n=1}^{n=\infty} u_n\alpha_n$ *is summable of the* r^{th} *order.*‡

* H. Bohr, Sur la série de Dirichlet, *Comptes rendus de l'Académie des Sciences, Paris*, vol. 148, 11 January 1909.

† By $\Delta^q\alpha_n$ we understand the q^{th} difference $\alpha_n - q\alpha_{n+1} \cdots + (-1)^q\alpha_{n+q}$.

‡ The conditions (1) and (2), which in the text are stated in the form in which they will be applied in the proof, may be simplified, since, as shown by Bromwich, *Mathematische Annalen*, vol. 65, 1908, p. 361, $\lim \alpha_n = 0$ and the convergence of $\sum n^r |\Delta^{r+1}\alpha_n|$ imply the convergence of the series $\sum n^q |\Delta^{q+1}\alpha_n|$ for all $q = 1, 2, \ldots, r-1$.

Proof. Putting

$$S_n^{(0)} = \sum_{m=1}^{m=n} u_m;\ S_n^{(1)} = \sum_{m=1}^{m=n} S_m^{(0)};\ \ldots;\ S_n^{(r)} = \sum_{m=1}^{m=n} S_m^{(r-1)}, \tag{3}$$

we have by hypothesis

$$\left|\frac{S_n^{(r)}\cdot r!}{n^r}\right| < K \text{ (const.)}. \tag{4}$$

From assumptions (1), (2), and (4), we have to prove that $\sum u_n\alpha_n$ is summable of the r^{th} order, or, in other words, on putting

$$T_n^{(0)} = \sum_{m=1}^{m=n} u_m\alpha_m;\ T_n^{(1)} = \sum_{m=1}^{m=n} T_m^{(0)};\ldots;\ T_n^{(r)} = \sum_{m=1}^{m=n} T_m^{(r-1)},$$

we shall prove that $\dfrac{T_n^{(r)}\cdot r!}{n^r}$ has a finite limit for $n=\infty$.

By partial summation, we find

$$T_n^{(0)} = \sum_{m=1}^{m=n} u_m\alpha_m = \sum_{m=1}^{m=n} (S_m^{(0)} - S_{m-1}^{(0)})\alpha_m$$

$$= S_n^{(0)}\alpha_n + [S_1^{(0)}\Delta^1\alpha_1 + S_2^{(0)}\Delta^1\alpha_2 + \cdots + S_{n-1}^{(0)}\Delta^1\alpha_{n-1}].$$

Forming now the expression

$$T_n^{(1)} = \sum_{m=1}^{m=n} T_m^{(0)},$$

we obtain two sums, namely

$$\sum_{m=1}^{m=n} S_m^{(0)}\alpha_m \text{ and } \sum_{m=2}^{m=n} [S_1^{(0)}\Delta^1\alpha_1 + \cdots + S_{m-1}^{(0)}\Delta^1\alpha_{m-1}].$$

Applying partial summation once more to these last expressions, we get

$$\sum_{m=1}^{m=n} S_m^{(0)}\alpha_m = S_n^{(1)}\alpha_n + \sum_{m=1}^{m=n-1} S_m^{(1)}\Delta^1\alpha_m,$$

and

$$\sum_{m=2}^{m=n} [S_1^{(0)}\Delta^1\alpha_1 + S_2^{(0)}\Delta^1\alpha_2 + \cdots + S_{m-1}^{(0)}\Delta^1\alpha_{m-1}]$$

$$= (n-1)S_1^{(0)}\Delta^1\alpha_1 + (n-2)S_2^{(0)}\Delta^1\alpha_2 + \cdots + 1\cdot S_{n-1}^{(0)}\Delta^1\alpha_{n-1}$$

$$= \sum_{m=1}^{m=n-1} S_m^{(1)}((n-m)\Delta^1\alpha_m - (n-m-1)\Delta^1\alpha_{m+1})$$

$$= \sum_{m=1}^{m=n-1} S_m^{(1)}\Delta^1\alpha_m + \sum_{m=1}^{m=n-2} S_m^{(1)}(n-m-1)\Delta^2\alpha_m.$$

We therefore find, by addition of these expressions,

$$T_n^{(1)} = S_n^{(1)}\alpha_n + 2\sum_{m=1}^{m=n-1} S_m^{(1)}\Delta^1\alpha_m + \sum_{m=1}^{m=n-2} S_m^{(1)}(n-m-1)\Delta^2\alpha_m\,. \tag{5}$$

We shall now verify by induction the following expression for $T_n^{(r)}$:

$$T_n^{(r)} = S_n^{(r)}\alpha_n + \binom{r+1}{1}\sum_{m=1}^{m=n-1} S_m^{(r)}\Delta^1\alpha_m + \cdots + \binom{r+1}{q}\sum_{m=1}^{m=n-q} S_m^{(r)}\binom{n-m-1}{q-1}\Delta^q\alpha_m$$

$$+\cdots+\binom{r+1}{r+1}\sum_{m=1}^{m=n-r-1} S_m^{(r)}\binom{n-m-1}{r}\Delta^{r+1}\alpha_m\,, \tag{6}$$

or in abbreviated notation,

$$T_n^{(r)} = \sum_{q=0}^{q=r+1}\binom{r+1}{q}P_{r,q,n}\,, \tag{7}$$

where

$$P_{r,0,n} = S_n^{(r)}\alpha_n \tag{8}$$

and, for $1 \leqq q \leqq r+1$,

$$P_{r,q,n} = \sum_{m=1}^{m=n-q} S_m^{(r)}\binom{n-m-1}{q-1}\Delta^q\alpha_m\,.^* \tag{9}$$

The expression (6) for $r = 1$ (and all n) coincides with the expression found above for $T_n^{(1)}$. Hence, its validity will be proved if it is possible, from the assumption that (6) is valid for a certain r and all n, to deduce the validity of the corresponding expression for $T_n^{(r+1)}$. This is done in the following way:

$$T_n^{(r+1)} = T_1^{(r)} + T_2^{(r)} + \cdots + T_n^{(r)}$$

$$= \sum_{q=0}^{q=r+1}\binom{r+1}{q}P_{r,q,1} + \sum_{q=0}^{q=r+1}\binom{r+1}{q}P_{r,q,2} + \cdots + \sum_{q=0}^{q=r+1}\binom{r+1}{q}P_{r,q,n}$$

$$= \sum_{q=0}^{q=r+1}\binom{r+1}{q}[P_{r,q,1} + P_{r,q,2} + \cdots + P_{r,q,n}]\,.$$

Since, as is well known,

$$\binom{r+1}{q-1} + \binom{r+1}{q} = \binom{r+2}{q},$$

we see that the proof by induction will be complete if we can show that the quantity

$$P_{r,q,1} + P_{r,q,2} + \cdots + P_{r,q,n} = \sum_{p=q+1}^{p=n} P_{r,q,p}$$

* If $n \leqq q$ (i.e., if $n-q$ is negative or 0) $P_{r,q,n}$ is to be interpreted as 0. Naturally, $n \leqq q$ can occur only when $n \leqq r+1$.

can be written in the form
$$P_{r+1,q,n}+P_{r+1,q+1,n};$$
for this will imply that
$$T_n^{(r+1)} = \sum_{q=0}^{q=r+1} \binom{r+1}{q} (P_{r+1,q,n}+P_{r+1,q+1,n})$$
$$= P_{r+1,0,n}+\sum_{q=1}^{q=r+1} P_{r+1,q,n}\left\{\binom{r+1}{q-1}+\binom{r+1}{q}\right\}+\binom{r+1}{r+1}P_{r+1,r+2,n}$$
$$= \sum_{q=0}^{q=r+2}\binom{r+2}{q}P_{r+1,q,n},$$
which last expression is just the one appearing in (7) when r is replaced by $r+1$.

Thus, the proof of the validity of (6) is reduced to a proof of the equation (or rather the $r+2$ equations)
$$\sum_{p=q+1}^{p=n} P_{r,q,p} = P_{r+1,q,n}+P_{r+1,q+1,n} \quad (q = 0, 1,\ldots, r+1).$$

First, we consider the equation corresponding to $q = 0$ and obtain by partial summation
$$\sum_{p=1}^{p=n} P_{r,0,p} = \sum_{p=1}^{p=n} S_p^{(r)}\alpha_p = \alpha_n S_n^{(r+1)}+\sum_{m=1}^{m=n-1} S_m^{(r+1)}\Delta^1\alpha_m = P_{r+1,0,n}+P_{r+1,1,n}.$$

Next, when q denotes any one of the numbers $1, 2,\ldots, r+1$, we get
$$\sum_{p=q+1}^{p=n} P_{r,q,p} = \sum_{p=q+1}^{p=n}\sum_{m=1}^{m=p-q} S_m^{(r)}\binom{p-m-1}{q-1}\Delta^q\alpha_m = \sum_{m=1}^{m=n-q} S_m^{(r)}\Delta^q\alpha_m \sum_{p=q+m}^{p=n}\binom{p-m-1}{q-1},$$
and hence, by the well-known equation $\sum_{l=q-1}^{l=t}\binom{l}{q-1} = \binom{t+1}{q}$,
$$\sum_{p=q+1}^{p=n} P_{r,q,p} = \sum_{m=1}^{m=n-q} S_m^{(r)}\Delta^q\alpha_m\binom{n-m}{q}.$$

Now, applying partial summation, we obtain
$$\sum_{p=q+1}^{p=n} P_{r,q,p} = \sum_{m=1}^{m=n-q-1} S_m^{(r+1)}\left(\binom{n-m}{q}\Delta^q\alpha_m-\binom{n-m-1}{q}\Delta^q\alpha_{m+1}\right)+S_{n-q}^{(r+1)}\Delta^q\alpha_{n-q},$$
and thus, using the identities $\binom{t}{q}-\binom{t-1}{q} = \binom{t-1}{q-1}$ and $\Delta^q\alpha_m-\Delta^q\alpha_{m+1} = \Delta^{q+1}\alpha_m$,
$$\sum_{p=q+1}^{p=n} P_{r,q,p} = \sum_{m=1}^{m=n-q} S_m^{(r+1)}\binom{n-m-1}{q-1}\Delta^q\alpha_m+\sum_{m=1}^{m=n-q-1} S_m^{(r+1)}\binom{n-m-1}{q}\Delta^{q+1}\alpha_m$$
$$= P_{r+1,q,n}+P_{r+1,q+1,n}, \qquad \text{q.e.d.}$$

Having written $T_n^{(r)}$ in the form (6), we can now prove the existence of $\lim\limits_{n=\infty} \frac{T_n^{(r)} \cdot r!}{n^r}$ by showing that each of the $r+2$ terms of which $\frac{T_n^{(r)} \cdot r!}{n^r}$ consists has a finite limit for $n = \infty$.

For the first term

$$\frac{r!}{n^r} S_n^{(r)} \alpha_n \tag{10}$$

we obtain immediately from (2) and (4)

$$\lim_{n=\infty} \frac{r!\, S_n^{(r)}}{n^r} \alpha_n = 0\,. \tag{11}$$

We now consider the term

$$\frac{r!}{n^r}\binom{r+1}{q} P_{r,q,n} = \frac{r!}{n^r}\binom{r+1}{q} \sum_{m=1}^{m=n-q} S_m^{(r)} \binom{n-m-1}{q-1} \Delta^q \alpha_m\,, \tag{12}$$

where q is assumed to be $\geqq 1$ and $\leqq r$; i.e., we consider an arbitrary term in the formula (6) for $T_n^{(r)}$ other than the first and the last terms. By the use of (1) and (4), and with the help of Theorem Va, § 1 (page 51), we can easily prove that the quantity (12) has the limit 0 for $n = \infty$.

Indeed, putting

$$v_m = m^{q-1} \Delta^q \alpha_m$$

and, for $m = 1, \ldots, n-q$,

$$\beta_{n,m} = \binom{r+1}{q} \frac{1}{(q-1)!} \frac{S_m^{(r)} \cdot r!}{m^r} \frac{(n-m-1)\ldots(n-m-q+1)}{n^{q-1}} \left(\frac{m}{n}\right)^{r-q+1},$$

so that

$$v_m \cdot \beta_{n,m} = \frac{r!}{n^r}\binom{r+1}{q} S_m^{(r)} \binom{n-m-1}{q-1} \Delta^q \alpha_m\,,$$

we find first that

$$\sum_{m=1}^{m=\infty} |v_m| = \sum_{m=1}^{m=\infty} m^{q-1} |\Delta^q \alpha_m|$$

is convergent; moreover, we have

$$|\beta_{n,m}| \leqq \binom{r+1}{q} \frac{1}{(q-1)!} \left|\frac{S_m^{(r)} \cdot r!}{m^r}\right| < \binom{r+1}{q} \frac{1}{(q-1)!} K = K_1\,,$$

and finally

$$\beta_m = \lim_{\substack{m \text{ const.} \\ n=\infty}} \beta_{n,m} = 0 \left(\text{on account of the factor} \left(\frac{m}{n}\right)^{r-q+1}\right).$$

Therefore, one infers from Theorem Va, § 1, that

$$\lim_{n=\infty} \frac{r!}{n^r}\binom{r+1}{q} P_{r,q,n} = \lim_{n=\infty} \sum_{m=1}^{m=n-q} v_m \beta_{n,m} = \sum_{m=1}^{m=\infty} v_m \beta_m = \sum 0 = 0\,. \tag{13}$$

It remains only to consider the quantity

$$\frac{r!}{n^r} P_{r,r+1,n} = \frac{r!}{n^r} \sum_{m=1}^{m=n-r-1} S_m^{(r)} \binom{n-m-1}{r} \Delta^{r+1} \alpha_m . \tag{14}$$

Of the quantity (14) we can prove, again using Theorem Va, § 1, that it has a finite limit for $n = \infty$. This limit is equal to the sum of the series

$$\sum_{n=1}^{n=\infty} S_n^{(r)} \Delta^{r+1} \alpha_n ,$$

which is absolutely convergent, as we shall now show.

Indeed, let us put

$$v_m = m^r \Delta^{r+1} \alpha_m$$

and, for $m = 1, \ldots, n-r-1$,

$$\beta_{n,m} = \frac{1}{r!} \frac{S_m^{(r)} \cdot r!}{m^r} \frac{(n-m-1) \cdots (n-m-r)}{n^r},$$

so that

$$v_m \cdot \beta_{n,m} = \frac{r!}{n^r} S_m^{(r)} \binom{n-m-1}{r} \Delta^{r+1} \alpha_m .$$

Then we have

$$|\beta_{n,m}| < \frac{1}{r!} K = K_2$$

and

$$\beta_m = \lim_{\substack{m \text{ const.} \\ n=\infty}} \beta_{n,m} = \frac{1}{r!} \frac{S_m^{(r)} \cdot r!}{m^r} = \frac{S_m^{(r)}}{m^r} .$$

In consequence of Theorem Va, § 1, we therefore obtain

$$\lim_{n=\infty} \frac{r!}{n^r} P_{r,r+1,n} = \lim_{n=\infty} \sum_{m=1}^{m=n-r-1} v_m \beta_{n,m} = \sum_{m=1}^{m=\infty} v_m \beta_m = \sum_{m=1}^{m=\infty} S_m^{(r)} \Delta^{r+1} \alpha_m , \tag{15}$$

where, again in consequence of Theorem Va, § 1, the last infinite series converges absolutely.

When we finally collect the results stated in (11), (13), and (15), we see that we have proved that, under the given hypotheses, $\dfrac{T_n^{(r)} \cdot r!}{n^r}$ has a finite limit, i.e., that $\sum u_n \alpha_n$ is summable of the r^{th} order, and that *the summability value of the series is equal to the sum of the absolutely convergent series*

$$\sum_{n=1}^{n=\infty} S_n^{(r)} \Delta^{r+1} \alpha_n .* \tag{16}$$

* Lemma Ia, which was proved by the author for use in his investigations concerning Dirichlet series, was communicated in the above-mentioned note: Sur la série de Dirichlet, *Comptes rendus de*

Lemma Ia now puts us in a position to prove the main theorem stated at the beginning of this section.

---

l'Académie des Sciences, Paris, 11 January 1909. As noted above, this theorem forms the natural extension of the following convergence lemma due to Dedekind: '*If* $\sum u_n$ *is convergent or oscillating between finite bounds, and if* $\sum |\Delta^1 \alpha_n|$ *is convergent and* $\lim \alpha_n = 0$, *then* $\sum u_n \alpha_n$ *is convergent with the sum* $\sum S_n \Delta^1 \alpha_n$.' Dedekind has also proved the following theorem, which is closely related to the preceding one: '*If* $\sum u_n$ *is convergent with the sum* U, *and if* $\sum |\Delta^1 \alpha_n|$ *converges, and* $\lim \alpha_n = A$, *then* $\sum u_n \alpha_n$ *is convergent with the sum* $AU + \sum S_n \Delta^1 \alpha_n$.' The natural extension of the latter theorem to summability of the r^{th} order may be stated as follows: '*If* $\sum u_n$ *is summable of the* r^{th} *order with the summability value* U, *and if* $\sum n^{q-1} |\Delta^q \alpha_n|$ *converges for* $q = 1, 2, \ldots, r+1$ *and if* $\lim \alpha_n = A$, *then the series* $\sum u_n \alpha_n$ *is summable of the* r^{th} *order and its summability value is equal to* $AU + \sum S_n^{(r)} \Delta^{r+1} \alpha_n$.' This extended theorem was used by the author to prove certain theorems concerning the simultaneous r^{th} order summability of a *Dirichlet series*, a *factorial series*, and a *binomial coefficient series*. These theorems are stated without proofs in a preliminary communication, Über die Summabilität Dirichletscher Reihen, *Nachrichten der Kgl. Gesellschaft der Wissenschaften zu Göttingen, math. phys. Klasse* 1909, pp. 247–262. If we use the theorem of Bromwich given in note ‡, page 53, then the conditions on the sequence $\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots$ can be reduced to *the two conditions*: $\sum n^r |\Delta^{r+1} \alpha_n|$ *converges, and* $\lim \alpha_n$ *exists and is equal to* A.

As the author was informed immediately before the completion of the present dissertation, Hardy, Generalisation of a Theorem in the Theory of Divergent Series, *Proceedings of the London Mathematical Society*, ser. 2, vol. 6, part 4 (10 July 1908), pp. 255–264, has already stated a very similar theorem which in a somewhat different notation runs as follows: *Theorem A*. '*If* $\sum u_n$ *is summable of the* r^{th} *order, and if* $\sum n^r |\Delta^{r+1} \alpha_n|$ *converges, then* $\sum u_n \alpha_n$ *is summable of the* r^{th} *order and its summability value is equal to the sum of the absolutely convergent series* $\sum S_n^{(r)} \Delta^{r+1} \alpha_n$.' But it must be remarked here that the theorem given by Hardy is not completely correct. First, a condition is lacking which, as one can easily see, is absolutely necessary for the general validity of the theorem, namely that $\lim \alpha_n$ should exist; secondly, in the expression given by Hardy for the summability value the term UA, where $A = \lim \alpha_n$, is lacking. The proof given in the text for Lemma Ia—and the proof of the modified lemma mentioned above is quite analogous—is essentially based on the expression for $T_n^{(r)}$ given in formula (6). On the other hand Hardy's proof of Theorem A is based on another formula for $T_n^{(r)}$, derived in an essentially different way. In this formula of Hardy, there appear (in contrast to what was the case in formula (6)) also differences $\Delta^q \alpha_m$ for which $m > n-q$; the quantities α_{n+k} $(k > 0)$ appearing in these differences are to be considered as identically 0 (consistent with the fact that $T_n^{(r)}$ cannot depend on quantities α_{n+k} for which $k > 0$). This fact necessitates a special investigation of certain terms in a later passage to the limit; however, Hardy has not taken account of this in the last part of his proof; and it is just these terms which on the one hand show the necessity of the existence of $\lim \alpha_n$, and on the other hand in the passage to the limit give rise to the term UA missing in Hardy's expression for the summability value of $\sum u_n \alpha_n$. Using formula (6) of the text, we obtain the term UA immediately as

$$\lim_{n=\infty} \frac{S_n^{(r)} \cdot r!}{n^r} \alpha_n .$$

Hardy gives an application of his theorem, which reads as follows when translated into the language of Dirichlet series: '*If* $\sum \frac{a_n}{n^s}$ *is summable of the* r^{th} *order at the point* $s_0 = \sigma_0 + it_0$, *then the series is also summable of the* r^{th} *order for values* $s = \sigma + it$ *such that* $t = t_0$ *and* $\sigma > \sigma_0$.' This theorem is included as a very special case in the author's Theorem I, which from the same assumption (or more generally, from the assumption $\left|\frac{S_n^{(r)}}{n^r}\right| <$ constant) states the summability of the r^{th} order *for all values* $s = \sigma + it$ *such that*

Proof of Theorem I: Let s be any complex number for which $\sigma > \sigma_0$. If we put $s-s_0 = \delta$, this means that $R(\delta) > 0$ (where $R(\delta)$ is the real part of δ). Let us further put

$$u_n = \frac{a_n}{n^{s_0}};\ \alpha_n = \frac{1}{n^{s-s_0}} = \frac{1}{n^{\delta}};\ u_n\alpha_n = \frac{a_n}{n^s}.$$

In view of the assumption that $\sum u_n = \sum \frac{a_n}{n^{s_0}}$ is summable of the r^{th} order or is summably oscillating of the r^{th} order between finite bounds, Theorem I will be proved if we can show that $\alpha_n = \frac{1}{n^{\delta}}$ satisfies conditions (1) and (2) in Lemma Ia.

Since $|\alpha_n| = \frac{1}{n^{R(\delta)}}$, where $R(\delta) > 0$, one sees immediately that (2) is satisfied. Before we pass to the proof that conditions (1) are also fulfilled, we interpolate the following general remark:

If $f(x)$ is a real or complex function of the real variable x which possesses finite (and hence continuous) derivatives of arbitrarily high order for $x \geqq N$, and if we put $f(n) = f_n$, then as is easily shown by induction (with respect to q) the following identity holds for every integral positive value of q, and $n \geqq N$:*

$\sigma > \sigma_0$, *no matter what is the value of t.* Naturally, it is impossible to deduce anything about the existence of a half-plane of summability from this last result of Hardy. Neither is it possible to draw conclusions about analytic continuation of Dirichlet series by summability, since it deals only with the behaviour on a straight line (and not in a two-dimensional region). Such analytic continuations will be treated in Theorems II and III (pages 64 and 65).

To his Theorem A Hardy has added the following *Theorem B*: '*If* $\sum u_n$ *is summable of the* r^{th} *order and if* α_n *is a function of a variable x, such that*

$$\sum_{n=1}^{n=N} |n^r\Delta^{r+1}\alpha_n| < \textit{constant}$$

for all N and x, then $\sum S_n^{(r)}\Delta^{r+1}\alpha_n$ *is uniformly convergent.*'

In proving this theorem, however, Hardy has at an essential point forgotten a factor having order of magnitude n^r, which oversight destroys the validity of the proof; as one can convince oneself by simple examples, Hardy's Theorem B is as it stands false.—But it is possible, as one immediately sees, to infer the uniform convergence of the series

$$\sum S_n^{(r)}\Delta^{r+1}\alpha_n$$

from the hypothesis that $\sum u_n$ is summable of the r^{th} order and that $\sum n^r\Delta^{r+1}\alpha_n$ *is uniformly convergent.*

We note, finally, that the assertion that the series $\sum S_n^{(r)}\Delta^{r+1}\alpha_n$ appearing in the expression

$$UA + \sum S_n^{(r)}\Delta^{r+1}\alpha_n$$

is uniformly convergent must not be confused with the assertion that $\sum u_n\alpha_n$ is uniformly summable of the r^{th} order.

* Cp. Jensen, Sur une expression simple du reste dans la formule d'interpolation de Newton, *Oversigt over det Kgl. Danske Videnskabernes Selskabs Forhandlinger*, 1894, p. 251.

$$\Delta^q f_n = (-1)^q \int_n^{n+1} dx_1 \int_{x_1}^{x_1+1} dx_2 \int_{x_2}^{x_2+1} dx_3 \ldots \int_{x_{q-2}}^{x_{q-2}+1} dx_{q-1} \int_{x_{q-1}}^{x_{q-1}+1} f^{(q)}(x_q) dx_q . \tag{17}$$

If we put $f(x)$ equal to $\frac{1}{x^\delta}$ in the identity (17), it follows immediately that, for every positive integer q,

$$\Delta^q \alpha_n = \delta(1+\delta)\cdots(q-1+\delta) \int_n^{n+1} dx_1 \int_{x_1}^{x_1+1} dx_2 \ldots \int_{x_{q-1}}^{x_{q-1}+1} \frac{dx_q}{x_q^{q+\delta}} \tag{18}$$

and hence, since $R(\delta)+q > 0$, that

$$|\Delta^q \alpha_n| \leqq |\delta| \cdot |1+\delta| \cdots |q-1+\delta| \frac{1}{n^{q+R(\delta)}}. \tag{19}$$

(As (18) shows, we have in all cases, i.e., whether $R(\delta) \gtreqless 0$,

$$\left| \Delta^q \frac{1}{n^\delta} \right| \leqq K \frac{1}{n^{q+R(\delta)}}, \tag{20}$$

where K is independent of n.)

From (19) it follows immediately that $\sum_{n=1}^{n=\infty} n^{q-1} |\Delta^q \alpha_n|$ converges $(q = 1, 2, \ldots, r+1)$, for the n^{th} term in this series is numerically less than a constant multiplied by $n^{-1-R(\delta)}$. The proof of Theorem I is thus complete.

It follows further from Lemma Ia *that the summability value of the series* $\sum \frac{a_n}{n^s}$, *which is summable of the* r^{th} *order for* $\sigma > \sigma_0$, *can be obtained for* $\sigma > \sigma_0$ *as the sum of the absolutely convergent series*

$$\sum_{n=1}^{n=\infty} S_n^{(r)} \Delta^{r+1} \left(\frac{1}{n^{s-s_0}} \right),$$

where $S_n^{(r)}$ *is formed by means of the equations* (3) *from the series* $\sum u_n = \sum \frac{a_n}{n^{s_0}}$.

From Theorem I we immediately infer:

To any given Dirichlet series $\sum \frac{a_n}{n^s}$, *there corresponds a straight line* $\sigma = \lambda_r$ $(-\infty \leqq \lambda_r \leqq +\infty)$* *orthogonal to the real axis such that the series is summable of the* r^{th} *order to the right of this line* (i.e., for $\sigma > \lambda_r$), *while the series is not summable of the* r^{th} *order at any point of the half-plane to the left* (i.e., for $\sigma < \lambda_r$).

The real number λ_r, which denotes the point of intersection of the real axis with

---

* $\lambda_r = +\infty$ denotes that the series is not summable of the r^{th} order for any value of s; $\lambda_r = -\infty$ denotes that the series is summable of the r^{th} order in the whole plane.

the boundary line of the region of r^{th} order summability, is called *the abscissa of summability of the* r^{th} *order.*

From Theorem I, it follows also *that if* $\sum \frac{a_n}{n^s}$ *is summably oscillating of the* r^{th} *order between finite bounds for* $s = s_0$, *then the point* s_0 *must necessarily lie on the boundary of summability* $\sigma = \lambda_r$.

A series which is summable of the r^{th} order is also summable of the $(r+1)^{\text{th}}$ order; it follows immediately from this that the straight line which, for a given Dirichlet series, bounds the region of $(r+1)^{\text{th}}$ order summability must lie to the left of or, in special cases, coincide with the line which bounds the region of r^{th} order summability. Expressed in terms of the abscissae of summability this means that *for any Dirichlet series we have*

$$\lambda_0 \geqq \lambda_1 \geqq \lambda_2 \ldots \geqq \lambda_r \geqq \lambda_{r+1} \ldots .$$

Before we begin to occupy ourselves with the analytic properties of the function represented by the summability value of a Dirichlet series, we interpolate the following lemma, which is a generalization of Lemma Ia:

Lemma Ib. *If* $\sum u_n$ *is summable of the* r^{th} *order or summably oscillating of the* r^{th} *order between finite bounds, and if* $\alpha_1(s), \alpha_2(s), \ldots, \alpha_n(s), \ldots$ *are functions of a complex variable* s *which, for* s *belonging to a certain domain* G, *fulfil the following conditions:*

1. $|\alpha_n(s)|$ *possesses a finite upper bound* K_n $(n = 1, 2, \ldots)$,

2. *the* $r+1$ *series* $\sum_{n=1}^{n=\infty} n^{q-1} |\Delta^q \alpha_n(s)|$ $(q = 1, 2, \ldots, r+1)$ *are all uniformly convergent,*

3. $\lim_{n=\infty} \alpha_n(s) = 0$;

then $\sum_{n=1}^{n=\infty} u_n \alpha_n(s)$ *is uniformly summable of the* r^{th} *order for* s *belonging to* G, *and the summability value of this last series is equal to the sum of the uniformly absolutely convergent series* $\sum_{n=1}^{n=\infty} S_n^{(r)} \Delta^{r+1} \alpha_n(s)$.

Proof. The proof of Lemma Ia was carried out by showing that each of the $r+2$ expressions (10), (12), and (14) of which $\frac{T_n^{(r)} \cdot r!}{n^r}$ consists has a finite limit for $n = \infty$. Similarly we can show under the assumptions of Lemma Ib that

$\frac{T_n^{(r)} \cdot r!}{n^r}$ converges uniformly to its limit for s belonging to the domain G by showing that (10), (12), and (14) all converge uniformly to their limits for $n = \infty$.

As noted earlier (page 7), the uniform convergence of $\sum_{n=1}^{n=\infty} |\Delta^1 \alpha_n(s)|$ implies the uniform convergence of $\alpha_n(s)$ to its limit value (here 0); we therefore immediately see that

$$\frac{S_n^{(r)} \cdot r!}{n^r} \cdot \alpha_n(s) \tag{10}$$

converges uniformly to the limit 0 for $n = \infty$.

We turn now to the consideration of the expressions

$$\frac{r!}{n^r}\binom{r+1}{q}\sum_{m=1}^{m=n-q} S_m^{(r)}\binom{n-m-1}{q-1}\Delta^q\alpha_m \quad (q = 1, 2, \ldots, r) \tag{12}$$

and

$$\frac{r!}{n^r}\sum_{m=1}^{m=n-r-1} S_m^{(r)}\binom{n-m-1}{r}\Delta^{r+1}\alpha_m\,. \tag{14}$$

Proceeding as in the proof of Lemma Ia, we put in the first case

$$v_m = v_m(s) = m^{q-1}\Delta^q\alpha_m(s)$$

$$\beta_{n,m} = \binom{r+1}{q}\frac{1}{(q-1)!}\,\frac{S_m^{(r)} \cdot r!}{m^r}\,\frac{(n-m-1)\ldots(n-m-q+1)}{n^{q-1}}\left(\frac{m}{n}\right)^{r-q+1};$$

and in the second case $\qquad v_m = v_m(s) = m^r\Delta^{r+1}\alpha_m(s)$

$$\beta_{n,m} = \frac{1}{r!}\,\frac{S_m^{(r)} \cdot r!}{m^r}\,\frac{(n-m-1)\ldots\,(n-m-r)}{n^r}.$$

As shown on pages 57—58 the quantities $\beta_{n,m}$, which are independent of s, fulfil in the first case the conditions:

$$|\beta_{n,m}| < \text{const.}, \quad \beta_m = \lim_{n=\infty}\beta_{n,m} = 0\,;$$

and in the second the conditions

$$|\beta_{n,m}| < \text{const.}, \quad \beta_m = \lim_{n=\infty}\beta_{n,m} = \frac{S_m^{(r)}}{m^r}.$$

Furthermore, it follows from our assumptions that in the first case

$$\sum_{n=1}^{n=\infty}|v_n(s)| = \sum_{n=1}^{n=\infty} n^{q-1}|\Delta^q\alpha_n(s)|$$

converges uniformly in G, while

$$|v_n(s)| < n^{q-1}(K_n + qK_{n+1} + \cdots + K_{n+q}) = v_n \text{ (independent of } s);$$

and that in the second case

$$\sum_{n=1}^{n=\infty} |v_n(s)| = \sum_{n=1}^{n=\infty} n^r |\Delta^{r+1}\alpha_n(s)|$$

converges uniformly in G, while

$$|v_n(s)| < n^r(K_n+(r+1)K_{n+1}+\cdots+K_{n+r+1}) = v_n .$$

Consequently, we can apply Theorem Vb, § 1, to both of the expressions (12) and (14); we thereby find that the expression (12) ($q = 1, 2, \dots, r$) converges uniformly to the limit $\sum_{m=1}^{m=\infty} v_m(s)\beta_m = 0$, that the expression (14) converges uniformly to the limit $\sum_{m=1}^{m=\infty} v_m(s)\beta_m = \sum_{m=1}^{m=\infty} S_m^{(r)}\Delta^{r+1}\alpha_m(s)$, and finally that the latter series is uniformly absolutely convergent in the domain G. The proof of Lemma Ib is hereby completed.

Lemma Ib now puts us in a position to prove the following important theorem:

Theorem II. *A Dirichlet series $\sum \frac{a_n}{n^s}$ with abscissa of r^{th} order summability λ_r is uniformly summable of the r^{th} order in every domain $G = G(\varepsilon, E)$ whose points $s = \sigma+it$ all satisfy the following two conditions:*

$$\sigma > \lambda_r+\varepsilon, \quad |s| < E .$$

Proof. Let $s_0 = \sigma_0+it_0$ be a point such that $\lambda_r < \sigma_0 < \lambda_r+\frac{\varepsilon}{2}$. If we put

$$u_n = \frac{a_n}{n^{s_0}}, \quad \alpha_n = \frac{1}{n^{s-s_0}}, \quad u_n\alpha_n = \frac{a_n}{n^s},$$

we find that the series $\sum u_n = \sum \frac{a_n}{n^{s_0}}$, which is independent of s, is summable of the r^{th} order. We further have

$$|\alpha_n| = |\alpha_n(s)| = \frac{1}{n^{\sigma-\sigma_0}} \leqq \frac{1}{n^{\frac{\varepsilon}{2}}} = K_n$$

and

$$\lim_{n=\infty} \alpha_n(s) = 0 .$$

In view of (19), for all $q = 1, 2, \dots, r+1$,

$$|n^{q-1}\Delta^q\alpha_n(s)| \leqq |s-s_0|\cdots|s-s_0+q-1| \frac{1}{n^{1+\sigma-\sigma_0}} < (E+|s_0|+q)^q \frac{1}{n^{1+\frac{\varepsilon}{2}}} .$$

From the last inequality, we infer immediately that the series $\sum_{n=1}^{n=\infty} n^{q-1} |\Delta^q \alpha_n(s)|$ is uniformly convergent in the domain G, since its n^{th} term is numerically less than a constant times $n^{-1-\frac{\varepsilon}{2}}$.

Since the conditions stated in Lemma Ib for $\sum u_n$ and the sequence $\alpha_1(s), \ldots, \alpha_n(s), \ldots$ are here all fulfilled, this lemma asserts that $\sum u_n \alpha_n(s) = \sum \frac{a_n}{n^s}$ is uniformly summable of the r^{th} order in the domain G. q.e.d.

Since the individual terms $\frac{a_n}{n^s}$ in a Dirichlet series are all integral functions of s and since one can obviously choose ε and E so that an arbitrary point of the half-plane $\sigma > \lambda_r$ belongs to the interior of the domain $G = G(\varepsilon, E)$, we infer immediately from Theorem II and Theorem I, § 1:

Theorem III. *A Dirichlet series $\sum \frac{a_n}{n^s}$ represents by its summability value of the r^{th} order a function $f(s)$ which is regular and analytic everywhere in the half-plane $\sigma > \lambda_r$. Furthermore, the series may be differentiated term by term an arbitrary number of times in this half-plane, i.e., the Dirichlet series $\sum \frac{a_n(-\log n)^p}{n^s}$ is also summable of the r^{th} order for $\sigma > \lambda_r$ and represents the function $f^{(p)}(s)$.*

Finally, since the series $\sum \frac{a_n}{n^s}$ in its half-plane of convergence (i.e., for $\sigma > \lambda_0$) represents identically the same function of s both by its sum (considered as a convergent series) and by its summability value (considered as a series summable of the r^{th} order), we find:

Theorem IV. *The analytic function which is defined beyond the boundary of convergence by r^{th} order summability* (unless $\lambda_0 = \lambda_r$) *is the analytic continuation of the analytic function defined as the sum of the series in its region of convergence.*

Since for a given Dirichlet series, the sequence of numbers $\lambda_0, \lambda_1, \ldots, \lambda_r, \ldots$ is monotonically decreasing, as noted above, there always exists a limit value Λ ($\Lambda \geqq -\infty$), which we shall call the *limit abscissa of summability* of the series. From Theorems III and IV, we immediately infer:

Theorem V. *A Dirichlet series $\sum \frac{a_n}{n^s}$ represents by summability* (i.e., summability of the 0^{th}, 1^{st}, 2^{nd}, ... orders) *a function which is regular and analytic everywhere to the*

right of the limit line of summability $\sigma = \Lambda$. In particular, if $\Lambda = -\infty$, the Dirichlet series is applicable in the whole plane upon introduction of summability, and the function defined by the series is an integral function.

§ 3.

Application of the general theorems found in § 2 to a special type of Dirichlet series; together with a comparison of the summability behaviour of the series

$$\sum \frac{a_n}{n^s} \text{ and } \sum \frac{a_n(\log n)^\alpha}{n^s}.$$

We shall determine in this section the summability behaviour of certain special Dirichlet series, among them certain series which play an important role in the analytic theory of numbers.

Suppose that the infinite sequence of real or complex numbers $a_1, a_2, \ldots, a_n, \ldots$, not all 0, satisfies the following two conditions:

1. *There exists an integer $k > 1$ such that when m is any of the numbers* $1, 2, \ldots, k$

$$a_m = a_{m+k} = a_{m+2k} = \cdots$$

(or, expressed in other words, such that $a_n = a_{n'}$ when $n \equiv n'$ (mod. k)).

2. $$a_1 + a_2 + \cdots + a_k = 0\,.$$

We shall then consider the Dirichlet series $\sum\limits_{n=1}^{n=\infty} \frac{a_n}{n^s}$ whose coefficients are just the elements of the sequence of numbers described above. The series $\sum \frac{a_n}{n^s}$ obviously has the abscissa of absolute convergence $l = 1$; for, when $\sigma > 1$, the series $\sum \left|\frac{a_n}{n^s}\right| = \sum \frac{|a_n|}{n^\sigma}$ converges, since for all n, $|a_n| \leq M$, where M denotes the maximum of the numbers $|a_1|, |a_2|, \ldots, |a_k|$; whereas, on the other hand, when $s = 1$, the series $\sum \left|\frac{a_n}{n^s}\right|$ diverges, since not all of the numbers $|a_m|$ $(m = 1, 2, \ldots, k)$ are equal to 0, and $\sum\limits_{n=1}^{n=\infty} \frac{1}{m+nk}$ diverges.

We shall now prove the following theorem, which determines the summability behaviour of a series $\sum \frac{a_n}{n^s}$ of the type defined above.

Theorem I. *The region of r^{th} order summability of $\sum \frac{a_n}{n^s}$ is the half-plane $\sigma > -r$, i.e., the equation*

$$\lambda_r = -r$$

holds for all $r = 0, 1, 2, \ldots$.

From this follows in particular

$$\Lambda = \lim_{r=\infty} \lambda_r = -\infty\,;$$

this last equation shows us that $\sum \frac{a_n}{n^s}$ is applicable in the whole plane upon introduction of summability and represents an integral function.

Before we pass to the proof of this theorem, we shall first show how the type of Dirichlet series under consideration contains in particular certain series which play an important role in the analytic theory of numbers.

If we assume first that $k = 2, a_1 = 1, a_2 = -1$, we obtain the series $\sum \frac{a_n}{n^s} = \sum \frac{(-1)^{n+1}}{n^s}$, which, as is known, represents the function $\zeta(s) \cdot (1-2^{1-s})$ in its region of convergence (for $\sigma > 0$).

The theorem thus shows *that this series is summable in the whole plane* and in particular *that the function $\zeta(s)(1-2^{1-s})$ is an integral function.**

Next, let k be an arbitrary positive integer greater than 2, and let us put $a_n = \chi(n)$ where $\chi(n)$ denotes an arbitrary real or complex character modulo k, distinct from the so-called principal character. Since the number-theoretic function $\chi(n)$, as is known, has the properties

$$\chi(n) = \chi(n') \text{ when } n \equiv n' \pmod{k}$$

and

$$\sum_{m=1}^{m=k} \chi(m) = 0\,,$$

it follows *that all of the series $L(s) = \sum_{n=1}^{n=\infty} \frac{\chi(n)}{n^s}$ belong to the type under consideration.*

* From the regularity everywhere of $\zeta(s)(1-2^{1-s})$ it follows immediately that $\zeta(s)$ is a function meromorphic in the whole plane which possibly has poles (of the first order) at the points $1 + 2p\pi i(\log 2)^{-1}$ (the zeros of $1-2^{1-s}$). (The author has incorrectly stated in his note referred to earlier that there could only be a pole at the point 1.) Of these points, the point 1 is a pole, while the points $1 + 2p\pi i(\log 2)^{-1}(p \neq 0)$ are not poles. This fact can (after de la Vallée Poussin) be proved quite easily.—That $\zeta(s)$ is a function meromorphic in the whole plane with the single pole $s = 1$ was, as is well known, first proved by Riemann; an elementary proof of this theorem (i.e., a proof which does not use Riemann's functional equation for $\zeta(s)$) was first given by Jensen, *Comptes rendus de l'Académie des Sciences, Paris*, vol. 104, 1887, p. 1156, who proved in an elementary way that $\zeta(s)(1-s)$ is an integral function.

These series, as is known, form the foundation of the analytic theory of arithmetic progressions. Thus, *these series are applicable in the whole plane upon introduction of summability*, from which it follows at once *that the represented functions $L(s)$ are all integral functions*; this last theorem is proved ordinarily by application of the functional equation for these functions; (a different proof was given by de la Vallée Poussin, based upon successive representations of $L(s)$ by definite integrals in regions extending farther and farther to the left).

Having made these preliminary remarks, we now pass to the proof of the general Theorem I stated above, and we shall first prove that the abscissa of convergence of the series $\sum \frac{a_n}{n^s}$ is equal to 0.

For this purpose, we consider $\sum \frac{a_n}{n^s}$ at the special point $s = 0$ (i.e., we consider the series $\sum \frac{a_n}{n^0} = \sum a_n$) and form the partial sums

$$S_1^{(0)} = a_1,\ S_2^{(0)} = a_1+a_2, \ldots, S_k^{(0)} = a_1+\cdots+a_k = 0, \ldots, S_{m+nk}^{(0)} = S_m^{(0)},$$

and

$$S_{m+nk}^{(1)} = \sum_{q=1}^{q=m+nk} S_q^{(0)} = n(S_1^{(0)}+S_2^{(0)}+\cdots+S_k^{(0)})+S_1^{(0)}+S_2^{(0)}+\cdots+S_m^{(0)}.$$

From the expressions found for $S_n^{(0)}$ and $S_n^{(1)}$, we see that the series $\sum a_n$ is not convergent, since $S_n^{(0)}$ does not converge to a limit for $n = \infty$; however, since $|S_n^{(0)}| <$ const., it oscillates between finite bounds, or, as we would rather say, is summably oscillating of the 0^{th} order between finite bounds. From this it follows at once that $\lambda_0 = 0$. We see further that $\sum a_n$ is summable of the first order since $\frac{S_n^{(1)}}{n}$ converges for $n = \infty$ to the finite limit

$$\frac{1}{k}(S_1^{(0)}+S_2^{(0)}+\cdots+S_k^{(0)}).$$

We shall now prove by induction that the theorem found here for $s = 0$ is valid for arbitrary r, i.e., *that* $\sum \frac{a_n}{n^s}$ *is for* $s = -r$ *summably oscillating of the* r^{th} *order between finite bounds and summable of the* $(r+1)^{\text{th}}$ *order*. From this it will follow immediately that $\lambda_r = -r$.

Let us suppose that the assertion under consideration has been proved for $s = -(r-1)$; we shall prove from this its validity for $s = -r$. For this purpose, we use formula (6), § 2 (page 55), and in this formula put

$$u_n = \frac{a_n}{n^{-(r-1)}} = a_n n^{r-1}\,;\ \alpha_n = n\,;\ u_n\alpha_n = \frac{a_n}{n^{-r}} = a_n n^r\,,$$

so that the quantity $S_n^{(r)}$ in this formula is to be formed from the series $\sum \frac{a_n}{n^s}$ at the point $s = -(r-1)$, while $T_n^{(r)}$ is to be formed from the series $\sum \frac{a_n}{n^s}$ at the point $s = -r$.

Since $\alpha_n = n$, it follows that

$$\Delta^1\alpha_n = -1\,;\ \Delta^2\alpha_n = 0\,;\ldots;\ \Delta^{r+1}\alpha_n = 0\,;$$

hence formula (6), § 2, here assumes the following simple form:

$$T_n^{(r)} = n\cdot S_n^{(r)} - (r+1)[S_1^{(r)} + \cdots + S_{n-1}^{(r)}] = n\cdot S_n^{(r)} - (r+1)\cdot S_{n-1}^{(r+1)}\,. \tag{1}$$

In the following, we shall use concerning the sum of powers $1^p + 2^p + \cdots + n^p$ only the fact that this quantity (when p is a positive integer) can be written as a polynomial of the $(p+1)^{\text{th}}$ degree in n with coefficients independent of n. Recalling the method of forming $T_n^{(0)}, T_n^{(1)}, \ldots, T_n^{(r)}$ from the series $\sum u_n\alpha_n = \sum a_n n^r$, we infer at once from the above remark, since $a_n = a_{n'}$ for $n \equiv n' \pmod{k}$, that all the k quantities $T_{(n-1)k+1}^{(r)}, T_{(n-1)k+2}^{(r)}, \ldots, T_{nk}^{(r)}$ must be expressible as polynomials in n with coefficients independent of n (the polynomials being in general all distinct).

We now consider the equation

$$\frac{T_n^{(r)}\cdot r!}{n^{r+1}} = \frac{S_n^{(r)}\cdot r!}{n^r} - \frac{S_{n-1}^{(r+1)}\cdot (r+1)!}{(n-1)^{r+1}}\cdot\left(\frac{n-1}{n}\right)^{r+1},$$

which is obtained from (1) by multiplication by $\frac{r!}{n^{r+1}}$. From this equation we infer, since by assumption $\sum u_n = \sum \frac{a_n}{n^{-(r-1)}}$ is summable of the r^{th} order (and hence also summable of the $(r+1)^{\text{th}}$ order with the same summability value), that

$$\lim_{n=\infty}\frac{T_n^{(r)}\cdot r!}{n^{r+1}} = \lim_{n=\infty}\frac{S_n^{(r)}\cdot r!}{n^r} - \lim_{n=\infty}\frac{S_{n-1}^{(r+1)}\cdot(r+1)!}{(n-1)^{r+1}}\cdot\lim_{n=\infty}\left(\frac{n-1}{n}\right)^{r+1} = 0\,. \tag{2}$$

Since all the k quantities $T_{m+(n-1)k}^{(r)}$ $(m = 1, 2, \ldots, k)$ are polynomials in n with coefficients independent of n, equation (2) shows that these polynomials can all be at most of the r^{th} degree in n. It follows at once that

$$\left|\frac{T_n^{(r)}\cdot r!}{n^r}\right| < \text{constant (for all } n),$$

i.e., that $\sum \frac{a_n}{n^{-r}}$ is either summable of the r^{th} order or summably oscillating of the r^{th} order between finite bounds.

However, we see indirectly as follows that $\sum \frac{a_n}{n^{-r}}$ cannot possibly be summable of the r^{th} order (and hence must be summably oscillating of the r^{th} order between finite bounds). Indeed, if it were summable of the r^{th} order, then either the k polynomials $T^{(r)}_{m+(n-1)k}$ would all be of degree lower than r (namely, if $\lim \frac{T^{(r)}_n \cdot r!}{n^r} = 0$), or they would all be of degree exactly r and have the same coefficient of the term of highest degree (namely, if $\lim \frac{T^{(r)}_n \cdot r!}{n^r} \neq 0$).

Since $T^{(r-1)}_n = T^{(r)}_n - T^{(r)}_{n-1}$, it would follow in both cases that all k of the polynomials $T^{(r-1)}_{m+(n-1)k}$ would be of at most degree $r-1$ in n, and hence that

$$\left| \frac{T^{(r-1)}_n \cdot (r-1)!}{n^{r-1}} \right| < \text{const.};$$

i.e., $\sum \frac{a_n}{n^{-r}}$ would be either summable or summably oscillating of the $(r-1)^{\text{th}}$ order between finite bounds; consequently, the Dirichlet series $\sum \frac{a_n}{n^s}$ would have to be summable of the $(r-1)^{\text{th}}$ order for $\sigma > -r$; however, this contradicts the assumption that $\sum \frac{a_n}{n^s}$ is summably oscillating of the $(r-1)^{\text{th}}$ order between finite bounds for $s = -(r-1)$.

We can now easily prove the second part of the assertion, namely that $\sum \frac{a_n}{n^{-r}}$ is summable of the $(r+1)^{\text{th}}$ order. In fact, if we form $T^{(r+1)}_n = \sum_{q=1}^{q=n} T^{(r)}_q$ there can only occur one of the following two cases:

1. The sum of the coefficients of the term n^r in the k polynomials $T^{(r)}_{m+(n-1)k}$ $(m = 1, 2, \ldots, k)$ is equal to 0; in this case, all k of the polynomials $T^{(r+1)}_{m+(n-1)k}$ are obviously of at most the r^{th} degree in n and consequently

$$\lim \frac{T^{(r+1)}_n \cdot (r+1)!}{n^{r+1}} = 0$$

(i.e., $\sum \frac{a_n}{n^{-r}}$ is summable of the $(r+1)^{\text{th}}$ order with the value 0).

2. The coefficient sum referred to has a value different from 0; in this case, all k of the polynomials $T^{(r+1)}_{m+(n-1)k}$ $(m = 1, \ldots, k)$ are of exactly the $(r+1)^{\text{th}}$ degree, but all have the same coefficient of the term of highest degree n^{r+1}; thus $\dfrac{T^{(r+1)}_n \cdot (r+1)!}{n^{r+1}}$ has a finite limit for $n = \infty$ (but here different from 0); i.e., $\sum \dfrac{a_n}{n^{-r}}$ in this case too is summable of the $(r+1)^{\text{th}}$ order.

The proof of Theorem I is thus complete.

We shall conclude this section by proving the following theorem, which shows, among other things, how one can immediately obtain from an arbitrary Dirichlet series a whole class of other Dirichlet series which all have exactly the same abscissae of summability as the original series.

Theorem II. *Let $\lambda^{(a)}_r$ denote the r^{th} abscissa of summability for the series*

$$\sum_{n=2}^{n=\infty} \frac{a_n}{n^s},*$$

and let $\lambda^{(b)}_r$ denote the r^{th} abscissa of summability for the series

$$\sum_{n=2}^{n=\infty} \frac{b_n}{n^s} = \sum_{n=2}^{n=\infty} \frac{a_n \cdot (\log n)^\alpha}{n^s},$$

where α denotes an arbitrary real or complex number. Then, we have for all $r = 0, 1, 2, \ldots$ the equation

$$\lambda^{(a)}_r = \lambda^{(b)}_r .$$

Theorem II will evidently be proved if we can show that $\lambda^{(b)}_r \leqq \lambda^{(a)}_r$. In fact, it follows from this, since $\sum \dfrac{a_n}{n^s}$ can be written as $\sum \dfrac{b_n \cdot (\log n)^{-\alpha}}{n^s}$ and the arbitrary number α can be replaced by the number $-\alpha$, that $\lambda^{(a)}_r \leqq \lambda^{(b)}_r$, and consequently that $\lambda^{(a)}_r = \lambda^{(b)}_r$.

In order to prove that $\lambda^{(b)}_r \leqq \lambda^{(a)}_r$, it evidently suffices to prove that if $\sum \dfrac{a_n}{n^s}$ is summable of the r^{th} order at the point $s = s_0 = \sigma_0 + it_0$, then $\sum \dfrac{b_n}{n^s}$ is summable of the r^{th} order for every s such that $\sigma > \sigma_0$, i.e., such that $\sigma = \sigma_0 + \delta$, where $\delta > 0$.

We see that this is the case by using Lemma Ia, § 2 (page 53) in the following way:

---

* We have omitted the first term a_1 in the series $\sum \dfrac{a_n}{n^s}$ in order to avoid the circumstance that the factor $(\log n)^\alpha$ entering in the quantity b_n becomes meaningless for $R(\alpha) < 0$.

If we put

$$u_n = \frac{a_n}{n^{s_0}},\quad \alpha_n = \frac{(\log n)^\alpha}{n^{s-s_0}},\quad u_n\alpha_n = \frac{a_n\cdot(\log n)^\alpha}{n^s} = \frac{b_n}{n^s},$$

then $\sum u_n = \sum \frac{a_n}{n^{s_0}}$ is by assumption summable of the r^{th} order, and since $\sigma-\sigma_0 = \delta>0$,

$$\lim_{n=\infty} \alpha_n = 0\,.$$

We further use the identity

$$\Delta^q\alpha_n = (-1)^q \int_n^{n+1} dx_1 \int_{x_1}^{x_1+1} dx_2 \ldots \int_{x_{q-1}}^{x_{q-1}+1} \left[\frac{(\log x_q)^\alpha}{x_q^{s-s_0}}\right]^{(q)} dx_q,$$

which is obtained from the general identity (17), § 2 (page 61), by putting $f(x)=\dfrac{(\log x)^\alpha}{x^{s-s_0}}$. It is easily seen by induction that the quantity

$$\left[\frac{(\log x)^\alpha}{x^{s-s_0}}\right]^{(q)}$$

appearing in the identity can be written as the sum of a finite number of terms of the form: constant $\cdot\dfrac{(\log x)^\beta}{x^{s-s_0+q}}$. Hence, from the identity we infer the inequality

$$|\Delta^q\alpha_n| < \text{const.}\frac{1}{n^{q+\frac{\delta}{2}}}\quad (n = 2, 3, \ldots),$$

from which it appears immediately that all the series $\sum_{n=2}^{n=\infty} n^{q-1}|\Delta^q\alpha_n|$ $(q = 1, 2, \ldots, r+1)$ converge.

Thus the conditions (1), § 2, (2), § 2, and (4), § 2 are satisfied. Hence Lemma Ia (page 53) shows that the series $\sum u_n\alpha_n = \sum\frac{b_n}{n^s}$ is summable of the r^{th} order, and Theorem II is thus proved.*

In the special case *where* $\alpha = +p$ (*p a positive integer*), Theorem II asserts *that*

---

* Theorem II shows that the series $\sum\frac{a_n}{n^s}$ and $\sum\frac{a_n(\log n)^\alpha}{n^s}$ have the same abscissae of summability, but not that these two series are either both summable of the r^{th} order or both not summable of the r^{th} order for a given value of s. That a theorem of this kind cannot exist follows at once from the fact that the series $\sum\frac{(-1)^{n-1}}{n^s}$, as one can easily convince oneself, is not summable of the r^{th} order for any value of $s = \sigma+it$ for which $\sigma = -r$ (i.e., is not summable of the r^{th} order at any point of the boundary line $\sigma = \lambda_r = -r$) while on the other hand the series $\sum\frac{(-1)^{n+1}(\log n)^\alpha}{n^s}$, for $R(\alpha) < 0$, is

the series $\sum \frac{a_n}{n^s}$ *and the series* $\sum \frac{b_n}{n^s} = (-1)^p \cdot \sum \frac{a_n(\log n)^p}{n^s}$, *obtained from it by* p *term by term differentiations, have exactly the same abscissae of summability.* Earlier (page 65) we proved only that the r^{th} abscissa of summability for the series $\sum \frac{b_n}{n^s}$ obtained by differentiation is less than or equal to the r^{th} abscissa of summability for the series $\sum \frac{a_n}{n^s}$.

Finally, *the case* $\alpha = -1$ possesses special interest.

Let the series $f(s) = \sum\limits_{n=2}^{n=\infty} \frac{a_n}{n^s}$ have its abscissa of r^{th} order summability equal to λ_r, and let $s_1 = \sigma_1 + it_1$ and $s_2 = \sigma_2 + it_2$ be two points such that $\sigma_1 > \lambda_r$ and $\sigma_2 > \lambda_r$.

From Theorem II, § 1 (page 47) it follows that the series

$$\sum_{n=2}^{n=\infty} \int_{s_1}^{s_2} \frac{a_n}{n^s} ds = \sum_{n=2}^{n=\infty} \left[\frac{a_n}{n^{s_2}(-\log n)} - \frac{a_n}{n^{s_1}(-\log n)} \right], \tag{3}$$

where the path of integration from s_1 to s_2 is supposed to lie entirely to the right of the line $\sigma = \lambda_r$, is summable of the r^{th} order with the value $\int_{s_1}^{s_2} f(s) ds$.

However, by virtue of Theorem II (with $\alpha = -1$), both of the series

$$\sum_{n=2}^{n=\infty} \frac{a_n}{n^{s_2}(-\log n)} \quad \text{and} \quad \sum_{n=2}^{n=\infty} \frac{a_n}{n^{s_1}(-\log n)}$$

are summable of the r^{th} order; in view of this, these two series can be separated from each other in the expression on the right-hand side of the identity (3). Consequently we find the following theorem of which we shall make important use in § 7.

Theorem III. *Let* $f(s) = \sum\limits_{n=2}^{n=\infty} \frac{a_n}{n^s}$ *be a Dirichlet series with abscissa of* r^{th} *order summability equal to* λ_r, *and let* $s_1 = \sigma_1 + it_1$ *and* $s_2 = \sigma_2 + it_2$ *be two points for which* $\sigma_1 > \lambda_r$ *and* $\sigma_2 > \lambda_r$; *then the following equation holds:*

---

summable of the r^{th} order for all s such that $\sigma = -r$ (i.e., is summable of the r^{th} order at all points of the line $\sigma = \lambda_r = -r$).

We note in this connection that the following theorem holds: *If* $R(\alpha) < 0$, the series $\sum \frac{a_n(\log n)^\alpha}{n^s}$ *is summable of the* r^{th} *order for all values of* s *such that* $\sum \frac{a_n}{n^s}$ *is either summable of the* r^{th} *order or summably oscillating of the* r^{th} *order between finite bounds.*

$$\int_{s_1}^{s_2} f(s)ds = \sum_{n=2}^{n=\infty} \frac{a_n}{-n^{s_2}\cdot\log n} - \sum_{n=2}^{n=\infty} \frac{a_n}{-n^{s_1}\cdot\log n}.$$

The integration in the integral on the left-hand side is to be taken along a path lying entirely to the right of $\sigma = \lambda_r$; and both of the infinite series on the right-hand side are summable of the r^{th} order.

§ 4.

Determination of the abscissa of summability λ_r $(r = 0, 1, 2, 3, \ldots)$ as a function of the coefficients of the series.

Corresponding to a theorem of Cahen (Theorem VII, page 16) and containing it as a special case ($r = 0$), we have the following theorem for summability of arbitrary order:

Theorem Ia. *Let $\sum \frac{a_n}{n^s}$ be a Dirichlet series with abscissa of r^{th} order summability equal to λ_r, and let us put*

$$S_n^{(0)} = \sum_{m=1}^{m=n} a_m;\ S_n^{(1)} = \sum_{m=1}^{m=n} S_m^{(0)};\ \ldots;\ S_n^{(r)} = \sum_{m=1}^{m=n} S_m^{(r-1)} \tag{1}$$

and

$$c_r = \limsup_{n=\infty} \frac{\log\left|\frac{S_n^{(r)}\cdot r!}{n^r}\right|}{\log n}. \tag{2}$$

Then, if $\lambda_r \geqq 0$, we have the equation

$$\lambda_r = c_r\,.^* \tag{3}$$

The proof of Theorem Ia will be carried out by proving the following somewhat stronger theorem:

---

* That the equation (3) loses its validity in general if the assumption $\lambda_r \geqq 0$ is removed, appears from the following remark: If $\lambda_r < 0$, the series $\sum \frac{a_n}{n^s}$ is summable of the r^{th} order at the point $s = 0$, i.e., $\sum \frac{a_n}{n^0} = \sum a_n$ is summable of the r^{th} order. Let us now assume that this last series has a summability value A different from 0, i.e., that $\lim \frac{S_n^{(r)}\cdot r!}{n^r} = A \neq 0$ as is naturally the case in general. Then

$$c_r = \limsup_{n=\infty} \frac{\log\left|\frac{S_n^{(r)}\cdot r!}{n^r}\right|}{\log n} = \lim_{n=\infty} \frac{\log|A|}{\log n} = 0, \text{ and consequently } \lambda_r \neq c_r.$$

Theorem Ib. *In all cases* (i.e., in any of the cases $\lambda_r \gtreqless 0$), *we have*

$$\lambda_r \leqq c_r. \tag{4}$$

If $c_r > 0$, *we also have*

$$\lambda_r \geqq c_r.^* \tag{5}$$

Before we pass to the proof of Theorem Ib, we shall make the following remark. As an abbreviation we have put

$$c_r = \limsup_{n=\infty} \frac{\log\left|\frac{S_n^{(r)} \cdot r!}{n^r}\right|}{\log n}.$$

This identity means that for arbitrary $\varepsilon > 0$ the two following inequalities hold:

$$\frac{\log\left|\frac{S_n^{(r)} \cdot r!}{n^r}\right|}{\log n} < c_r + \varepsilon \ \left(\text{for } n > N = N(\varepsilon)\right)$$

and

$$\frac{\log\left|\frac{S_n^{(r)} \cdot r!}{n^r}\right|}{\log n} > c_r - \varepsilon \ \text{(for infinitely many } n).$$

As one sees immediately, these two inequalities can also be written in the following form, which is particularly convenient for our purpose:

$$\left|\frac{S_n^{(r)} \cdot r!}{n^r}\right| < n^{c_r+\varepsilon} \ \left(\text{for } n > N = N(\varepsilon)\right),$$

or equivalently:

$$\left|\frac{S_n^{(r)} \cdot r!}{n^r}\right| < K \cdot n^{c_r+\varepsilon} \ \text{(for all } n = 1, 2, \ldots) \tag{6}$$

($K = K(\varepsilon)$ denotes a constant independent of n),

and

$$\left|\frac{S_n^{(r)} \cdot r!}{n^r}\right| > n^{c_r-\varepsilon} \ \text{(for infinitely many } n). \tag{7}$$

We now turn to the proof of the first part of Theorem Ib, i.e., to the proof that $\lambda_r \leqq c_r$ in all cases.

---

* As one immediately sees, Theorem Ib can also be expressed in the following way: *If* $c_r > 0$, *then* $\lambda_r = c_r$; *if* $c_r \leqq 0$, *then* $\lambda_r \leqq c_r$. That Theorem Ia (which can be stated as $\lambda_r = c_r$ if $\lambda_r \geqq 0$) is contained in Theorem Ib, is seen as follows: 1. If $\lambda_r > 0$, then also, in consequence of (4), $c_r > 0$, hence $\lambda_r = c_r$; 2. If $\lambda_r = 0$, then, in consequence of (4), $c_r \geqq 0$; however, c_r cannot possibly be > 0, for this would imply $\lambda_r = c_r > 0$ in contradiction to the assumption $\lambda_r = 0$; hence we have $c_r = 0 = \lambda_r$.

We do this by showing that when $s_1 = \sigma_1 + it_1$ denotes an arbitrary number such that $\sigma_1 > c_r$, i.e., such that $\sigma_1 = c_r + \delta$, where $\delta > 0$, then the series $\sum \frac{a_n}{n^{s_1}}$ is summable of the r^{th} order.

In the proof of this part, of the two inequalities (6) and (7) which define c_r we shall use only (6); in the following we shall assume that the arbitrarily small positive number ε occurring in (6) is chosen smaller than the given number $\frac{\delta}{2}$.

We now apply formula (6), § 2 (page 55), putting therein

$$u_n = a_n;\ \alpha_n = \frac{1}{n^{s_1}};\ u_n\alpha_n = \frac{a_n}{n^{s_1}},$$

whereby the quantity designated as $S_n^{(r)}$ in (6), § 2 becomes identical with the quantity designated as $S_n^{(r)}$ in (1) of this section, i.e., with the $S_n^{(r)}$ formed from the series $\sum a_n$, while $T_n^{(r)}$ in formula (6), § 2 is to be formed from the series $\sum \frac{a_n}{n^{s_1}}$.

By the aid of (6), § 2, we can now easily prove that $\sum \frac{a_n}{n^{s_1}}$ is summable of the r^{th} order, i.e., that $\frac{T_n^{(r)} \cdot r!}{n^r}$ has a finite limit for $n = \infty$. In fact, as was done in the investigations of § 2, we can show that the $r+2$ terms (10), § 2, (12), §2, and (14), § 2 of which $\frac{T_n^{(r)} \cdot r!}{n^r}$ consists, all have finite limits for $n = \infty$. For the expression

$$\alpha_n \frac{S_n^{(r)} \cdot r!}{n^r}$$

we obtain at once

$$\left|\alpha_n \frac{S_n^{(r)} \cdot r!}{n^r}\right| = \left|\frac{1}{n^{s_1}}\right| \cdot \left|\frac{S_n^{(r)} \cdot r!}{n^r}\right| < \frac{1}{n^{\sigma_1}} \cdot K \cdot n^{c_r+\varepsilon} < \frac{1}{n^{c_r+\delta}} \cdot K \cdot n^{c_r+\frac{\delta}{2}} = K \cdot n^{-\frac{\delta}{2}},$$

and consequently
$$\lim_{n=\infty} \alpha_n \frac{S_n^{(r)} \cdot r!}{n^r} = 0\,. \tag{8}$$

We next consider the expression

$$\frac{r!}{n^r}\binom{r+1}{q} P_{r,q,n} = \frac{r!}{n^r}\binom{r+1}{q} \sum_{m=1}^{m=n-q} S_m^{(r)} \binom{n-m-1}{q-1} \Delta^q \alpha_m$$
$$(q = 1, 2, \ldots, r)\,.$$

If we separate the general term

$$\frac{r!}{n^r}\binom{r+1}{q} S_m^{(r)} \binom{n-m-1}{q-1} \Delta^q \alpha_m$$

into the two factors

$$v_m = m^{q-1}\Delta^q \alpha_m \cdot m^{c_r+\frac{\delta}{2}}$$

and

$$\beta_{n,m} = \binom{r+1}{q} \frac{1}{(q-1)!} \left(\frac{S_m^{(r)} \cdot r!}{m^r} \cdot \frac{1}{m^{c_r+\frac{\delta}{2}}}\right) \left(\frac{(n-m-1)\cdots(n-m-q+1)}{n^{q-1}}\right) \left(\frac{m}{n}\right)^{r-q+1}$$

we obtain

$$|v_m| = (m^{q-1}|\Delta^q \alpha_m|) \cdot \left(m^{c_r+\frac{\delta}{2}}\right) < \frac{K_1}{m^{c_r+\delta+1}} \cdot m^{c_r+\frac{\delta}{2}} = \frac{K_1}{m^{1+\frac{\delta}{2}}}$$

(in view of (20), § 2 we have, in fact, $m^{q-1}|\Delta^q \alpha_m| = m^{q-1} \cdot \left|\Delta^q \frac{1}{m^{s_1}}\right| < \frac{K_1}{m^{1+\sigma_1}}$, where K_1 denotes a constant independent of m and $q = 1, 2, \ldots, r+1$).

Hence

$$\sum |v_m| \text{ converges.}$$

We also have

$$|\beta_{n,m}| < K_2 \quad (K_2 \text{ independent of } m \text{ and } n)$$

since

$$\left|\frac{S_m^{(r)} \cdot r!}{m^r} \cdot \frac{1}{m^{c_r+\frac{\delta}{2}}}\right| < K \cdot m^{c_r+\varepsilon} \cdot \frac{1}{m^{c_r+\frac{\delta}{2}}} < \frac{K}{m^{\frac{\delta}{2}-\varepsilon}} < K\,,$$

and

$$\beta_m = \lim_{n=\infty} \beta_{n,m} = 0$$

on account of the factor $\left(\frac{m}{n}\right)^{r-q+1}$.

Consequently, we can apply Theorem Va, § 1 (page 51) and infer therefrom

$$\lim_{n=\infty} \frac{r!}{n^r}\binom{r+1}{q} P_{r,q,n} = \lim_{n=\infty} (v_1 \cdot \beta_{n,1} + \cdots + v_{n-q} \cdot \beta_{n,n-q})$$
$$= \sum_{n=1}^{n=\infty} v_n \beta_n = \sum_{n=1}^{n=\infty} 0 = 0\,. \tag{9}$$

We now turn to the consideration of the last term in (6), § 2:

$$\frac{r!}{n^r} P_{r,r+1,n} = \frac{r!}{n^r} \sum_{m=1}^{m=n-r-1} S_m^{(r)} \binom{n-m-1}{r} \Delta^{r+1} \alpha_m\,.$$

If we put

$$v_m = m^r \Delta^{r+1} \alpha_m \cdot m^{c_r+\frac{\delta}{2}}$$

and

$$\beta_{n,m} = \frac{1}{r!}\left(\frac{S_m^{(r)} \cdot r!}{m^r} \cdot \frac{1}{m^{c_r+\frac{\delta}{2}}}\right)\left(\frac{(n-m-1)\cdots(n-r)}{n^r}\right),$$

we obtain

$$|v_m| = m^r \cdot |\Delta^{r+1}\alpha_m| \cdot m^{c_r+\frac{\delta}{2}} < \frac{K_1}{m^{c_r+\delta+1}} \cdot m^{c_r+\frac{\delta}{2}} < \frac{K_1}{m^{1+\frac{\delta}{2}}};$$

hence

$$\sum |v_m| \text{ converges.}$$

Further, we have as previously

$$|\beta_{n,m}| < \text{constant (independent of } m \text{ and } n)$$

and

$$\beta_m = \lim_{n=\infty} \beta_{n,m} = \frac{S_m^{(r)}}{m^r} \frac{1}{m^{c_r+\frac{\delta}{2}}}.$$

In consequence, we find upon applying Theorem V a, § 1

$$\lim_{n=\infty} \frac{r!}{n^r} P_{r,r+1,n} = \lim_{n=\infty} \sum_{m=1}^{m=n-r-1} v_m \beta_{n,m} = \sum_{n=1}^{n=\infty} v_n \beta_n$$

$$= \sum_{n=1}^{n=\infty} S_n^{(r)} \Delta^{r+1}\alpha_n = \sum_{n=1}^{n=\infty} S_n^{(r)} \Delta^{r+1}\left(\frac{1}{n^{s_1}}\right). \tag{10}$$

The last infinite series, also as a consequence of Theorem V a, § 1, converges absolutely.

When we finally assemble the results expressed in equations (8), (9), and (10), we see that we have proved that $\frac{T_n^{(r)} \cdot r!}{n^r}$ has a finite limit for $n = \infty$, i.e., that $\sum \frac{a_n}{n^{s_1}}$ is summable of the r^{th} order, and we have shown that the summability value of the series is equal to the sum of the absolutely convergent series

$$\sum_{n=1}^{n=\infty} S_n^{(r)} \Delta^{r+1}\left(\frac{1}{n^{s_1}}\right), \tag{11}$$

where $S_n^{(r)}$ is formed from the coefficients of the given Dirichlet series $\sum \frac{a_n}{n^s}$.*

---

* If we assume for the moment that Theorem I b (and hence also Theorem I a) has been proved, then we can infer with the aid of the expression (11) the following theorem, which we shall use in a subsequent section: *Let* $\sum \frac{a_n}{n^s}$ *be a Dirichlet series with abscissa of* r^{th} *order summability equal to* λ_r, *and let* $s_0 = \sigma_0 + it_0$ *be a number such that* $\sigma_0 \leq \lambda_r$; *if we now put*

We now turn to the proof of the last part of Theorem Ib, i.e., to the proof that $\lambda_r \geqq c_r$ if $c_r > 0$.

The proof of this fact will obviously be completed when we have shown that, if $\sum \frac{a_n}{n^s}$ is summable of the r^{th} order for $s = s_1 = \sigma_1 + it_1$, where $\sigma_1 > 0$, then the inequality $\sigma_1 \geqq c_r$ holds. (Indeed, since $c_r > 0$, there would, if $\lambda_r < c_r$, necessarily exist points $s = \sigma + it$ such that $\sigma > 0$ and also $\lambda_r < \sigma < c_r$.) The inequality $\sigma_1 \geqq c_r$ will be proved if we can show that for sufficiently large n (i.e., for $n > N = N(\varepsilon)$)

$$\left| \frac{S_n^{(r)} \cdot r!}{n^r} \right| < n^{\sigma_1 + \varepsilon} \quad (\varepsilon \text{ an arbitrarily small positive number}), \tag{12}$$

for, from this it will follow that

$$\frac{\log \left| \frac{S_n^{(r)} \cdot r!}{n^r} \right|}{\log n} < \sigma_1 + \varepsilon \quad (\text{for } n > N),$$

and hence

$$c_r = \limsup_{n=\infty} \frac{\log \left| \frac{S_n^{(r)} \cdot r!}{n^r} \right|}{\log n} \leqq \sigma_1.$$

In order to prove the inequality (12), we use once more formula (6), § 2 (page

---

$$S_n^{(0)} = \sum_{m=1}^{m=n} \frac{a_m}{m^{s_0}}; \ldots ; S_n^{(r)} = \sum_{m=1}^{m=n} S_m^{(r-1)},$$

the summability value of the series $\sum \frac{a_n}{n^s}$, *which is summable of the* r^{th} *order for* $\sigma > \lambda_r$, *is represented in the whole half-plane* $\sigma > \lambda_r$ *by the absolutely convergent series* $\sum S_n^{(r)} \Delta^{r+1} \left(\frac{1}{n^{s-s_0}} \right)$.

In the proof of this theorem, we can obviously assume $s_0 = 0$, when we assume at the same time $\lambda_r \geqq 0$. (In fact, if the theorem is proved in this special case, the general theorem can be inferred at once by the use of a transformation of variable.) We therefore assume $\lambda_r \geqq 0$. Then, in view of Theorem Ia, we have $c_r = \lambda_r$. If we now consider an arbitrary number $s_1 = \sigma_1 + it_1$ such that $\sigma_1 > \lambda_r$, we have also $\sigma_1 > c_r$, and the validity of the theorem follows at once from the expression (11), which for $\sigma_1 > c_r$ gives the summability value of the series at the point s_1.

If we omit the condition $\sigma_0 \leqq \lambda_r$ in the above general theorem, the theorem loses its validity. Indeed, if $\sigma_0 > \lambda_r$ (in which case the series $\sum \frac{a_n}{n^s}$ is certainly summable of the r^{th} order at the point s_0), we find that the series

$$\sum S_n^{(r)} \Delta^{r+1} \left(\frac{1}{n^{s-s_0}} \right)$$

is in general not absolutely convergent (or even convergent) for all s such that $\sigma > \lambda_r$, but only for s such that $\sigma > \sigma_0$, as we have already proved in § 1.

55). Our task here being that of drawing conclusions about $S_n^{(r)}$ (formed from the series $\sum a_n$) from assumptions about $\sum \frac{a_n}{n^{s_1}}$, we write here, in contrast to what we did in the proof of the first part of Theorem I b,

$$u_n = \frac{a_n}{n^{s_1}};\ \alpha_n = n^{s_1};\ u_n \alpha_n = a_n .$$

The quantity $T_n^{(r)}$ in formula (6), § 2 then becomes identical with what we have denoted by $S_n^{(r)}$ in the present section, i.e., with the $S_n^{(r)}$ formed from the series $\sum a_n$, while $S_n^{(r)}$ in formula (6), § 2—this $S_n^{(r)}$ we shall denote by $\overset{*}{S}{}_n^{(r)}$ in the following—is to be formed from the series $\sum \frac{a_n}{n^{s_1}}$, which by assumption is summable of the r^{th} order.

Formula (6), § 2 gives us immediately the following inequality:

$$\left|\frac{T_n^{(r)} \cdot r!}{n^r}\right| \equiv \left|\frac{S_n^{(r)} \cdot r!}{n^r}\right| \leqq \frac{r!}{n^r} \sum_{q=0}^{q=r+1} \binom{r+1}{q} |P_{r,q,n}| .$$

If we consider first the term

$$\frac{r!}{n^r} \cdot P_{r,0,n} = \frac{r!}{n^r} \alpha_n \overset{*}{S}{}_n^{(r)} ,$$

we obtain

$$\frac{r!}{n^r} |P_{r,0,n}| = |\alpha_n| \cdot \left|\frac{\overset{*}{S}{}_n^{(r)} \cdot r!}{n^r}\right| < n^{\sigma_1} \cdot K ; \tag{13}$$

for, since $\sum \frac{a_n}{n^{s_1}}$ is summable of the r^{th} order, we have *a fortiori*

$$\left|\frac{\overset{*}{S}{}_n^{(r)} \cdot r!}{n^r}\right| < K \text{ (const.)}.$$

In the foregoing investigations, besides being compelled to consider separately the first term $P_{r,0,n}$ in formula (6), § 2, we had to give special attention to the last term $P_{r,r+1,n}$ as well. This step is unnecessary here. We therefore turn to the consideration of any one of the quantities

$$\frac{r!}{n^r} \binom{r+1}{q} P_{r,q,n} = \frac{r!}{n^r} \binom{r+1}{q} \sum_{m=1}^{m=n-q} \overset{*}{S}{}_m^{(r)} \binom{n-m-1}{q-1} \Delta^q \alpha_m$$
$$(q = 1, 2, \ldots, r+1)$$

and find

$$\left|\frac{r!}{n^r}\binom{r+1}{q}\overset{*}{S}{}_m^{(r)}\binom{n-m-1}{q-1}\Delta^q\alpha_m\right|$$

$$=\left|\frac{r!\overset{*}{S}{}_m^{(r)}}{m^r}\right|\binom{r+1}{q}\frac{1}{(q-1)!}\left(\frac{(n-m-1)\cdots(n-m-q+1)}{n^{q-1}}\right)\cdot\left(\frac{m}{n}\right)^{r-q+1}|m^{q-1}\Delta^q\alpha_m|$$

$$< K\cdot\binom{r+1}{q}\frac{1}{(q-1)!}\cdot K_1\cdot m^{\sigma_1-1}{}^{*} < K_2\cdot m^{\sigma_1-1},$$

where K_2 denotes a constant independent of m, n and q $(q = 1, 2,\ldots, r+1)$. Consequently we have

$$\left|\frac{r!}{n^r}\binom{r+1}{q}P_{r,q,n}\right| < K_2\sum_{m=1}^{m=n-q} m^{\sigma_1-1} \leqq K_2\sum_{m=1}^{m=n-1} m^{\sigma_1-1}$$

$$< K_2\int_0^n x^{\sigma_1-1}dx = \frac{K_2}{\sigma_1}\cdot n^{\sigma_1} = K_3\cdot n^{\sigma_1}.\dagger \tag{14}$$

Assembling the results obtained in (13) and (14), we obtain

$$\left|\frac{S_n^{(r)}\cdot r!}{n^r}\right| \leqq \sum_{q=0}^{q=r+1}\left|\frac{r!}{n^r}\binom{r+1}{q}P_{r,q,n}\right| < K\cdot n^{\sigma_1}+(r+1)K_3\cdot n^{\sigma_1} < K_4\cdot n^{\sigma_1}$$

and hence, ε denoting an arbitrarily small positive number, we find

$$\left|\frac{S_n^{(r)}\cdot r!}{n^r}\right| < n^{\sigma_1+\varepsilon}\ \bigl(\text{for } n > N = N(\varepsilon)\bigr). \qquad \text{q.e.d.}$$

Hereby Theorem I b, and consequently also Theorem I a, is completely proved.

As one immediately sees, the following theorem, which is especially convenient for later applications, is included in Theorem I b:

Theorem Ic. *If it is known only that one of the quantities λ_r and c_r is greater than 0, then always $\lambda_r = c_r$.*

We shall conclude this section by applying the results obtained above to prove the existence of a Dirichlet series—or, more accurately, of an entire class of Dirichlet

---

* In consequence of formula (20), § 2 (page 61), we have

$$|m^{q-1}\Delta^q\alpha_m| = |m^{q-1}\Delta^q m^{s_1}| < K_1 m^{\sigma_1-1}.$$

† Only at this point it is used that $\sigma_1 > 0$;—in fact, the inequality

$$\sum_{m=1}^{m=n-1} m^{\sigma_1-1} < \text{const.}\ n^{\sigma_1}$$

is only true if $\sigma_1 > 0$.

series—*whose abscissae of summability* λ_r *for all* $r = 0, 1, 2, \ldots$ *satisfy the condition*

$$\lambda_r - \lambda_{r+1} = \theta\,,$$

where θ *denotes an arbitrary number lying between* 0 *and* 1.

The existence of such series—and to have established this existence will turn out to be of importance in our subsequent investigations—is proved by the following theorem:

Theorem II. *Let* $p_1, p_2, \ldots, p_m, \ldots$ *be an arbitrary sequence of positive integers which satisfy the following condition for all* m:

$$p_{m+1} \geqq (m+1)\cdot p_m \tag{15}$$

$$(e.g.\ \ p_m = m! \ \ or \ \ p_m = m^m)$$

and let q_m *denote the largest integer less than* $p_m^{1-\theta}$ $(0 < \theta < 1)$.

The Dirichlet series

$$\sum \frac{a_n}{n^s} = \frac{1}{p_1^s} - \frac{1}{(p_1+q_1)^s} + \frac{1}{p_2^s} - \frac{2}{(p_2+q_2)^s} + \frac{1}{(p_2+2q_2)^s} + \cdots$$

$$+ \frac{1}{p_m^s} - \frac{\binom{m}{1}}{(p_m+q_m)^s} + \frac{\binom{m}{2}}{(p_m+2q_m)^s} + \cdots + (-1)^m \frac{\binom{m}{m}}{(p_m+mq_m)^s} + \cdots$$

then has its abscissae of summability λ_r *determined by the equation*

$$\lambda_r = -r\cdot\theta \ \ (r = 0, 1, 2, \ldots)\,.$$

Proof. Since the terms in $\sum \frac{a_n}{n^s}$ do not converge to 0 as n tends to infinity, when $s < 0$, the series cannot converge to the left of the imaginary axis; consequently, we have $\lambda_0 \geqq 0$.

Since, on the other hand, for $s > 0$,

$$\left|\frac{1}{p_m^s}\right| + \left|\frac{-\binom{m}{1}}{(p_m+q_m)^s}\right| + \cdots + \left|\frac{(-1)^m\binom{m}{m}}{(p_m+mq_m)^s}\right| < \frac{1}{p_m^s}\left(1+\binom{m}{1}+\cdots+\binom{m}{m}\right) = \frac{2^m}{p_m^s}\,,$$

and since $\sum\limits_{m=1}^{m=\infty} \frac{2^m}{p_m^s}$ is seen at once to be convergent for $s > 0$ (because

$$\lim_{n=\infty}\left(\frac{2^{m+1}}{(p_{m+1})^s} : \frac{2^m}{p_m^s}\right) = 0$$

for all positive values of s, however small, as (15) shows), it follows that $\sum\frac{a_n}{n^s}$ converges (indeed, converges absolutely) for $s > 0$; hence we have $\lambda_0 = 0$.

We now pass to the proof that the equation $\lambda_r = -r\theta$ holds for arbitrary $r = 1, 2, 3, \ldots$. In order to apply Theorem Ic to this proof, let us first transform the series $\sum\frac{a_n}{n^s}$ into the series

$$\sum\frac{b_n}{n^s} = \sum\frac{a_n}{n^{s-r}} = \sum\frac{a_n \cdot n^r}{n^s}.$$

We shall then prove that the last series $\sum\frac{b_n}{n^s}$ has its r^{th} abscissa of summability equal to the *positive* quantity $r-r\theta = r(1-\theta)$.

Instead of considering the transformed series $\sum\frac{b_n}{n^s}$ itself, it will, however, be convenient to consider the series

$$\sum\frac{d_n}{n^s} = \sum_{m=2r+1}^{m=\infty}\left(\frac{p_m^r}{p_m^s} - \frac{\binom{m}{1}(p_m+q_m)^r}{(p_m+q_m)^s}\cdots+(-1)^m\frac{\binom{m}{m}(p_m+mq_m)^r}{(p_m+mq_m)^s}\right),$$

where the series $\sum\frac{b_n}{n^s}$ and $\sum\frac{d_n}{n^s}$ are identical, except that, in the latter series, the 1st, 2nd, …, $2r^{\text{th}}$ 'term groups' are omitted. As one can see at once, the alteration from $\sum\frac{b_n}{n^s}$ to $\sum\frac{d_n}{n^s}$ cannot change the summability behaviour of the series.

We shall accordingly prove that $\sum\frac{d_n}{n^s}$ has its r^{th} abscissa of summability equal to $r(1-\theta)$; by virtue of Theorem Ic, this will be proved if we show that

$$c_r \equiv \limsup_{n=\infty}\frac{\log\left|\dfrac{S_n^{(r)}\cdot r!}{n^r}\right|}{\log n},$$

where $S_n^{(r)}$ is formed from the series $\sum d_n$, has the value $r(1-\theta)$.

The coefficients $d_1, d_2, \ldots, d_n, \ldots$ fall naturally into separated groups of the form

$$d_{p_m} = p_m^r,\ d_{p_m+1} = \cdots = d_{p_m+q_m-1} = 0\,,$$

$$d_{p_m+q_m} = -\binom{m}{1}(p_m+q_m)^r, \ldots, d_{p_m+mq_m} = (-1)^m\binom{m}{m}(p_m+mq_m)^r;$$

before the first term group (corresponding to $m = 2r+1$) and between two successive groups, all the d's are equal to 0.

If we now form $S_n^{(0)}, S_n^{(1)}, \ldots, S_n^{(r)}$ from the series $\sum d_n$, we can first show that one can handle each of the above-mentioned term groups by itself in forming these sums. By this, we understand that, in the first place,

$$S_n^{(k)} = 0 \text{ for } p_m + mq_m < n < p_{m+1} \text{ and } k = 0, 1, 2, \ldots, r,$$

and, in the second place, in determining

$$S_{p_m+l}^{(k)} \; (0 \leqq l \leqq mq_m, \; k = 0, 1, 2, \ldots, r)$$

one need not consider the d's with index less than p_m (i.e., in determining $S_{p_m+l}^{(k)}$ one can operate as if the equations $d_1 = \cdots = d_{p_m-1} = 0$ were valid).

We shall now establish the legitimacy of this procedure. We consider a single one of the term groups and for brevity's sake, we put $d_{p_m+n-1} = u_n$. Then we have

$$u_1 = p_m^r, \; u_2 = \cdots = u_{q_m} = 0,$$

$$u_{q_m+1} = -\binom{m}{1}(p_m+q_m)^r, \ldots, u_{mq_m+1} = (-1)^m \binom{m}{m}(p_m+mq_m)^r .$$

We put

$$T_n^{(0)} = \sum_{m=1}^{m=n} u_m ; \; T_n^{(1)} = \sum_{m=1}^{m=n} T_m^{(0)}; \ldots ; T_n^{(r)} = \sum_{m=1}^{m=n} T_m^{(r-1)} .$$

Then it will suffice to prove that the equation

$$T_{mq_m+1}^{(k)} = 0 \tag{16}$$

holds for all $k = 0, 1, \ldots, r$ and $m = 2r+1, 2r+2, \ldots$.

Indeed, if one imagines the successive formation of $S_n^{(0)}, \ldots, S_n^{(r)}$ one sees that when (16) holds one will always leave a term group with the value 0 and arrive at the following term group with the value 0. From this it follows that one can construct the quantities $S_n^{(k)}$ by considering the individual term groups without worrying about the preceding ones.

In order to prove (16), we now use the formula stated above (page 47):

$$T_n^{(k)} = u_1\binom{n+k-1}{k} + u_2\binom{n+k-2}{k} + \cdots + u_n\binom{k}{k}.$$

If we here put $n = mq_m+1$ and substitute the values for $u_1, u_2, \ldots$, we obtain

$$T_{mq_m+1}^{(k)} = p_m^r \binom{mq_m+k}{k} -$$

$$\binom{m}{1}(p_m+q_m)^r \binom{(m-1)q_m+k}{k} + \cdots + (-1)^m \binom{m}{m}(p_m+mq_m)^r \binom{k}{k}$$

$$= \sum_{\alpha=0}^{\alpha=r}\left[\left(\binom{r}{\alpha}q_m^\alpha p_m^{r-\alpha}\right)\cdot\left(0^\alpha\binom{mq_m+k}{k}-\binom{m}{1}1^\alpha\binom{(m-1)q_m+k}{k}\dots+(-1)^m\binom{m}{m}m^\alpha\binom{k}{k}\right)\right]^*$$

$$= \sum_{\alpha=0}^{\alpha=r}\left[\left(\binom{r}{\alpha}q_m^\alpha p_m^{r-\alpha}\right)\left(\sum_{\beta=0}^{\beta=k}\left\{q_m^\beta K_\beta\left(0^\alpha m^\beta-\binom{m}{1}1^\alpha(m-1)^\beta\dots+(-1)^m\binom{m}{m}m^\alpha 0^\beta\right)\right\}\right)\right]$$

(where K_β ($\beta = 0, 1, \dots, k$) denotes a constant depending only on β).

However, for $\alpha+\beta < m$ (which is here always the case, since $\alpha+\beta \leqq 2r$ and $m \geqq 2r+1$), we have

$$0^\alpha m^\beta-\binom{m}{1}1^\alpha(m-1)^\beta+\dots+(-1)^{m-1}\binom{m}{m-1}(m-1)^\alpha 1^\beta+(-1)^m\binom{m}{m}m^\alpha 0^\beta$$

$$= \Bigg\{\underbrace{-z\frac{d}{dz}\Big(\dots\Big(-z\frac{d}{dz}}_{\beta\text{ times}}\Big(z^{-m}\Big[\underbrace{z\frac{d}{dz}\Big(\dots\Big(z\frac{d}{dz}\Big(z\frac{d}{dz}}_{\alpha\text{ times}}[(1-z)^m]\Big)\Big)\dots\Big)\Big]\Big)\Big)\dots\Big)\Bigg\}_{z=1} = 0;$$

hence we obtain
$$T^{(k)}_{mq_m+1} = 0 \text{ (for } k = 0, 1, \dots, r). \qquad \text{q.e.d. (16)}$$

It follows immediately from equation (16), as emphasized above, that

$$S_n^{(r)} = 0 \text{ for } p_m+mq_m < n < p_{m+1} \tag{17}$$

and

$$S^{(r)}_{p_m+l} = T^{(r)}_{l+1} \text{ for } 0 \leqq l \leqq mq_m. \tag{18}$$

From (18) we now infer, when l is any of the numbers $0, 1, \dots, mq_m$, that

$$|S^{(r)}_{p_m+l}| = |T^{(r)}_{l+1}|$$

$$\leqq p_m^r\binom{mq_m+r}{r}+(p_m+q_m)^r\binom{m}{1}\binom{(m-1)q_m+r}{r}+\dots+(p_m+mq_m)^r\binom{m}{m}\binom{r}{r}$$

$$< (p_m+mq_m)^r\binom{mq_m+r}{r}\left(1+\binom{m}{1}+\dots+\binom{m}{m}\right)$$

$$< K(p_m+mp_m^{1-\theta})^r(mq_m)^r 2^m < K_1 p_m^r(mp_m^{1-\theta})^r 2^m < K_2 p_m^{r+r(1-\theta)}m^r 2^m < p_m^{r+r(1-\theta)+\varepsilon_m}$$

$$\leqq (p_m+l)^{r+r(1-\theta)+\varepsilon_m} \quad (\lim_{m=\infty}\varepsilon_m = 0).$$

From the inequality
$$|S^{(r)}_{p_m+l}| < (p_m+l)^{r+r(1-\theta)+\varepsilon_m},$$

valid for $0 \leqq l \leqq mq_m$, in conjunction with (17), it follows immediately that

$$c_r \equiv \limsup_{n=\infty}\frac{\log\left|S_n^{(r)}\dfrac{r!}{n^r}\right|}{\log n} \leqq r(1-\theta). \tag{19}$$

---

* 0^α (and in the sequel 0^β) is to be considered as 1 when α (or β) is equal to 0.

We now pass to the proof that c_r cannot be less than $r(1-\theta)$; that this is the case may be seen in the following manner.

If we consider $S_n^{(r)}$ for $n = p_m+q_m-1$ $(m = 2r+1, 2r+2,\ldots)$, we obtain

$$S^{(r)}_{p_m+q_m-1} = T^{(r)}_{q_m} = p_m^r \binom{q_m+r-1}{r} > \frac{1}{r!}\, p_m^r \cdot q_m^r$$

$$> K \cdot p_m^{r+r(1-\theta)} > K_1 (p_m+q_m-1)^{r+r(1-\theta)} \text{ (where } K_1 > 0);$$

from this, it follows immediately that

$$c_r \geqq r(1-\theta)\,. \tag{20}$$

Finally, it follows from (19) and (20) that $c_r = r(1-\theta)$. q.e.d.

We have thus proved that, corresponding to an arbitrary number θ lying between 0 and 1, there exist Dirichlet series $\sum \frac{a_n}{n^s}$ whose abscissae of summability are equi-distant, with the distance θ.

The question naturally arises whether or not for other values of θ there exist Dirichlet series such that $\lambda_r - \lambda_{r+1}$ is constant and equal to θ. *For the cases* $\theta = 0$ *and* $\theta = 1$, *this question must be answered affirmatively.* Thus, the series $\zeta(s) = \sum \frac{1}{n^s}$ (or any other series with positive coefficients for which $\lambda_0 \neq \pm\infty$) belongs to the first category, while the series $\zeta(s)(1-2^{1-s}) = \sum \frac{(-1)^{n+1}}{n^s}$, for example, is of the latter type, as was shown in § 3. (Incidentally, Theorem II holds also in the case $\theta = 1$, as one can easily show.)

On the other hand, as will be shown in the next section, *there cannot exist Dirichlet series corresponding to any value of* θ *greater than* 1.

§ 5.

Distribution of the abscissae of summability.

We shall introduce this section by deducing some general theorems concerning the difference $\lambda_r - \lambda_{r+1}$ between two successive abscissae of summability. For this purpose we shall utilize the expression found in the preceding section for the abscissa of summability λ_r as a function of the coefficients of the series.

In the proof of these theorems, the results obtained in the preceding section will come into play in the form expressed by Theorem Ic, § 4:

If $c_r > 0$, *then* $\lambda_r = c_r$; *if* $\lambda_r > 0$, *then* $\lambda_r = c_r$ $(r = 0, 1, 2,\ldots)$. Here, c_r is deter-

mined by (1), § 4 and (2), § 4, or, what is equivalent to (2), § 4, by the inequalities

$$\left|\frac{S_n^{(r)}\cdot r!}{n^r}\right| < n^{c_r+\delta} \quad \text{for } n > N = N(\delta)$$

and

$$\left|\frac{S_n^{(r)}\cdot r!}{n^r}\right| > n^{c_r-\delta} \quad \text{for infinitely many } n.$$

Since the quantity $r!$, which is independent of n, can obviously be omitted, these inequalities may also be written in the following form, particularly convenient for our purpose, in which, for brevity, we have put $c_r+r = d_r$ $(r = 0, 1, 2, \ldots)$:

$$|S_n^{(r)}| < n^{r+c_r+\delta} = n^{d_r+\delta} \quad \text{for } n > N = N(\delta) \tag{1}$$

and

$$|S_n^{(r)}| > n^{r+c_r-\delta} = n^{d_r-\delta} \quad \text{for infinitely many } n. \tag{2}$$

Theorem I. *The breadth of the strip of r^{th} order summability is less than or equal to* 1 *for all* $r = 1, 2, \ldots$, *i.e., expressed in terms of the abscissae of summability, we have*

$$\lambda_r - \lambda_{r+1} \leqq 1 \ \textit{or}\ \lambda_{r+1} \geqq \lambda_r - 1 \ \ (r = 0, 1, 2, \ldots).^*$$

Proof. If we take $\lambda_r > 1$ (as we can always do, if $\lambda_r \neq -\infty$, by a simple transformation of variable which obviously leaves the differences $\lambda_r - \lambda_{r+1}$ unchanged), then $\lambda_r = c_r$, and the proof of Theorem I will plainly be completed if we can show that

$$c_{r+1} \geqq c_r - 1\,; \tag{3}$$

for, from this it will also follow that $c_{r+1} > 0$, and hence $\lambda_{r+1} = c_{r+1}$.

Since $d_r = r + c_r$ and $d_{r+1} = (r+1) + c_{r+1}$, the inequality (3) can also be written in the form

$$d_{r+1} \geqq d_r\,. \tag{4}$$

The validity of this last inequality—and therewith the validity of Theorem I—may be seen by the following indirect consideration.

---

* It follows immediately from Theorem I that if the series $\sum \frac{a_n}{n^s}$ is summable of the r^{th} order at the point $s_0 = \sigma_0 + it_0$, then it is summable of the $(r-1)^{\text{th}}$ order for all $s = \sigma + it$ such that $\sigma > \sigma_0 + 1$. This theorem has been communicated by the author in the note cited above: Sur la série de Dirichlet, *Comptes rendus de l'Académie des Sciences, Paris*, vol. 148, 1909 (11 January). In a note published half a year later: Sur les séries de Dirichlet, *Comptes rendus de l'Académie des Sciences, Paris*, vol. 148, 1909 (21 June), M. Riesz has communicated the addition to this theorem that one can also infer from the above assumption that $\sum \frac{a_n}{n^s}$ is summable of the $(r-1)^{\text{th}}$ order for values of s such that $\sigma = \sigma_0 + 1$.

If we assume that (4) is incorrect, i.e., $d_{r+1} < d_r$, then we can determine two numbers α and β such that $d_{r+1} < \alpha < \beta < d_r$.

Since $\alpha > d_{r+1}$, we infer from (1) (when we put $r+1$ instead of r) that

$$|S_n^{(r+1)}| < n^\alpha \text{ for } n > N_1 . \tag{5}$$

Upon applying the identity

$$S_n^{(r)} = S_n^{(r+1)} - S_{n-1}^{(r+1)} ,$$

we infer with the aid of (5) that

$$|S_n^{(r)}| \leqq |S_n^{(r+1)}| + |S_{n-1}^{(r+1)}| < n^\alpha + (n-1)^\alpha < n^\beta \text{ for } n > N_2 . \tag{6}$$

However, since $\beta < d_r$, it follows from (2) that

$$|S_n^{(r)}| > n^\beta \text{ for infinitely many } n . \tag{7}$$

Our assumption that $d_{r+1} < d_r$ has thus led us to the two contradictory inequalities (6) and (7). Therefore we have $d_{r+1} \geqq d_r$. q.e.d.*

Whereas Theorem I considered only one of the differences $\lambda_r - \lambda_{r+1}$ $(r = 0, 1, 2, \ldots)$, the following, much deeper, theorem shows a relation between two successive differences of abscissae of summability.

Theorem II. *The breadths of the strips of summability of the* $1^{\text{st}}, 2^{\text{nd}}, \ldots, r^{\text{th}}, \ldots$ *orders form a monotonically decreasing sequence of numbers, i.e.*

$$\lambda_r - \lambda_{r+1} \geqq \lambda_{r+1} - \lambda_{r+2} \quad (r = 0, 1, 2, \ldots).$$

Proof. We shall assume $\lambda_r > 2$, so that we can be sure that the three abscissae of summability λ_r, λ_{r+1}, and λ_{r+2} are all positive. If we put $\lambda_{r+1} = \lambda_r - (1-\alpha)$, we can further assume in the proof that $1 \geqq \alpha > 0$; for, if $\alpha = 0$ (i.e., $\lambda_r - \lambda_{r+1} = 1$), Theorem II follows immediately from Theorem I.

Finally, for the sake of brevity, we put

$$d_r = \lambda_r + r = \beta , \tag{8}$$

---

* When we assume that λ_r is greater than 1, we have seen that the theorem $\lambda_r - \lambda_{r+1} \leqq 1$ is identical with the theorem $d_{r+1} \geqq d_r$. If we assume, always under the assumption $\lambda_r > 1$, that $d_{r+1} = d_r + \alpha$, then this equation becomes identical with the equation $c_{r+1} = c_r - (1-\alpha)$, and hence with the equation $\lambda_r - \lambda_{r+1} = 1 - \alpha$. As stated in § 2, it follows immediately from the definition of summability, that $\lambda_r \geqq \lambda_{r+1}$, and hence α must obviously be $\leqq 1$. That this is the case can also be seen directly as follows. From $|S_n^{(r)}| < K n^{d_r+\delta}$ $(n = 1, 2, \ldots)$, it follows that

$$|S_n^{(r+1)}| \leqq |S_1^{(r)}| + |S_2^{(r)}| + \ldots + |S_n^{(r)}| < K(1^{d_r+\delta} + 2^{d_r+\delta} + \ldots + n^{d_r+\delta}) < K_1 n^{1+d_r+\delta} ,$$

and from this we immediately infer that $d_{r+1} \leqq d_r + 1$ and thus $\alpha \leqq 1$. q.e.d.

so that
$$d_{r+1} = \lambda_{r+1}+(r+1) = (\lambda_r-(1-\alpha))+(r+1) = (\lambda_r+r)+\alpha = \beta+\alpha\,. \tag{9}$$

We shall prove that $\lambda_r-\lambda_{r+1} \geqq \lambda_{r+1}-\lambda_{r+2}$, i.e., that
$$\lambda_{r+2} \geqq 2\lambda_{r+1}-\lambda_r = 2(\lambda_r-(1-\alpha))-\lambda_r = \lambda_r-2+2\alpha\,,$$
or, equivalently, that
$$d_{r+2} = \lambda_{r+2}+(r+2) \geqq \lambda_r+r+2\alpha = \beta+2\alpha\,. \tag{10}$$

The proof of Theorem II is thus reduced to proving from the assumptions (8) and (9) that
$$d_{r+2} \geqq \beta+2\alpha\,, \tag{10}$$
or, equivalently, to proving that if ε is an arbitrarily small number (we choose it smaller than the given number $\alpha > 0$), and if E is an arbitrarily large positive number independent of ε, then there exist integers $n > E$ such that
$$|S_n^{(r+2)}| > n^{\beta+2\alpha-\varepsilon}. \tag{11}$$

For this purpose we first determine three positive numbers δ_1, δ_2, and δ_3, which satisfy the following conditions:
$$\delta_1+\delta_2 < \frac{2\varepsilon}{3}\,,\ 2\delta_3-\delta_1 > \varepsilon,\ \delta_2 > \delta_3$$
$$\left(\text{e.g. } \delta_1 = \frac{\varepsilon}{19}, \delta_2 = \frac{11\varepsilon}{18}, \delta_3 = \frac{5\varepsilon}{9}\right).$$

After fixing these three numbers, we determine an integer m such that for $n > m$, the following three conditions are satisfied:
$$|S_n^{(r)}|\,,\ |S_{n+1}^{(r)}|\,,\ldots,\ |S_{2n}^{(r)}| \text{ are all } < n^{d_r+\delta_1} = n^{\beta+\delta_1}\,,$$
$$n^{\alpha-\delta_3}-n^{\alpha-\delta_2} > 1\,, \tag{12}$$
and
$$1-2\cdot n^{-\frac{\varepsilon}{3}} > 2^{\beta+2\alpha-\varepsilon}\cdot n^{-\frac{\varepsilon}{3}}\,.$$

After fixing the number m, we finally determine an integer N which satisfies the following conditions:
$$N > m,\ N > E,\ |S_N^{(r+1)}| > N^{d_{r+1}-\delta_1} = N^{\beta+\alpha-\delta_1}\,.$$

If now
$$|S_N^{(r+2)}| > N^{\beta+2\alpha-\varepsilon}\,, \tag{13}$$
we have already found a number n (namely the number N) which satisfies the original conditions, i.e., which is $> E$ and for which (11) is satisfied. The theorem will therefore obviously be proved if we can show, under the assumption that (13) *fails*, i.e., under the assumption that

$$|S_N^{(r+2)}| \leqq N^{\beta+2\alpha-\varepsilon},$$

that there must certainly exist a number $N_1 > N$ (hence also $> E$) which satisfies (11), i.e., for which

$$|S_{N_1}^{(r+2)}| > N_1^{\beta+2\alpha-\varepsilon}.$$

The existence of such a number N_1 is proved in the following way.

In view of (12), we can determine an integer p, for which

$$(N >)\; N^{\alpha-\delta_3} > p > N^{\alpha-\delta_2}.$$

I now assert that $N_1 = N+p$ has the desired property, i.e., that $n = N_1$ satisfies the inequality (11).

In order to prove this assertion, we start from the identity

$$\begin{aligned} S_{N_1}^{(r+2)} = S_{N+p}^{(r+2)} &= S_N^{(r+2)}+[S_{N+1}^{(r+1)}+S_{N+2}^{(r+1)}+\cdots+S_{N+p}^{(r+1)}] \\ &= S_N^{(r+2)}+[(S_N^{(r+1)}+S_{N+1}^{(r)})+(S_N^{(r+1)}+S_{N+1}^{(r)}+S_{N+2}^{(r)})+\cdots \\ &\qquad +(S_N^{(r+1)}+S_{N+1}^{(r)}+\cdots+S_{N+p}^{(r)})] \\ &= S_N^{(r+2)}+p\cdot S_N^{(r+1)}+[p\cdot S_{N+1}^{(r)}+(p-1)\cdot S_{N+2}^{(r)}+\cdots+1\cdot S_{N+p}^{(r)}]. \end{aligned} \tag{14}$$

From this identity, we infer at once

$$|S_{N_1}^{(r+2)}| \geqq p\,|S_N^{(r+1)}|-|S_N^{(r+2)}|-[p\,|S_{N+1}^{(r)}|+(p-1)\,|S_{N+2}^{(r)}|+\cdots+1\,|S_{N+p}^{(r)}|],$$

and from this

$$\begin{aligned} |S_{N_1}^{(r+2)}| &\geqq N^{\alpha-\delta_2}\cdot N^{\beta+\alpha-\delta_1}-N^{\beta+2\alpha-\varepsilon}-p^2\cdot N^{\beta+\delta_1} > N^{\beta+2\alpha-(\delta_1+\delta_2)}-N^{\beta+2\alpha-\varepsilon}-N^{\beta+2\alpha-(2\delta_3-\delta_1)} \\ &> N^{\beta+2\alpha-\frac{2\varepsilon}{3}}-2N^{\beta+2\alpha-\varepsilon} = N^{\beta+2\alpha-\frac{2\varepsilon}{3}}\left\{1-2N^{-\frac{\varepsilon}{3}}\right\} \\ &> N^{\beta+2\alpha-\frac{2\varepsilon}{3}}\left\{2^{\beta+2\alpha-\varepsilon}\cdot N^{-\frac{\varepsilon}{3}}\right\} = (2N)^{\beta+2\alpha-\varepsilon} > N_1^{\beta+2\alpha-\varepsilon}. \end{aligned}$$

q.e.d.

Theorem II includes the following note-worthy theorem as a special case:

Theorem III. *If $\lambda_r = \lambda_{r+1}$ for some value of r, then $\lambda_r = \lambda_{r+m}$ for all $m = 1, 2, 3, \ldots$, and hence also $\lambda_r = \Lambda$. Expressed in other words: if one cannot get beyond the boundary of summability $\sigma = \lambda_r$ by using summability of the $(r+1)^{\text{th}}$ order, then one cannot get beyond this line by using summability of any order, however high.*

Summarizing briefly the results which we have found above regarding the distribution of the abscissae of summability of a Dirichlet series, we come to the following theorem:

Theorem IV. *Let $\sum \frac{a_n}{n^s}$ be a Dirichlet series with abscissae of summability*

$\lambda_0, \lambda_1, \ldots, \lambda_r, \ldots$, and let us put $\mu_r = \lambda_{r-1} - \lambda_r$ $(r = 1, 2, \ldots)$. Then, for all $r = 1, 2, \ldots$, the following inequalities hold:

$$\mu_r \geqq 0; \mu_r \leqq 1; \mu_r \geqq \mu_{r+1}. \tag{15}$$

Theorem IV, in particular the inequality $\mu_r \geqq \mu_{r+1}$, exhibits an extraordinary regularity in the distribution of the abscissae of summability; one might expect that this regularity would go even further, for instance that it would appear also in the second differences of the abscissae of summability or the like.

However, as will appear from the following, no such phenomenon occurs. It will be shown—and hereby we shall also give a complete solution to the problem of the distribution of the abscissae of summability—that the conditions (15) appearing in Theorem IV are not only *necessary* but also *sufficient*. In other words, we have the converse theorem corresponding to Theorem IV:

Theorem V. *Let*

$$\Lambda_0, \Lambda_1, \ldots, \Lambda_r, \ldots \tag{16}$$

be an arbitrary sequence of real numbers which, when $M_r = \Lambda_{r-1} - \Lambda_r$, satisfy the following conditions for all $r = 1, 2, \ldots$:

$$M_r \geqq 0; M_r \leqq 1; M_r \geqq M_{r+1}. \tag{17}$$

Then there always exists at least one Dirichlet series $\sum \frac{a_n}{n^s}$ which has as its abscissae of summability $\lambda_0, \lambda_1, \ldots, \lambda_r, \ldots$ precisely the elements of the given sequence (16), i.e., for which $\lambda_r = \Lambda_r$ for all $r = 0, 1, 2, \ldots$.*

Proof. Let the integers

$$r_1 = 1, r_2, \ldots, r_p, \ldots \tag{18}$$

be determined in such a way that

$$M_{r_1} = \cdots = M_{r_2-1} > M_{r_2} \cdots M_{r_p-1} > M_{r_p} = \cdots = M_{r_{p+1}-1} > M_{r_{p+1}} \cdots$$

and let us put

$$M_{r_1} = \theta_1, M_{r_2} = \theta_2, \ldots, M_{r_p} = \theta_p, \ldots$$
$$(\text{thus } \theta_1 > \theta_2 \ldots > \theta_p \ldots; 1 \geqq \theta_p \geqq 0).$$

---

* The existence of one such Dirichlet series implies the existence of infinitely many. This follows immediately from the fact that changing a finite number of coefficients cannot change the summability behaviour of the series. If one prefers, one may also note for instance that the series $\sum \frac{a_n}{n^s}$ and $\sum \frac{a_n (\log n)^\alpha}{n^s}$ have the same abscissae of summability for arbitrary α. Incidentally, the proof of Theorem V will itself give us at once not a single Dirichlet series, but a whole class of them, satisfying the stated conditions.

We prove Theorem V by starting with certain Dirichlet series which have very simple summability behaviour and whose existence we have already established. From these series, we are able to build up series of the required type. The Dirichlet series which serve us as a starting point in this process are the series

$$f_1(s) = \sum_{n=1}^{n=\infty} \frac{a_{1,n}}{n^s}; f_2(s) = \sum_{n=1}^{n=\infty} \frac{a_{2,n}}{n^s}; \ldots ; f_p(s) = \sum_{n=1}^{n=\infty} \frac{a_{p,n}}{n^s}; \ldots ,$$

which are determined in such a way that all differences $\lambda_r - \lambda_{r+1}$ of abscissae of summability of $f_p(s)$ are the same and equal to θ_p, while the r_pth abscissa of summability λ_{r_p} coincides with the r_pth element Λ_{r_p} in the given sequence of numbers (16). Since

$$\theta_p = M_{r_p} = \cdots = M_{r_{p+1}-1} ,$$

it follows that

$$\lambda_r = \Lambda_r \text{ for } r_p - 1 \leqq r \leqq r_{p+1} - 1 ,$$

while $\lambda_r < \Lambda_r$ for $r > r_{p+1} - 1$ as well as for $r < r_p - 1$.

We shall first prove the following

Lemma. *If $k_1, k_2, \ldots, k_q$ are arbitrary constants different from 0, the abscissae of summability $\lambda_r^{(b)}$ for the Dirichlet series*

$$\sum_{n=1}^{n=\infty} \frac{b_n}{n^s} = \sum_{p=1}^{p=q} k_p f_p(s) \quad \left(b_n = \sum_{p=1}^{p=q} k_p a_{p,n}\right)$$

are determined by the equations

$$\lambda_r^{(b)} = \Lambda_r \text{ for } r \leqq r_{q+1} - 1 \tag{19}$$

and

$$\lambda_r^{(b)} - \lambda_{r+1}^{(b)} = \theta_q \text{ for } r \geqq r_{q+1} - 1 . \tag{20}$$

The proof of this lemma is most easily carried through by induction. Since the lemma is obvious for $q = 1$ $\left(\text{in which case the series} \sum \frac{b_n}{n^s} \text{ reduces to the series}\right.$ $k_1 f_1(s)$, which naturally has the same abscissae of summability as the series $\left. f_1(s) = \sum \frac{a_{1,n}}{n^s}\right)$, the lemma will be proved for arbitrary q if we can prove it for q under the assumption that it is valid for $q-1$. That is, we assume that the series

$$\sum_{n=1}^{n=\infty} \frac{c_n}{n^s} = \sum_{p=1}^{p=q-1} k_p f_p(s) \quad \left(c_n = \sum_{p=1}^{p=q-1} k_p a_{p,n}\right)$$

has its abscissae of summability $\lambda_r^{(c)}$ determined by the equations

$$\lambda_r^{(c)} = \Lambda_r \text{ for } r \leqq r_q - 1$$

and

$$\lambda_r^{(c)} - \lambda_{r+1}^{(c)} = \theta_{q-1} \text{ for } r \geqq r_q - 1\,,$$

and then have to prove that the series

$$\sum_{n=1}^{n=\infty} \frac{b_n}{n^s} = \sum_{p=1}^{p=q} k_p f_p(s)$$

has its abscissae of summability $\lambda_r^{(b)}$ determined by the equations (19) and (20).

This is done in the following way.

The series $\sum \frac{b_n}{n^s}$ is formed by term by term addition of the two series

$$\sum_{n=1}^{n=\infty} \frac{c_n}{n^s} = \sum_{p=1}^{p=q-1} k_p f_p(s) \text{ and } \sum_{n=1}^{n=\infty} \frac{d_n}{n^s} = k_q f_q(s)\,.$$

If we denote the abscissae of summability of the three series under consideration by $\lambda_r^{(b)}$, $\lambda_r^{(c)}$, and $\lambda_r^{(d)}$, respectively, we have

$$\lambda_{r_q-1}^{(c)} = \lambda_{r_q-1}^{(d)} = \Lambda_{r_q-1}\,, \tag{21}$$

while

$$\lambda_r^{(c)} > \lambda_r^{(d)} \text{ for } r < r_q - 1 \tag{22}$$

and

$$\lambda_r^{(c)} < \lambda_r^{(d)} \text{ for } r > r_q - 1\,. \tag{23}$$

From (22) and (23), it now follows at once* that

$$\lambda_r^{(b)} = \lambda_r^{(c)} \text{ for } r < r_q - 1 \tag{24}$$

and

$$\lambda_r^{(b)} = \lambda_r^{(d)} \text{ for } r > r_q - 1\,, \tag{25}$$

while we can infer from equation (21) only that

$$\lambda_{r_q-1}^{(b)} \leqq \Lambda_{r_q-1}\,.$$

However, $\lambda_{r_q-1}^{(b)}$ cannot be less than Λ_{r_q-1}, for in such a case, the difference $\lambda_{r_q-1}^{(b)} - \lambda_{r_q}^{(b)}$ would be less than $\lambda_{r_q}^{(b)} - \lambda_{r_q+1}^{(b)} = \theta_q$, which would contradict Theorem II (page 88); hence we obtain

$$\lambda_{r_q-1}^{(b)} = \Lambda_{r_q-1}\,. \tag{26}$$

---

* Namely, with the help of the following theorem, which can be immediately inferred from Theorem IV, § 1 (page 51): *If we denote the abscissae of summability of the* r^{th} *order of the series*

$$\sum \frac{\alpha_n}{n^s}, \quad \sum \frac{\beta_n}{n^s}, \quad \textit{and} \quad \sum \frac{\gamma_n}{n^s} = \sum \frac{\alpha_n}{n^s} + \sum \frac{\beta_n}{n^s}$$

by $\lambda_r^{(\alpha)}$, $\lambda_r^{(\beta)}$, *and* $\lambda_r^{(\gamma)}$, *respectively, then, if* $\lambda_r^{(\alpha)} > \lambda_r^{(\beta)}$, *we have* $\lambda_r^{(\gamma)} = \lambda_r^{(\alpha)}$, *while, if* $\lambda_r^{(\alpha)} = \lambda_r^{(\beta)}$, *we can infer only that* $\lambda_r^{(\gamma)} \leqq \lambda_r^{(\alpha)}$.

But equations (24), (25), and (26) are equivalent to equations (19) and (20), as one sees, so that the inductive proof of the lemma is complete.

In continuing with the proof of Theorem IV, we are naturally led to distinguish between two different possible cases.

Case I: The differences $M_r = \Lambda_{r-1} - \Lambda_r$ are all equal from a certain point on, or, in the notation introduced above, the sequence of numbers $r_1, r_2, \ldots, r_p, \ldots$ breaks off, i.e., has a last element r_P.

In this case, the following theorem holds, which is immediately inferred from the above lemma when we put $q = P$:

If $k_1, k_2, \ldots, k_P$ denote arbitrary constants different from 0, the Dirichlet series

$$\sum_{n=1}^{n=\infty} \frac{a_n}{n^s} = \sum_{p=1}^{p=P} k_p f_p(s)$$

has as its abscissae of summability $\lambda_r^{(a)}$ precisely the elements of the given sequence of numbers (16).

Case II: Here we have sequences (16), in which the differences $M_r = \Lambda_{r-1} - \Lambda_r$ are not all the same from a certain point on, or in other words the sequence $r_1, r_2, \ldots, r_p, \ldots$ does not break off.

Concerning this case we have the following theorem:

There exists an infinite sequence of positive constants $e_1, e_2, \ldots, e_p, \ldots$ such that the Dirichlet series

$$\sum_{n=1}^{n=\infty} \frac{a_n}{n^s}, \quad \textit{where } a_n = \sum_{p=1}^{p=\infty} \varepsilon_p a_{p,n}$$

(i.e., which is constructed by formal computation from the series $\sum_{p=1}^{p=\infty} \varepsilon_p f_p(s)$), *has as its abscissae of summability precisely the elements $\Lambda_0, \Lambda_1, \ldots, \Lambda_p, \ldots$ in the given sequence* (16) *whenever the numbers $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_p, \ldots$ are different from* 0 *and satisfy the conditions*

$$|\varepsilon_1| < e_1, \ |\varepsilon_2| < e_2, \ldots, |\varepsilon_p| < e_p, \ldots.$$

Proof. Let q denote any one of the numbers $0, 1, 2, \ldots$. As one easily sees, all the Dirichlet series $f_p(s) = \sum \frac{a_{p,n}}{n^s}$ corresponding to values of $p > q+1$ (i.e., for $p = q+2, q+3, \ldots$) have their abscissae of q^{th} order summability less than $\Lambda_q - C_q$, where C_q denotes a positive constant depending only on q (we use the fact that

$q < r_p - 1$ for such p). Hence the series in question are all summable of the q^{th} order at the point $\Lambda_q - C_q$, i.e., the series $\sum\limits_{n=1}^{n=\infty} \frac{a_{p,n}}{n^{\Lambda_q - C_q}}$ $(p = q+2, q+3, \ldots)$ are all summable of the q^{th} order.

In view of Theorem III, § 1 (page 48), we can therefore find positive constants $e_{q,p}$ $(p = q+2, q+3, \ldots)$ such that, if $|\varepsilon_p| < e_{q,p}$, the series

$$\sum_{n=1}^{n=\infty} \beta_n, \quad \text{where} \quad \beta_n = \sum_{p=q+2}^{p=\infty} \varepsilon_p \frac{a_{p,n}}{n^{\Lambda_q - C_q}},$$

is summable of the q^{th} order. We can express this in other words by saying that, if $|\varepsilon_p| < e_{q,p}$, the series

$$\sum_{n=1}^{n=\infty} \frac{b_n}{n^{\Lambda_q - C_q}}, \quad \text{where} \quad b_n = \sum_{p=q+2}^{p=\infty} \varepsilon_p a_{p,n},$$

is summable of the q^{th} order. It follows immediately from this that the Dirichlet series $\sum \frac{b_n}{n^s}$ must have its q^{th} abscissa of summability less than Λ_q.

In this way, we determine sequences of numbers

$$\begin{array}{llll}
e_{0,2}, e_{0,3}, \ldots, & e_{0,m}, & e_{0,m+1}, & \ldots\ldots \\
e_{1,3}, \ldots, & e_{1,m}, & e_{1,m+1}, & \ldots\ldots \\
\ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\
 & e_{m-2,m}, & e_{m-2,m+1}, & \ldots \\
 & & \ldots\ldots & \ldots\ldots
\end{array}$$

corresponding to $q = 0, 1, 2, \ldots, m, \ldots$.

Now, let e_m $(m = 2, 3, \ldots)$ denote the smallest of the $m-1$ numbers $e_{0,m}, e_{1,m}, \ldots, e_{m-2,m}$, and let e_1 be an arbitrary positive number. Then the numbers $e_1, e_2, \ldots, e_m, \ldots$ found in this way will satisfy the conditions stated in the above theorem.

Indeed, let $0 < |\varepsilon_1| < e_1$, $0 < |\varepsilon_2| < e_2, \ldots, 0 < |\varepsilon_p| < e_p, \ldots$ and let us consider the Dirichlet series

$$\sum_{n=1}^{n=\infty} \frac{a_n}{n^s}, \quad \text{where} \quad a_n = \sum_{p=1}^{p=\infty} \varepsilon_p a_{p,n}.$$

We shall then prove that when q is any one of the numbers $0, 1, 2, \ldots$, the q^{th} abscissa of summability of this series is just the quantity Λ_q. That this is the case may be seen as follows.

The series $\sum \frac{a_n}{n^s}$ can be written as the term by term sum of the two Dirichlet

series $\sum \frac{c_n}{n^s}$ and $\sum \frac{d_n}{n^s}$, where

$$c_n = \sum_{p=1}^{p=q+1} \varepsilon_p a_{p,n} \text{ and } d_n = \sum_{p=q+2}^{p=\infty} \varepsilon_p a_{p,n} .$$

Of these two series, $\sum \frac{d_n}{n^s}$ has its q^{th} abscissa of summability less than Λ_q, since $|\varepsilon_p| < e_p \leqq e_{q,p}$ for $p \geqq q+2$. On the other hand, the series

$$\sum_{n=1}^{n=\infty} \frac{c_n}{n^s} = \sum_{p=1}^{p=q+1} \varepsilon_p f_p(s) ,$$

in view of the lemma proved above, has its q^{th} abscissa of summability equal to Λ_q, since $q < r_{q+2}-1$. The series $\sum \frac{a_n}{n^s} = \sum \frac{c_n}{n^s} + \sum \frac{d_n}{n^s}$ consequently has its q^{th} abscissa of summability equal to Λ_q. q.e.d.

The proof of Theorem V is thus complete.

For the sake of a later application, we add the following remark to the proof of Theorem V. In Case I, where the constructed series $\sum \frac{a_n}{n^s}$ is formed only from a finite number of series $f_p(s) = \sum \frac{a_{p,n}}{n^s}$, one sees immediately that the function $f(s)$ represented by the series $\sum \frac{a_n}{n^s}$ is equal to $\sum_{p=1}^{p=P} k_p f_p(s)$. In Case II, the analogous conclusion cannot be made immediately, since $\sum \frac{a_n}{n^s}$ here is formed from an infinite number of series $f_p(s)$. We shall prove, however, *that it is possible to choose the positive numbers $e_1, e_2, \ldots, e_p, \ldots$ occurring in the proof of Case* II *so small that $f(s)$ is equal to the absolutely convergent series*

$$\sum_{p=1}^{p=\infty} \varepsilon_p f_p(s)$$

for sufficiently large values of σ, e.g. for $\sigma \geqq \Lambda_0+2$. This is seen in the following way.

Since the series $f_p(s) = \sum \frac{a_{p,n}}{n^s}$ all have their abscissae of convergence less than or equal to Λ_0, as one sees immediately from their definition, it follows that these series, like the series $\sum \frac{a_n}{n^s}$, converge absolutely at the point Λ_0+2.

If we now put

$$K_p = \sum_{n=1}^{n=\infty} \left| \frac{a_{p,n}}{n^{\Lambda_0+2}} \right|$$

and choose e_p less than $\dfrac{C_p}{K_p}$, where $\sum C_p$ converges, we have, if $\sigma \geqq \Lambda_0+2$,

$$F(s) = \sum_{n=1}^{n=\infty} \frac{a_n}{n^s} = \sum_{n=1}^{n=\infty} \frac{1}{n^s} \sum_{p=1}^{p=\infty} \varepsilon_p a_{p,n} = \sum_{p=1}^{p=\infty} \varepsilon_p \sum_{n=1}^{n=\infty} \frac{a_{p,n}}{n^s} = \sum_{p=1}^{p=\infty} \varepsilon_p f_p(s) .$$

The interchange of summation is permissible, since for $\sigma \geqq \Lambda_0+2$,

$$\sum_{p=1}^{p=\infty} |\varepsilon_p| \sum_{n=1}^{n=\infty} \left| \frac{a_{p,n}}{n^s} \right| \leqq \sum_{p=1}^{p=\infty} e_p K_p < \sum_{p=1}^{p=\infty} C_p \quad \text{(which is convergent)} .$$

§ 6.

The behaviour of the sum function upon convergence to certain points on the boundary of summability $\sigma = \lambda_r$.

Theorem I. *If* $f(s) = \sum_{n=1}^{n=\infty} \frac{a_n}{n^s}$ *is summable of the* r^{th} *order at the point* $s_0 = \sigma_0 + it_0$ *with the summability value* A, *then we have*

$$\lim_{s=s_0} f(s) = A \tag{1}$$

when s *in converging to* s_0 *is limited to an angular region with vertex at* s_0 *and with boundary lines which form angles lying between* $-\frac{\pi}{2}$ *(excl.) and* $+\frac{\pi}{2}$ *(excl.) with the horizontal line* $t = t_0$ *going to the right from the point* $s_0 = \sigma_0 + it_0$.

In the proof of Theorem I, we may obviously assume $s_0 = 0$ without loss of generality.

Since $\sum \frac{a_n}{n^s}$ is summable of the r^{th} order with the value A for $s = s_0 = 0$, we have, when

$$S_n^{(0)} = \sum_{m=1}^{m=n} a_m ; \; S_n^{(1)} = \sum_{m=1}^{m=n} S_m^{(0)} ; \; \ldots ; \; S_n^{(r)} = \sum_{m=1}^{m=n} S_m^{(r-1)} ,$$

the relation

$$\lim_{n=\infty} \frac{S_n^{(r)} \cdot r!}{n^r} = A . \tag{2}$$

Further, as shown in § 2 (page 61), the series $f(s) = \sum \frac{a_n}{n^s}$, which is summable of the r^{th} order for $\sigma > 0$, can be represented by the absolutely convergent series

$$f(s) = \sum_{n=1}^{n=\infty} S_n^{(r)} \Delta^{r+1}\left(\frac{1}{n^s}\right).$$

In § 2 (page 61) we proved the identity

$$\Delta^{r+1}\left(\frac{1}{n^s}\right) = s(s+1)\cdots(s+r)\int_n^{n+1} dx_1 \int_{x_1}^{x_1+1} dx_2 \cdots \int_{x_{r-1}}^{x_{r-1}+1} dx_r \int_{x_r}^{x_r+1} \frac{dx_{r+1}}{x_{r+1}^{s+r+1}}.$$

We subtract the quantity

$$s(s+1)\cdots(s+r)\,\frac{1}{n^{s+r+1}}$$

from both sides of the equation, and obtain the following equation:

$$F_n(s) = \Delta^{r+1}\left(\frac{1}{n^s}\right) - s(s+1)\cdots(s+r)\,\frac{1}{n^{s+r+1}}$$

$$= s(s+1)\cdots(s+r)\int_n^{n+1} dx_1 \int_{x_1}^{x_1+1} dx_2 \cdots \int_{x_{r-1}}^{x_{r-1}+1} dx_r \int_{x_r}^{x_r+1} \left(\frac{1}{x_{r+1}^{s+r+1}} - \frac{1}{n^{s+r+1}}\right) dx_{r+1}$$

$$= -s(s+1)\cdots(s+r+1)\int_n^{n+1} dx_1 \int_{x_1}^{x_1+1} dx_2 \cdots \int_{x_{r-1}}^{x_{r-1}+1} dx_r \int_{x_r}^{x_r+1} dx_{r+1} \int_n^{x_{r+1}} \frac{dx_{r+2}}{x_{r+2}^{s+r+2}}. \quad (3)$$

From this, it follows immediately that $F_n(0) = 0$, and that, for example for $\sigma > -1$,

$$|F_n(s)| \leqq |s|\cdots|s+r+1|\,\frac{1}{n^{\sigma+r+2}} \int_n^{n+1} dx_1 \cdots \int_{x_r}^{x_r+1} dx_{r+1} \int_n^{n+r+1} dx_{r+2}$$

$$= (r+1)\cdot|s|\cdots|s+r+1|\,\frac{1}{n^{\sigma+r+2}}. \quad (4)$$

From the last inequality, together with (2), it follows immediately that the series $\sum_{n=1}^{n=\infty} S_n^{(r)} F_n(s)$ is uniformly absolutely convergent for $\sigma > -1+\varepsilon$ and $|s| <$ const. In particular, it is uniformly convergent within a circle of radius $\frac{1}{2}$ around the origin.

In addition, since $F_n(s)$ (for every $n = 1, 2, \ldots$) is an integral function, as its definition (3) shows, the series $\sum S_n^{(r)} F_n(s)$ represents a regular analytic function $g(s)$ within the circle $|s| = \frac{1}{2}$. From this we infer that

$$\lim_{s=0} g(s) = g(0) = \sum_{n=1}^{n=\infty} S_n^{(r)} F_n(0) = 0.$$

Now we have, in view of (3), for $\sigma > 0$

$$f(s) = \sum_{n=1}^{n=\infty} S_n^{(r)} \Delta^{r+1}\left(\frac{1}{n^s}\right) = s \cdots (s+r) \sum_{n=1}^{n=\infty} \frac{S_n^{(r)}}{n^{s+r+1}} + \sum_{n=1}^{n=\infty} S_n^{(r)} F_n(s) . \tag{5}$$

Since, as shown above,

$$\lim_{s=0} \sum_{n=1}^{n=\infty} S_n^{(r)} F(s) = 0 ,$$

in order to prove (1) we need only show that the sum of the series

$$s(s+1) \cdots (s+r) \sum_{n=1}^{n=\infty} \frac{S_n^{(r)}}{n^{s+r+1}},$$

or, equivalently, since $\lim\limits_{s=0} \dfrac{(s+1)\cdots(s+r)}{r!} = 1$, that the sum of the series

$$r! \cdot s \cdot \sum_{n=1}^{n=\infty} \frac{S_n^{(r)}}{n^{s+r+1}} = s \cdot \sum_{n=1}^{n=\infty} \frac{\dfrac{S_n^{(r)} \cdot r!}{n^r}}{n^{1+s}},$$

has the limit A when s converges to 0 within the angular region described in the theorem. This, however, follows at once from Theorem IX in Part One (page 17), when we take account of (2). The proof of Theorem I is thus complete.

Since it is assumed in Theorem I that $\sum \frac{a_n}{n^s}$ is summable of the r^{th} order at $s_0 = \sigma_0 + it_0$, the point s_0 must lie either on the boundary of summability itself or to the right of this line.

In the latter case (i.e., if $\sigma_0 > \lambda_r$), Theorem I, even without the limitation to a bounded angular region, is also an immediate consequence of the fact that $f(s)$ represents for $\sigma > \lambda_r$ a regular analytic and *a fortiori* continuous function by its summability value of the r^{th} order. Also, even if s_0 lies on the boundary of summability $\sigma = \lambda_r$, Theorem I is included in the results of § 2, provided that $\lambda_{r+1} < \lambda_r$ (i.e. provided that, as is generally the case, $\sum \frac{a_n}{n^s}$ is summable of the $(r+1)^{\text{th}}$ order beyond the boundary of summability $\sigma = \lambda_r$). To see this, we need only recall that a series which is summable of the r^{th} order at the points s_0 is also summable of the $(r+1)^{\text{th}}$ order with the same summability value. Only in the case when $\lambda_{r+1} = \lambda_r$, which implies $A = \lambda_r$ in view of Theorem III, § 5, does Theorem I yield a really new result, i.e., a result which is not contained as a special case in the theorems proved previously.—Let it be noted finally that Theorem I includes as a very special case Theorem X of Part One, concerning summability on the boundary of convergence.

§ 7.

The behaviour for infinitely large values of the ordinate t of the analytic function represented by a Dirichlet series in its regions of summability.

Corresponding to Theorem XII, Part One (page 21) and including this theorem as a special case ($r = 0$), we have the following important theorem concerning summability of arbitrary order r.

Theorem I. *Let* $f(s) = \sum \frac{a_n}{n^s}$ *be a Dirichlet series with abscissa of summability of the* r^{th} *order equal to* λ_r *and abscissa of absolute convergence equal to* $l > \lambda_r$; *then, for* $l+\varepsilon \geqq \sigma \geqq \lambda_r+\varepsilon$ (ε an arbitrarily small positive number) *we have*

$$f(s) = O\left(|t|^{(r+1)\frac{l-\sigma+\varepsilon}{l-\lambda_r}}\right). \tag{1}$$

Proof. Since $f(s) = \sum \frac{a_n}{n^s}$ is summable of the r^{th} order at the point $s = \lambda_r+\frac{\varepsilon}{2}$, the function $f(s)$ (as shown on page 61, § 2) can be represented for $\sigma > \lambda_r+\frac{\varepsilon}{2}$ as the sum of the absolutely convergent series

$$\sum_{n=1}^{n=\infty} S_n^{(r)} \Delta^{r+1}\left(\frac{1}{n^{s-\left(\lambda_r+\frac{\varepsilon}{2}\right)}}\right); \tag{2}$$

here $S_n^{(r)}$ is built up in the usual way from the series $\sum u_n = \sum \frac{a_n}{n^{\lambda_r+\frac{\varepsilon}{2}}}$, which is summable of the r^{th} order. From the expression (2) for $f(s)$, together with the inequality (19), § 2 (page 61), it now follows for $\sigma \geqq \lambda_r+\varepsilon$ that

$$|f(s)| \leqq \sum_{n=1}^{n=\infty} \left|\frac{S_n^{(r)}}{n^r}\right| \cdot \left|n^r \Delta^{r+1}\left(\frac{1}{n^{s-\left(\lambda_r+\frac{\varepsilon}{2}\right)}}\right)\right| < K|s|^{r+1} \cdot \sum_{n=1}^{n=\infty} \frac{1}{n^{1+\frac{\varepsilon}{2}}} < K_1 \cdot |s|^{r+1}.$$

Hence for $l+\varepsilon \geqq \sigma \geqq \lambda_r+\varepsilon$

$$f(s) = O(|t|^{r+1}).$$

Since $f(s) = O(1)$ for $\sigma = l+\varepsilon$, it follows at once from the theorem of Lindelöf stated in Part One (page 22) on putting $\sigma_1 = \lambda_r+\varepsilon$, $\sigma_2 = l+\varepsilon$, $k = c = r+1$ that for $l+\varepsilon \geqq \sigma \geqq \lambda_r+\varepsilon$ we have the equation

$$f(s) = O\left(|t|^{(r+1)\frac{l-\sigma+\varepsilon}{l-\lambda_r}}\right). \qquad \text{q.e.d.*} \tag{1}$$

* It follows immediately from Theorem I, which was communicated by the author in a lecture to Mathematische Gesellschaft in Göttingen, 29 July 1909 (which is printed in essentially unchanged form

From Theorem I we can infer in particular the following theorem:

Theorem II. *If* $f(s) = \sum \frac{a_n}{n^s}$ *is a Dirichlet series with the limit abscissa of summability* Λ $(\geqq -\infty)$, *then for* $\sigma > \Lambda + \varepsilon$ (if $\Lambda = -\infty$, for $\sigma > -E$), *we have*

$$f(s) = O(|t|^K), \tag{3}$$

where $K = K(\varepsilon)$ *denotes a constant independent of* σ *and* t.

Proof. Since $\Lambda = \lim_{r=\infty} \lambda_r$, there is certainly an integer $R = R(\varepsilon)$ such that $\Lambda+\varepsilon > \lambda_R$, and hence $\Lambda+\varepsilon > \lambda_R+\delta$ where $\delta > 0$. If we now put $K = R+1$, equation (3) will be satisfied by virtue of Theorem I, and Theorem II is proved.

In Part One, it was shown in Landau's Theorem XIII (page 23) that the function $f(s)$ represented by a Dirichlet series $\sum \frac{a_n}{n^s}$ has a mean value equal to the first coefficient a_1 of the series, for arguments s lying on an infinite straight line $\sigma = \sigma_0$ to the right of the boundary of convergence.

We shall now show how Landau's theorem can be included as a special case in a more general theorem. This theorem shows that the function $f(s)$ also has a mean value equal to the first coefficient a_1 of the series for arguments s belonging to a straight line $\sigma = \sigma_0$ lying in the region of r^{th} order summability for arbitrary r. One

---

in the paper: Über die Summabilität Dirichletscher Reihen, *Nachrichten der Kgl. Gesellschaft der Wissenschaften zu Göttingen, math. phys. Klasse* 1909, pp. 247–262) that $\frac{f(s)}{s^{r+1}}$ converges uniformly to the limit value 0 for s tending to infinity in the half-plane $\sigma > \lambda_r+\varepsilon$. (It is to be recalled that $|f(s)| < K$ for $\sigma \geqq l+\varepsilon$.) As the author noticed after his lecture, this last somewhat more special theorem has also been found by M. Riesz and communicated in a note: Sur les séries de Dirichlet, *Comptes rendus de l'Académie des Sciences, Paris*, 21 June 1909, vol. 148, p. 1658.

If we apply Theorem I for example to the special Dirichlet series $\sum \frac{(-1)^{n+1}}{n^s} = \zeta(s)(1-2^{1-s})$ studied in § 3, we immediately obtain *upper bounds for the absolute value of the function* $\zeta(s)$ *corresponding to numerically large values of the ordinate* t *in the entire plane* (for we have here $\Lambda = \lim \lambda_r = -\infty$). Previously, i.e., before the introduction of summability, investigations of the series $\sum \frac{(-1)^{n+1}}{n^s}$ could only give such upper bounds for the ζ-function *in the half-plane to the right of the line of convergence* $\sigma = \lambda_0 = 0$. Nevertheless, we shall not pause to give the bounds, which incidentally are fairly exact, obtainable for the special function $\zeta(s)$ by application of this general method (which is applicable to all Dirichlet series to the right of $\sigma = \Lambda$). This is because Riemann's functional equation for the ζ-function presents a means of giving even more exact—indeed, outside the strip $1 > \sigma > 0$ the exact—upper bounds for the order of magnitude of the ζ-function with respect to the ordinate t.

must, as might be expected from Theorem I, merely employ stronger and stronger smoothing processes for larger and larger values of the order of summability r.

Theorem III. *Let* $\sum\limits_{n=1}^{n=\infty} \frac{a_n}{n^s}$ *be a Dirichlet series with abscissa of summability of the* r^{th} *order equal to* λ_r. *Then, for every* $\sigma_0 > \lambda_r$ *the following equation holds:*

$$a_1 = \lim_{T=\infty} \frac{(r+1)!}{T^{r+1}} \int_0^T dt_r \int_0^{t_r} dt_{r-1} \cdots \int_0^{t_2} dt_1 \int_0^{t_1} f(\sigma_0+it)dt\,.$$

Proof. Upon applying Theorem III, § 3 (page 73) concerning the permissibility of term by term integration of a Dirichlet series in its region of summability of the r^{th} order for an arbitrary r, we obtain

$$\int_0^{t_1} f(\sigma_0+it)dt = \int_0^{t_1} \sum_{n=1}^{n=\infty} \frac{a_n}{n^{\sigma_0+it}}\,dt = \int_0^{t_1} a_1 dt + \int_0^{t_1} \sum_{n=2}^{n=\infty} \frac{a_n}{n^{\sigma_0+it}}\,dt$$

$$= a_1 t_1 + \left[\, i \sum_{n=2}^{n=\infty} \frac{a_n}{\log n \cdot n^{\sigma_0+it_1}} - i \sum_{n=2}^{n=\infty} \frac{a_n}{\log n \cdot n^{\sigma_0}} \right] = a_1 t_1 + i \sum_{n=2}^{n=\infty} \frac{\dfrac{a_n}{\log n}}{n^{\sigma_0+it_1}} + K_1\,.$$

Here the Dirichlet series $\sum\limits_{n=2}^{n=\infty} \frac{\frac{a_n}{\log n}}{n^s}$ (in view of Theorem II, § 3) is summable of the r^{th} order for $\sigma > \lambda_r$, and K_1 $\left(\text{equal to the summability value of } -i\sum \frac{a_n}{\log n \cdot n^{\sigma_0}}\right)$ denotes a constant, i.e., a quantity independent of t_1.

We then integrate with respect to t_1 and obtain

$$\int_0^{t_2} dt_1 \int_0^{t_1} f(\sigma_0+it)\,dt = a_1 \frac{t_2^2}{1\cdot 2} + i^2 \sum_{n=2}^{n=\infty} \frac{\dfrac{a_n}{(\log n)^2}}{n^{\sigma_0+it_2}} + K_1 t_2 + K_2\,.$$

We continue this process $r-1$ times and obtain finally

$$\int_0^T dt_r \int_0^{t_r} dt_{r-1} \cdots \int_0^{t_2} dt_1 \int_0^{t_1} f(\sigma_0+it)dt$$

$$= a_1 \frac{T^{r+1}}{(r+1)!} + i^{r+1} \sum_{n=2}^{n=\infty} \frac{\dfrac{a_n}{(\log n)^{r+1}}}{n^{\sigma_0+iT}} + \frac{K_1}{r!} T^r + \frac{K_2}{(r-1)!} T^{r-1} + \cdots + K_r T + K_{r+1}\,. \quad (4)$$

Since, however, the series $g(s) = \sum \frac{a_n (\log n)^{-r-1}}{n^s}$ is summable of the r^{th} order

for $\sigma > \lambda_r$ (in view of Theorem II, § 3), and since $\sigma_0 > \lambda_r$, it follows at once from Theorem I (page 100) that

$$\lim_{T=\infty} \frac{g(\sigma_0+iT)}{T^{r+1}} = \lim_{T=\infty} \frac{1}{T^{r+1}} \sum_{n=2}^{n=\infty} \frac{\frac{a_n}{(\log n)^{r+1}}}{n^{\sigma_0+iT}} = 0\,.$$

This equation, in conjunction with (4), gives us immediately

$$\lim_{T=\infty} \frac{(r+1)\,!}{T^{r+1}} \int_0^T dt_r \int_0^{t_r} dt_{r-1} \cdots \int_0^{t_2} dt_1 \int_0^{t_1} f(\sigma_0+it)dt = a_1\,. \qquad \text{q.e.d.}$$

If we apply Theorem III in particular to the series

$$\zeta(s)(1-2^{1-s}) = \sum \frac{(-1)^{n+1}}{n^s}$$

or to any of the series $L(s) = \sum \frac{\chi(n)}{n^s}$, we see *that the function* $\zeta(s)(1-2^{1-s})$, *as well as all of the functions* $L(s)$, *have mean value* $+1$ *in the above sense along an arbitrary ordinate* $\sigma = \sigma_0$ $(-\infty < \sigma_0 < +\infty)$.

§ 8.

Determination of the limit abscissa of summability Λ from the mere knowledge of the analytic properties of the function represented by the series.

Theorem I. *Let* $\sum \frac{a_n}{n^s}$ *be a Dirichlet series with abscissa of summability of the* r^{th} *order* $\lambda_r = \alpha$, *about which we assume that the analytic function* $f(s)$ *represented by the series is regular and satisfies the condition*

$$f(s) = O(|t|^{r+1+k})$$

for $\sigma > \alpha-1+\eta$ $(0 < \eta < 1)$. *Then*

$$\lambda_{r+1} \leqq (\alpha-1)+\frac{\eta+c}{1+c} < \alpha\,, \tag{1}$$

where c *denotes the larger of the numbers* k *and* $1-\eta$.

Theorem I, which provides an analogue of the Landau-Schnee Theorem XIV mentioned in Part One (page 27), shows how $\sum \frac{a_n}{n^s}$ is summable of the $(r+1)^{\text{th}}$ order beyond the r^{th} boundary of summability $\sigma = \lambda_r$, under the given assumptions.

However, the theorem does not merely state that $\lambda_{r+1} < \lambda_r$ but even gives a lower bound for the difference $\lambda_r - \lambda_{r+1}$, namely the quantity

$$1 - \frac{\eta + c}{1 + c} = \frac{1 - \eta}{1 + c} > 0 .$$

By using a special device, the author succeeded in proving this rather deep theorem directly from the Landau-Schnee convergence theorem. Therefore we need not go through such penetrating and difficult arguments as those used by the above-mentioned mathematicians in the proof of Theorem XIV (page 27). This device consists essentially in considering, instead of the given series $\sum \frac{a_n}{n^s}$, a different Dirichlet series $\sum \frac{b_n}{n^s}$, which stands in a close relation to the original series. This series can be shown to converge just so far as the original series is summable of the $(r+1)^{\text{th}}$ order. The Landau-Schnee theorem is then immediately applied to the series $\sum \frac{b_n}{n^s}$. Before we turn to the proof of Theorem I we give the following lemma:

If the Dirichlet series $\sum \frac{a_n}{n^s}$ *has its* r^{th} *abscissa of summability* $\lambda_r \geqq 1$, *and if we put*

$$S_n^{(0)} = \sum_{q=1}^{q=n} a_q \,; \quad S_n^{(1)} = \sum_{q=1}^{q=n} S_q^{(0)} \,; \ldots ; \quad S_n^{(r)} = \sum_{q=1}^{q=n} S_q^{(r-1)}, \tag{2}$$

then the abscissa of convergence μ_0 *of the series* $\sum_{n=1}^{n=\infty} \dfrac{\frac{S_n^{(r)}}{n^{r+1}}}{n^s}$ *is equal to the* $(r+1)^{\text{th}}$ *abscissa of summability* λ_{r+1} *of the given series* $\sum \frac{a_n}{n^s}$; *equivalently, we have* $v_0 = \lambda_{r+1} + r + 1$, *where* $v_0 = \mu_0 + r + 1$ *denotes the abscissa of convergence of the series* $\sum \frac{S_n^{(r)}}{n^s}$.

Proof. Since $\lambda_r \geqq 1$ implies $\lambda_{r+1} \geqq 0$, the abscissa λ_{r+1} can be determined, as shown in Theorem Ia, § 4, by the formula

$$\lambda_{r+1} = \limsup_{n=\infty} \frac{\log \left| \frac{S_n^{(r+1)}}{n^{r+1}} \right|}{\log n} = \limsup_{n=\infty} \frac{\log |S_n^{(r+1)}|}{\log n} - (r+1) . \tag{3}$$

Since $\lambda_{r+1} \geqq 0$ and $r+1 > 0$ in equation (3), we necessarily have

$$\limsup_{n=\infty} \frac{\log |S_n^{(r+1)}|}{\log n} > 0 .$$

It follows from this, upon applying Theorem I c, § 4, that the abscissa of convergence (abscissa of summability of the 0th order) v_0 of the series $\sum\limits_{n=1}^{n=\infty} \frac{S_n^{(r)}}{n^s}$ can be determined by the equation

$$v_0 = \limsup_{n=\infty} \frac{\log \left| \sum\limits_{m=1}^{m=n} S_m^{(r)} \right|}{\log n} = \limsup_{n=\infty} \frac{\log |S_n^{(r+1)}|}{\log n}. \tag{4}$$

Equations (3) and (4) show precisely that

$$v_0 = \lambda_{r+1} + r + 1 . \qquad \text{q.e.d.}$$

We now turn to the proof of Theorem I, in which we can obviously assume $\alpha = 1$ (hence $\alpha - 1 + \eta = \eta$).

If we determine $S_n^{(r)}$ by the formulae (2) and note that $\lambda_r = 1$, we have

$$\limsup_{n=\infty} \frac{\log \left| \frac{S_n^{(r)}}{n^r} \right|}{\log n} = 1 ,$$

from which we infer at once

$$\lim_{n=\infty} \frac{S_n^{(r)}}{n^{r+1+\delta}} = 0 \quad (\text{for all } \delta > 0) . \tag{5}$$

Furthermore, in view of the theorem in note*, page 78, the function $f(s) = \sum \frac{a_n}{n^s}$ can be represented for $\sigma > \lambda_r = 1$ by the sum of the absolutely convergent series

$$\sum_{n=1}^{n=\infty} S_n^{(r)} \Delta^{r+1} \left(\frac{1}{n^s} \right). \tag{6}$$

If we now put

$$F_n(s) = \Delta^{r+1} \left(\frac{1}{n^s} \right) - s(s+1) \cdots (s+r) \frac{1}{n^{s+r+1}} \tag{7}$$

we have, as shown on page 98 (formula (4)), for $\sigma > -1$,

$$|F_n(s)| \leqq (r+1) \cdot |s| \cdot |s+1| \cdots |s+r+1| \cdot \frac{1}{n^{\sigma+r+2}}. \tag{8}$$

We now consider the following identity, known at present only for $\sigma > 1$:

$$f(s) = \sum_{n=1}^{n=\infty} S_n^{(r)} \Delta^{r+1} \left(\frac{1}{n^s} \right) = s(s+1) \cdots (s+r) \sum_{n=1}^{n=\infty} \frac{S_n^{(r)}}{n^{s+r+1}} + \sum_{n=1}^{n=\infty} S_n^{(r)} F_n(s) . \tag{9}$$

Here the first series $f(s) = \sum S_n^{(r)} \Delta^{r+1} \left(\frac{1}{n^s} \right)$ converges absolutely for $\sigma > 1$ and is

equal to $O(|t|^{r+1})$ for $\sigma \geqq 1+\varepsilon$ as shown by Theorem I, § 7 (page 100). Furthermore, the Dirichlet series $g(s) = \sum \frac{S_n^{(r)}}{n^{s+r+1}}$, in view of (5), converges absolutely for $\sigma > 1$, and hence is equal to $O(1)$ for $\sigma \geqq 1+\varepsilon$. Finally, the last series $\sum S_n^{(r)} F_n(s)$ converges absolutely for $\sigma > 0$ in view of (5) and (8). It also follows from (8) that this series converges uniformly in every finite region for which $\sigma > \varepsilon$; since $F_n(s)$ is an integral function for all n, it follows that the series $\sum S_n^{(r)} F_n(s)$ represents a regular analytic function for $\sigma > 0$ (hence *a fortiori* for $\sigma > \eta$). In the following we shall denote this function by $h(s)$. We also have for $\varepsilon \leqq \sigma \leqq 1+\varepsilon$, as appears from (8),

$$h(s) = O(|t|^{r+2}) .$$

On the other hand, as one can read off directly from (9), we have for $\sigma = 1+\varepsilon$

$$h(s) = f(s) - s(s+1)\cdots(s+r)g(s) = O(|t|^{r+1}) .$$

By applying Lindelöf's theorem (page 22) to the function $h(s)/s^{r+1}$, we therefore find that

$$h(s) = O(|t|^{r+2-\sigma+\varepsilon}) \quad (\varepsilon \text{ arbitrarily small})$$

for $\varepsilon \leqq \sigma \leqq 1+\varepsilon$, and *a fortiori* that the equation

$$h(s) = O(|t|^{r+2-\eta+\varepsilon}) \tag{10}$$

holds for $\eta \leqq \sigma \leqq 1+\varepsilon$.

Solving equation (9) with respect to $g(s) = \sum \frac{S_n^{(r)}}{n^{s+r+1}}$ we obtain

$$g(s) = \big(f(s) - h(s)\big)\frac{1}{s(s+1)\cdots(s+r)}. \tag{11}$$

We now infer that $g(s)$ is a regular analytic function for $\sigma > \eta$ and that, for $\eta < \sigma \leqq 1+\varepsilon$,

$$g(s) = \big(O(|t|^{r+1+k}) + O(|t|^{r+2-\eta+\varepsilon})\big)\frac{1}{s\cdots(s+r)} = O(|t|^{k}) + O(|t|^{1-\eta+\varepsilon}) = O(|t|^{c+\varepsilon}) .$$

The last equation, together with the equation $g(s) = O(1)$, valid for $\sigma \geqq 1+\varepsilon$, shows at once that $g(s)$ for $\sigma > \eta$ is equal to

$$O(|t|^{c+\varepsilon}) \quad (\varepsilon \text{ arbitrarily small}).$$

We have thus shown that the Dirichlet series

$$\sum_{n=1}^{n=\infty} \frac{S_n^{(r)}}{n^{r+1+s}} = \sum_{n=1}^{n=\infty} \frac{\frac{S_n^{(r)}}{n^{r+1}}}{n^{s}}$$

satisfies the following two conditions:

1. $\lim\limits_{n=\infty} \frac{S_n^{(r)}}{n^{r+1+\delta}} = 0$ for all $\delta > 0$, and

2. The analytic function $g(s)$ represented by the series is regular and equal to $O(|t|^{c+\varepsilon})$ for $\sigma > \eta$.

We can therefore apply the Landau-Schnee Theorem XIV (page 27) to the series $g(s) = \sum\limits_{n=1}^{n=\infty} \frac{\frac{S_n^{(r)}}{n^{r+1}}}{n^s}$ and in this way infer that the series converges for $\sigma > \frac{\eta+c+\varepsilon}{1+c+\varepsilon}$; thus, if μ_0 denotes the abscissa of convergence of the series, we have $\mu_0 \leqq \frac{\eta+c+\varepsilon}{1+c+\varepsilon}$. Consequently, if we let the arbitrarily small ε converge to 0, we have

$$\mu_0 \leqq \frac{\eta+c}{1+c}.$$

However, in view of the lemma proved above, we have $\mu_0 = \lambda_{r+1}$; we thus obtain

$$\lambda_{r+1} \leqq \frac{\eta+c}{1+c}. \qquad \text{q.e.d.*}$$

From Theorem I, we can immediately obtain the following two Theorems II and III. For the first of these theorems, we have already given a proof in a previous

---

* Theorem I has been communicated by the author in his paper: Über die Summabilität Dirichletscher Reihen, l.c. It should be noted here that M. Riesz, Sur les séries de Dirichlet, l.c., has stated without proof the following theorem which goes in a somewhat different direction: '*If $f(s)$ is regular for $\sigma > \sigma_0$ and equal to $o(|t|^{r+1})$, then $\lambda_{r+1} \leqq \sigma_0$.*' Theorem I and the theorem communicated by Riesz are independent of each other, i.e., neither can be deduced from the other. While with either of these two theorems one can prove Theorem III of the text, one cannot obtain Theorem II of the text from Riesz's theorem.—Theorem I can be stated in a more general form; thus the author has for $r = 1$ and $r = 2$ proved the following theorem, which forms a complete analogue of the Landau-Schnee theorem (i.e., coincides with this theorem for $r = 0$): '*If $\lambda_r \leqq 1$ and if $f(s)$ is regular and equal to $O(|t|^{r+1+k})$ for $\sigma > \eta$ $\left(\eta \gtreqless 0\right)$, then $\lambda_{r+1} \leqq \frac{\eta+k}{1+k}$.*' This theorem contains Theorem I and Riesz's theorem in the cases which have been proved.—The different theorems cited above seem to suggest the existence of a positive number K_r *such that every Dirichlet series is summable of the* r^{th} *order just so far as the represented function is regular and equal to* $o(|t|^{K_r})$. The author has succeeded, however, in showing *that this is false for every* r, by making use of considerations which are very similar to those used in Part One in the treatment of the convergence problem for Dirichlet series and which we shall therefore not discuss in detail here. More specifically, it has been shown *that the individual abscissae of summability* (like the abscissa of convergence in particular) *cannot be determined from the mere knowledge of the regularity or singularity of the function and the order of magnitude of the function with respect to the ordinate t.*

section (§ 5). While the proof referred to was obtained by a direct investigation of the formal expressions which represent the abscissae of summability of a Dirichlet series as functions of the coefficients, the proof which will be given below rests upon the connection which was shown in Theorem I to exist between the summability behaviour of a Dirichlet series and analytic properties of the function represented by it. This proof will cast new light on the theorem under consideration.

Theorem II. *If* $\lambda_r = \lambda_{r+1}$, *then also* $\lambda_{r+m} = \lambda_r$ *for all* $m = 1, 2, \ldots$, i.e., *the limit abscissa of summability* Λ *is equal to* λ_r.

Proof. From the assumption $\lambda_r = \lambda_{r+1}$ it follows first that there cannot exist any positive quantity ε, however small, and a corresponding constant k, such that the function $f(s)$ represented by the series is regular and equal to $O(|t|^k)$ for $\sigma > \lambda_r - \varepsilon$. Indeed, if such a pair of numbers ε, k existed, one could infer by Theorem I that $\lambda_{r+1} < \lambda_r$, and this inequality contradicts the assumption $\lambda_r = \lambda_{r+1}$.

From this remark it follows at once that $\Lambda = \lambda_r$. Indeed, Λ in the first place can not be larger than λ_r, and if $\Lambda < \lambda_r$, the function $f(s)$ would have to be regular and equal to $O(|t|^{\text{const.}})$ for $\sigma > \lambda_r - \varepsilon$ (where ε is chosen less than $\lambda_r - \Lambda$). in view of Theorem II, § 7. This, however, would contradict the result obtained above from the assumption $\lambda_r = \lambda_{r+1}$.

Furthermore, the following theorem can be immediately inferred from Theorem I.

Theorem III. *Let the function* $f(s)$ *be represented in some half-plane by a convergent Dirichlet series* $\sum \frac{a_n}{n^s}$ *and let* $f(s)$ *be regular and equal to* $O(|t|^K)$ ($K = const.$) *for* $\sigma > \alpha$. *Then, if* Λ *denotes the limit abscissa of summability of* $\sum \frac{a_n}{n^s}$, *we have*

$$\Lambda \leqq \alpha .$$

Proof. Let us assume that $\Lambda = \beta > \alpha$, and let γ be an arbitrary positive number less than 1 and less than $\beta - \alpha$. Since $\lambda_r \geqq \Lambda = \beta$, we can choose for all r the number $1-\eta$ occurring in Theorem I equal to the number γ and the number $r+1+k$ occurring in Theorem I equal to the given number K. The number c occurring in Theorem I will then for all $r > K-1$ be equal to $1-\eta = \gamma$. The lower bound given in Theorem I for the difference $\lambda_r - \lambda_{r+1}$ will therefore be a positive constant independent of r for all $r > K-1$, which shows at once that $\Lambda = \lim \lambda_r$ is equal to $-\infty$, and conse-

quently that $\Lambda \neq \beta$. Our assumption that $\Lambda = \beta > \alpha$ has thus led us to a contradiction, and Theorem III is proved.

From Theorem III in conjunction with Theorem V, § 2 (page 65) and Theorem II, § 7 (page 101) one immediately obtains the following theorem, fundamental for the theory of summability for Dirichlet series.

A Dirichlet series $\sum \frac{a_n}{n^s}$ *is summable just so far as the function* $f(s)$ *represented by the series is regular and of finite order of magnitude with respect to the ordinate* t.

Or, formulated more precisely: Let $f(s)$ be a function which is represented by a convergent Dirichlet series $\sum \frac{a_n}{n^s}$ in a certain half-plane; then the limit abscissa of summability $\sigma = \Lambda$ of this series can always be determined from the mere knowledge of simple analytic properties of the function $f(s)$, namely through one of the three theorems given below, which correspond to the three possible distinct cases.

The first theorem treats the case in which summability of the series is halted by the fact that, on going to the left, one encounters a singularity. This theorem is stated as follows.

Theorem IVa. *If* $f(s)$ *is regular for* $\sigma > \gamma$ *but not regular everywhere for* $\sigma > \gamma - \varepsilon$ (ε *an arbitrarily small positive number*), *and if for* $\sigma > \gamma + \varepsilon$

$$f(s) = O(|t|^k),$$

where $k = k(\varepsilon)$ *is a positive constant independent of* σ *and* t (which may tend to infinity when ε approaches 0), *then*

$$\Lambda = \gamma.$$

The second theorem treats the case in which summability of the series is halted by the fact that, on going to the left, without encountering a singularity one reaches a region where the function $f(s)$ is not of finite order of magnitude with respect to the ordinate t. This theorem is stated as follows.

Theorem IVb. *Let* $f(s)$ *be regular for* $\sigma > \beta$, *but let there exist a number* $\alpha > \beta$ *such that for* $\sigma > \alpha$, $f(s)$ *is not equal to* $O(|t|^k)$ *for any value of the constant* k, *however large; in this case there must exist a finite number* $\gamma > \beta$ (determined by a so-called Dedekind cut) *such that for* $\sigma > \gamma + \varepsilon$

$$f(s) = O(|t|^k) = O(|t|^{k(\varepsilon)})$$

while for $\sigma > \gamma-\varepsilon$ such an equation holds for no k. Then

$$\varLambda = \gamma .$$

The third and last theorem, which incidentally can be considered as a limiting case both of Theorem IVa and of Theorem IVb, treats the case in which, on going to the left, one neither encounters a singularity nor reaches a region where $f(s)$ is no longer of finite order of magnitude with respect to the ordinate t. This theorem is stated as follows.

Theorem IVc. *If $f(s)$ is an integral function and if, corresponding to every real number α, there exists a constant $k = k(\alpha)$ such that*

$$f(s) = O(|t|^k) = O(|t|^{k(\alpha)})$$

for $\sigma > \alpha$, then

$$\varLambda = -\infty ,$$

i.e., $\sum \frac{a_n}{n^s}$ *is summable in the whole plane.*

While it has been proved above that there cannot exist a Dirichlet series not falling under any of the three cases considered in Theorem IV, it has not yet been determined whether or not there really exist Dirichlet series corresponding to all three cases.

Before the investigation of the summability problem for Dirichlet series can be considered as complete, we must therefore first subject this existence question to a closer examination. This examination will result in the proof of three theorems given below, which correspond to the three cases considered in Theorem IV, and which show the existence of all the types of Dirichlet series that can possibly exist in view of the foregoing (i.e., the results of the present section and § 5).

Theorem Va. *Let*

$$\lambda_0, \lambda_1, \ldots, \lambda_r, \ldots \tag{12}$$

be an arbitrary sequence of real numbers which satisfy the conditions

$$0 \leqq \lambda_r - \lambda_{r+1} \leqq \lambda_{r-1} - \lambda_r \leqq 1 \tag{13}$$

(for all $r = 1, 2, \ldots$) and

$$\varLambda = \lim_{r=\infty} \lambda_r \neq -\infty . \tag{14}$$

Then there exist a Dirichlet series $f(s) = \sum \frac{a_n}{n^s}$ and a Dirichlet series $g(s) = \sum \frac{b_n}{n^s}$ such that both of these series have precisely the elements of the given sequence (12) *as*

their abscissae of summability, and such that $f(s)$ possesses a singularity on the boundary of summability $\sigma = \Lambda$, while $g(s)$ is regular everywhere on this line but possesses singularities arbitrarily close to the left of this line.

Theorem Vb. *Let* (12) *be an arbitrary sequence of numbers which satisfy the conditions* (13) *and* (14). *Then there exists a Dirichlet series* $\sum \frac{a_n}{n^s}$ *which has precisely the elements of the sequence* (12) *as its abscissae of summability, and for which the represented function $f(s)$ is regular a finite distance beyond the boundary of summability $\sigma = \Lambda$* (i.e., where $f(s)$ possesses singularities neither on the line $\sigma = \Lambda$ nor arbitrarily close to the left of this line).

Theorem Vc. *There exist a Dirichlet series* $\sum \frac{a_n}{n^s}$ *whose abscissae of summability are all equal to* $-\infty$ (i.e., which converges in the whole plane) *and, when* (12) *is an arbitrary sequence subject only to the condition* (13) *and the condition* $\lim \lambda_r = \Lambda = -\infty$, *a Dirichlet series* $\sum \frac{b_n}{n^s}$ *which has as its abscissae of summability precisely the elements of the sequence* (12).

Theorem Vc has already been proved in the foregoing, since the first half follows at once from the example $\sum \frac{1}{n!\, n^s}$ considered in Part One, and the second half follows from Theorem V, § 5.*

We now turn to the proof of Theorem Vb, in which one is naturally led to consider two different cases.

---

* As the simplest example of a special Dirichlet series for which λ_r is finite for all r but for which $\Lambda = -\infty$ (i.e., which is not convergent but is nevertheless summable in the whole plane), may be reckoned the series $\sum \frac{(-1)^{n+1}}{n^s}$, which has already often been discussed. We have considered in § 3 the summability behaviour of this series and we have mentioned how from the results obtained one can immediately infer that the function $\zeta(s)(1-2^{1-s})$ is an integral function and also how with the aid of Theorem I, § 7 one can immediately obtain upper bounds for the numerical values of this function corresponding to arbitrarily large values of the ordinate t.—By using the results of the present section we are now in a position to go in the opposite direction; i.e., if we assume as known (for instance from Riemann's functional equation) that the function $\zeta(s)(1-2^{1-s})$ is an integral function and that the function is equal to $O(|t|^K) = O(|t|^{K(\alpha)})$ for $\sigma > \alpha$, where α is an arbitrary real number, then with the aid of Theorem IVc, we can conclude at once *that the series* $\sum \frac{(-1)^{n+1}}{n^s}$, which we know beforehand is convergent and represents the function $\zeta(s)(1-2^{1-s})$ in a certain half-plane (namely for $\sigma > \lambda_0 = 0$), *must necessarily be summable in the whole plane.*

Case 1. Here we consider sequences of numbers (12) such that $\lambda_r-\lambda_{r+1}$ is different from 0 for all r, or, equivalently, $\lambda_r \neq \Lambda$ for all r.

As shown in § 5 (where we considered the problem of the distribution of the abscissae of summability, but without paying attention to the regularity or singularity of the analytic functions represented by the series), there exists an infinite sequence of Dirichlet series $f_p(s) = \sum \frac{a_{p,n}}{n^s}$ $(p = 1, 2, 3, \ldots)$ (all summable in the whole plane, which implies that the functions $f_p(s)$ are all integral functions) and an infinite sequence of positive numbers e_p $(p = 1, 2, \ldots)$ such that if the non-vanishing numbers ε_p satisfy the conditions $|\varepsilon_p| < e_p$, then the Dirichlet series $\sum \frac{a_n}{n^s}$, where

$$a_n = \sum_{p=1}^{p=\infty} \varepsilon_p a_{p,n},$$

has as its abscissae of summability precisely the quantities $\lambda_0, \lambda_1, \ldots, \lambda_r, \ldots$, and the function $f(s)$ represented by the series $\sum \frac{a_n}{n^s}$ is for $\sigma \geqq \lambda_0+2$ equal to the sum of the absolutely convergent series $\sum\limits_{p=1}^{p=\infty} \varepsilon_p f_p(s)$.

Now let K_p be determined so that $|f_p(s)| < K_p$ for $|s| \leqq p$. Let the non-vanishing numbers ε_p be so chosen that they satisfy not only the conditions $|\varepsilon_p| < e_p$ but also the conditions $|\varepsilon_p| < \frac{C_p}{K_p}$, where $\sum\limits_{p=1}^{p=\infty} C_p$ is convergent. I now assert that the analytic function $f(s)$ represented by the series $\sum \frac{a_n}{n^s}$ is regular beyond the boundary of summability and is, in fact, an integral function.

That this is so may be seen in the following way. Since $|\varepsilon_p| < \frac{C_p}{K_p}$, where $\sum C_p$ is convergent, one sees immediately that the series $\sum \varepsilon_p f_p(s)$ converges uniformly in every finite region (i.e., for $|s| <$ const.) and consequently represents an integral function $F(s)$. Since $f(s)$ is equal to $F(s)$ for $\sigma \geqq \lambda_0+2$, it follows that $f(s)$ must be identical with $F(s)$ and therefore must also be an integral function. q.e.d.

Case 2. We consider here sequences (12) for which there exists a number R such that $\lambda_r = \Lambda$ for $r \geqq R$.

For a sequence $\lambda_0, \lambda_1, \ldots, \lambda_r, \ldots$ of this type, we have proved in § 5 that there exist a finite number of series $f_p(s) = \sum\limits_{n=1}^{n=\infty} \frac{a_{p,n}}{n^s}$ $(p = 1, 2, \ldots, P)$ such that the

series $\sum \frac{a_n}{n^s}$, where
$$a_n = \sum_{p=1}^{p=P} k_p a_{p,n} \quad (k_p \neq 0),$$
as its abscissae of summability has precisely the quantities $\lambda_0, \lambda_1, \ldots, \lambda_r, \ldots$, and where the function $f(s)$ represented by $\sum \frac{a_n}{n^s}$ is equal to the function $\sum_{p=1}^{p=P} k_p f_p(s)$.

Whereas, in Case 1, all of the series $f_p(s) = \sum \frac{a_{p,n}}{n^s}$ were summable in the whole plane and hence represented functions which were regular a finite distance beyond the boundary of summability $\sigma = \Lambda$, in the present case, exactly one of these series $\left(\text{for example } f_P(s) = \sum \frac{a_{P,n}}{n^s}\right)$ is not summable in the whole plane, since it is determined so that it has all of its abscissae of summability equal to Λ. As an example of such a series, we have hitherto only known (and hence have used exclusively) series which are not regular a finite distance beyond the boundary of summability $\sigma = \Lambda$. Therefore, in order to prove Theorem Vb when the given sequence (12) falls under Case 2, it is necessary and sufficient to prove the existence of a Dirichlet series $f(s) = \sum \frac{a_n}{n^s}$ whose abscissae of summability are all equal to Λ, but whose represented function $f(s)$ nevertheless is regular a finite distance beyond the boundary line of summability $\sigma = \Lambda$.* The existence of such a series can be shown by application of the following device. Let $g(s) = \sum \frac{b_n}{n^s}$ be a Dirichlet series with the abscissa of absolute convergence $l = \Lambda$, about which we assume that $g(s)$ is regular for $\sigma > \Lambda - C$ $(C > 0)$, and that $g(s)$ is not equal to $O(\log |t|)$ for $\sigma > \Lambda - \varepsilon$ (ε arbitrarily small). (The author has shown that such a series $g(s) = \sum \frac{b_n}{n^s}$ exists, for instance in Theorem XVII, page 32).

Among the four functions $+g(s)$, $-g(s)$, $+ig(s)$, and $-ig(s)$, there must exist at least one, which we shall denote by $h(s) = \sum \frac{c_n}{n^s}$, such that the function $f(s) = e^{h(s)}$ for $\sigma > \Lambda - \varepsilon$ (ε arbitrarily small) is not equal to $O(|t|^K)$ for any value of K.

The function $f(s)$ will then be of the desired kind. First, the function $f(s)$ is represented for $\sigma > \Lambda$ by a convergent (indeed, an absolutely convergent) Dirichlet

* As one can easily see, none of the special Dirichlet series met in the literature are of this kind; indeed, it can be remarked that none of these special series, so far as one can determine their summability behaviour, belong to the case considered under b).

series $\sum \frac{a_n}{n^s}$, namely the Dirichlet series obtained by formal computations from the expression

$$1+\frac{1}{1!}\sum_{n=1}^{n=\infty}\frac{c_n}{n^s}+\frac{1}{2!}\left(\sum_{n=1}^{n=\infty}\frac{c_n}{n^s}\right)^2+\cdots$$

(this follows immediately from the fact that $\sum \frac{c_n}{n^s}$ converges absolutely for $\sigma > \Lambda$, while $\sum \frac{x^n}{n!}$ converges absolutely for all x). Secondly, $\sum \frac{a_n}{n^s}$ cannot be summable beyond the line $\sigma = \Lambda$, since this would imply that $f(s) = O(|t|^{\text{const.}})$ for $\sigma > \Lambda-\varepsilon$. Thirdly, $f(s)$ is regular for $\sigma > \Lambda-C$, since $f(s)$ is equal to $e^{h(s)}$, and $h(s)$ is regular for $\sigma > \Lambda-C$.

The proof of Theorem Vb is thus complete.

We now turn to the proof of Theorem Va, which we can easily carry through by using Theorem Vb.

Let $h(s) = \sum \frac{c_n}{n^s}$ be a Dirichlet series with abscissae of summability equal to the given numbers $\lambda_0, \lambda_1, \ldots, \lambda_r, \ldots$ and for which $h(s)$ is regular for $\sigma > \Lambda-C$ $(C > 0)$. Such a series certainly exists in view of Theorem Vb.

Further, let $j(s) = \sum \frac{d_n}{n^s}$ and $k(s) = \sum \frac{e_n}{n^s}$ be two Dirichlet series which both converge for $\sigma > \Lambda$ and such that $j(s)$ possesses a singularity on the line $\sigma = \Lambda$, while $k(s)$ is regular on this line but possesses singularities arbitrarily near to the left of it.

The series $f(s) = \sum \frac{a_n}{n^s} = \sum \frac{c_n+d_n}{n^s}$ and $g(s) = \sum \frac{b_n}{n^s} = \sum \frac{c_n+e_n}{n^s}$ then satisfy the conditions described in Theorem Va, as one sees immediately, and this theorem is consequently proved.*

---

* As a case of a special Dirichlet series of the type considered in Theorems IVa and Va which seems especially interesting to the author, one can mention the series $\sum \frac{\mu_n}{n^s}$, where μ_n denotes the so-called Möbius factor (i.e., where μ_n is 0 when n is divisible by the square of a prime, while $\mu_n = +1$ or -1 for all square-free n, according as the number of prime factors in n is even or odd). As one sees immediately, this series converges at least for $\sigma > 1$ and represents here, as is known, the function $1/\zeta(s)$. Since the function $\zeta(s)$ is a meromorphic function in the whole plane which is different from 0 for $\sigma \geqq 1$ and is known to possess complex zeros whose real parts lie between $+1$ (excluded) and $\frac{1}{2}$ (included), the function $1/\zeta(s)$ is likewise a meromorphic function in the whole plane, regular for $\sigma \geqq 1$, but possessing poles whose real parts lie between $+1$ (excluded) and $\frac{1}{2}$ (included). Let θ ($\frac{1}{2} \leqq \theta \leqq 1$) denote the least upper bound of the real parts of the zeros of the ζ-function, or, equivalently, of the poles of the $(1/\zeta)$-function. Since it is known that $1/\zeta(s) = O(|t|^k)$ for $\sigma \geqq \theta+\varepsilon$ (ε arbitrarily small), where

We shall conclude this part and therewith the present dissertation by showing briefly how introduction of Cesàro summability also brings essential advantages in the theory of multiplication of Dirichlet series.

Let $f(s) = \sum \frac{a_n}{n^s}$ and $g(s) = \sum \frac{b_n}{n^s}$ be two Dirichlet series which both converge for $\sigma > \sigma_0$. The product series $h(s) = f(s) \cdot g(s) = \sum \frac{c_n}{n^s}$ need *not* converge for $\sigma > \sigma_0$, as noted in Part One (it was shown that in general one can conclude only that $\sum \frac{c_n}{n^s}$ converges for $\sigma > \sigma_0 + \frac{1}{2}$). However, as communicated independently by M. Riesz* and the author†, $\sum \frac{c_n}{n^s}$ *is always summable of the first order for* $\sigma > \sigma_0$.

Here we shall not investigate further what one can say about the product series $\sum \frac{c_n}{n^s}$ for $\sigma > \sigma_0$ when it is known that $\sum \frac{a_n}{n^s}$ is summable of the p^{th} order for $\sigma > \sigma_0$ and that $\sum \frac{b_n}{n^s}$ is summable of the q^{th} order for $\sigma > \sigma_0$. We shall content ourselves with proving the following general theorem, which shows *that summability of a Dirichlet series for* $\sigma > \sigma_0$ (in contrast to convergence for $\sigma > \sigma_0$) *is invariant under the operation of multiplication.*

Theorem VI. *If* $f(s) = \sum \frac{a_n}{n^s}$ *and* $g(s) = \sum \frac{b_n}{n^s}$ *are both summable for* $\sigma > \sigma_0$ (i.e., if $\Lambda \leqq \sigma_0$ for both series), *then the product series* $h(s) = \sum \frac{c_n}{n^s}$ *is likewise summable for* $\sigma > \sigma_0$.

$k = k(\varepsilon)$ is a positive constant, we immediately infer from Theorem IVa *that the series* $\sum \frac{\mu_n}{n^s}$ *is summable just so far as the function* $1/\zeta(s)$ *is regular, i.e., that the limit abscissa of summability* Λ *of the series* $\sum \frac{\mu_n}{n^s}$ *is precisely the quantity* θ.—*Thus, if the Riemann hypothesis* $\theta = \frac{1}{2}$ *is correct, then* $\Lambda = \frac{1}{2}$, *and conversely.* It is known from a theorem of Landau, Beiträge zur analytischen Zahlentheorie, l.c., p. 259, that *if* $\theta < 1$, *then also* $\lambda_0 < 1$, where λ_0 denotes the abscissa of convergence of $\sum \frac{\mu_n}{n^s}$. However, one has no idea whether (in case $\theta < 1$) $\lambda_0 > \theta$ or $\lambda_0 = \theta$. Only upon introducing summability, therefore, are the problem of determining the least upper bound for the real parts of the zeros of the ζ-function and the study of the series $\sum \frac{\mu_n}{n^s}$ brought into complete correspondence with each other. The problems, incidentally, are probably equally difficult.

* Sur la sommation des séries de Dirichlet, *Comptes rendus de l'Académie des Sciences, Paris*, vol. 149, 5 July 1909.

† Über die Summabilität Dirichletscher Reihen, l.c., p. 255.

Proof. *Since* $\sum \frac{a_n}{n^s}$ and $\sum \frac{b_n}{n^s}$ are summable for $\sigma > \sigma_0$, it follows from Theorem V, § 2 and Theorem II, § 7 that

1. $f(s)$ and $g(s)$ are both regular for $\sigma > \sigma_0$;
2. for $\sigma > \sigma_0+\varepsilon$, we have $f(s) = O(|t|^{k_1(\varepsilon)})$ and $g(s) = O(|t|^{k_2(\varepsilon)})$.

Since $h(s) = f(s) \cdot g(s)$, it follows from this that $h(s)$ is regular for $\sigma > \sigma_0$ and also that the equation

$$h(s) = O(|t|^{k_1(\varepsilon)+k_2(\varepsilon)}) = O(|t|^{k_3(\varepsilon)})$$

holds for $\sigma > \sigma_0+\varepsilon$. This immediately shows, in view of Theorem III (page 108), that the Dirichlet series $\sum \frac{c_n}{n^s}$ which represents $h(s)$ is summable for $\sigma > \sigma_0$. q.e.d.

THESES

1. The opinion that there exists a very essential and fundamental difference between convergent series and series which fall under a generalized definition of convergence, which one sometimes meets, appears untenable to the author.—The question as to where one shall draw the line, in a given investigation, between series which are 'usable to represent a number' and those which are 'not usable', is entirely one of convenience.

2. Just as for a Dirichlet series, the line of convergence $\sigma = \lambda_0$ for a factorial series is a line which seems not to stand in any simple relation to the analytic properties of the function represented by the series. On the basis of the close analogy between the behaviour of a factorial series and a Dirichlet series, it must be considered probable that the boundary line of summability $\sigma = \Lambda$ for a factorial series (just as for a Dirichlet series) is a line which may be determined in a simple way from the mere knowledge of the analytic properties of the function represented by the series.

3. At several places in the theory of infinite series, one can obtain essential advantages through the introduction of a certain notion: 'series with arbitrary indices', a notion which forms a bridge between the usual infinite series and the infinite integrals.

Let $x_0, x_1, \ldots, x_n, \ldots$ be a sequence of numbers which tends steadily to infinity, and let u_{x_n} be a function of x_n; the series

$$u_{x_1}(x_1-x_0)+\cdots+u_{x_n}(x_n-x_{n-1})+\cdots$$

—where x_n-x_{n-1} is to be considered as a summation factor corresponding to dx in the integral $\int u(x)dx$—will be said to belong to the index system

$$x_0, x_1, \ldots, x_n, \ldots.$$

Upon introducing this notion, one is in a position to perform transformations on infinite

series corresponding exactly to the usual transformations for definite integrals. Among the fields in which the concept of series with arbitrary indices can be applied with special advantage, we shall mention here, in addition to the theory of multiplication of series, only the theory of so-called *general* convergence criteria. As in the case of definite integrals, there is no difference between special and general criteria when one operates with the extended concept of a series, where the transformation is at our disposal. More correctly, from a special criterion, one can immediately obtain a corresponding general criterion by a transformation. Thus, for example, corresponding to the special Cauchy criterion

$$\frac{u_n}{u_{n-1}} \leqq a < 1$$

we obtain immediately the general criterion

$$\sqrt[x_n - x_{n-1}]{\frac{u_n}{u_{n-1}} \cdot \frac{x_{n-1}-x_{n-2}}{x_n - x_{n-1}}} \leqq a < 1\,,$$

which is equivalent to a criterion found in another way by Pringsheim.

4. The Picard-Landau theorem on integral functions contains the following algebraic theorem as a special case: To every polynomial $f(x)$ for which the coefficients of x^0 and x^1 are a_0 and a_1, respectively, there exists a constant $K = K(a_0, a_1)$ such that at least one of the equations $f(x) = 0$ and $f(x) = 1$ has a root within the circle $|x| = K$.

It would be of great interest if one could find an elementary proof of this theorem, or merely—which might perhaps be easier—of the special theorem that there exists such a constant $K = K(a_0, a_1, n)$ corresponding to all the polynomials of a *given* degree n.

5. The general theory of limits can be closely linked to the theory of fundamental sequences (convergent sequences of numbers) by the following theorem: 'Let X, with the elements x, be an arbitrary 'fundamental set' (i.e., an ordered set which, taken in its order, 'tends to a definite limit'), and let Y be a set whose elements y are single-valued functions of the elements x. The necessary and sufficient condition that the set Y (ordered so that $y(x_1)$ follows $y(x_2)$ when x_1 follows x_2) be a fundamental set is that all sequences of numbers $y_1, \ldots, y_n, \ldots$ in the set Y which correspond to steadily proceeding sequences of numbers $x_1, \ldots, x_n, \ldots$ in X which ultimately pass every given element x, are fundamental sequences.'

This theorem allows one to transfer the theorems holding for fundamental sequences which are ordinarily proved in the introductions to textbooks on analysis, to the corresponding theorems holding for fundamental sets (i.e., for passages to the limit in general). These theorems are often used in textbooks without sufficient justification for their validity.

6. Let $f(x) = \sum a_n x^n$ be a power series with radius of convergence 1. Abel has proved that if $\sum a_n$ is convergent with the sum A, then the equation $\lim\limits_{x=1-0} f(x) = A$ holds. On the other hand, the converse theorem does not hold in general. However, Tauber has proved that when the assumption $\lim\limits_{n=\infty} n a_n = 0$ is added to the assumption $\lim\limits_{x=1-0} f(x) = A$, the convergence of $\sum a_n$ can be inferred.

One can show that Tauber's theorem cannot be essentially improved, since one can prove

that there does not exist any sequence g_n, tending to infinity however slowly, such that the condition $\lim na_n = 0$ in Tauber's theorem can be replaced by the condition $\lim \frac{n}{g_n} \cdot a_n = 0$. (For the case $n = g_n$, this contains a result communicated by Pringsheim, namely, that one cannot replace the condition $\lim na_n = 0$ by the condition $\lim a_n = 0$.)

A theorem proved by Fatou, which stands in interesting contrast to the above, is to the effect that when the assumption '$\lim\limits_{x=1-0} f(x)$ exists' is replaced by the stronger assumption '$f(x)$ is regular at the point 1', one needs only the assumption $\lim a_n = 0$ in order to infer the convergence of $\sum a_n$.

7. As first shown by C. Jordan, the fundamental theorem holds that every continuous closed curve without double points divides the plane into two parts. It must, however, as yet be considered an open question whether or not there exists a general theorem corresponding to Jordan's theorem which holds for ordinary three-dimensional space and in general for n-dimensional space.

8. Corresponding exactly to the ordinary theory of differentials, one can construct what might be called a theory of quotientials, which considers the two inverse operations defined by the equations

$$'u(x) = \lim_{\Delta x = 0} \sqrt[\Delta x]{\frac{u(x+\Delta x)}{u(x)}}$$

and

$$\prod_a^b u(x)^{dx} = \lim_{\Delta x = 0} \prod_{x=a}^{x=b} u(x)^{\Delta x}.$$

Although such a theory of quotientials is closely related to the theory of differentials (for instance by the equation $'u(x) = e^{(\log u(x))'}$), still, the introduction into analysis of special notations for such operations would be advantageous in many respects. Thus one is thereby immediately led to establish what might be called integral criteria of convergence of the second kind for infinite integrals

$$\int_0^\infty u(x)dx \quad (u(x) > 0).$$

(In fact, the relative measure of growth $'u(x)$ corresponds exactly to $\frac{u_{n+1}}{u_n}$, in the same way as the absolute measure of growth $u'(x)$ corresponds to $u_{n+1} - u_n$.) Thus, for example, corresponding to Bertrand's convergence criterion for infinite series, we get the integral convergence criterion

$$'u(x) \leqq 1 - \frac{1}{x} - \frac{1}{x \log x} \cdot \cdot \cdot - \frac{1+\varepsilon}{x \log x \cdot \cdot \cdot \log_m x} \quad (\varepsilon > 0).$$

SOME REMARKS ON THE UNIFORM CONVERGENCE OF DIRICHLET SERIES

Summary of Nogle Bemærkninger om de Dirichletske Rækkers ligelige Konvergens.
Matematisk Tidsskrift B 1921, 51—55.

It is pointed out that in many theorems on general Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}, \quad s = \sigma + it, \quad 0 \leqq \lambda_1 < \lambda_2 < \cdots \quad (\lambda_n \to \infty),$$

containing the condition that the series should possess a half-plane of *absolute* convergence, this condition may be replaced by the weaker condition that the series should possess a half-plane of *uniform* convergence.

The theorems in question are those in which the existence of a half-plane of absolute convergence is merely used to show that for 'large' σ the first terms of the series will dominate the remainder, in the sense that if for an arbitrary N we write

$$f(s) = \sum_{n=1}^{N} a_n e^{-\lambda_n s} + e^{-\lambda_N s} R(s),$$

then $R(s) \to 0$ as $\sigma \to \infty$, uniformly in t. But this follows from the existence of a half-plane of uniform convergence. Indeed, the function $R(s)$ is represented by the Dirichlet series

$$e^{\lambda_N s} \sum_{n=N+1}^{\infty} a_n e^{-\lambda_n s} = \sum_{n=N+1}^{\infty} a_n e^{-(\lambda_n - \lambda_N)s}.$$

It is sufficient to show that this series possesses a half-plane of uniform convergence. For then we may pass to the limit term by term and find that $R(s) \to 0$ as $\sigma \to \infty$, since all $\lambda_n - \lambda_N$ are positive. Evidently, if the original series converges uniformly in a half-plane, the series representing $R(s)$ will converge uniformly *on every vertical line* $\sigma = \sigma_0$ in this half-plane (the factor $e^{\lambda_N s}$ being bounded on the line). However, by a lemma previously proved by the author,* the uniform convergence of a Dirichlet series on a line $\sigma = \sigma_0$ implies the uniform convergence in the half-plane $\sigma > \sigma_0$.

* H. Bohr, Darstellung der gleichmässigen Konvergenzabszisse einer Dirichletschen Reihe $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ als Funktion der Koeffizienten der Reihe, *Archiv der Mathematik und Physik* (3) 21 (1913), 326–330. The theorem is here proved for ordinary Dirichlet series but the proof is valid in the general case.

A REMARK ON THE UNIFORM CONVERGENCE OF DIRICHLET SERIES

Summary of En Bemærkning om Dirichletske Rækkers ligelige Konvergens.
Matematisk Tidsskrift B 1951, 1—8.

1. For a somewhere convergent Dirichlet series

$$f(s) = f(\sigma+it) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s} \quad (0 \leqq \lambda_1 < \lambda_2 < \cdots \to \infty)$$

let σ_L denote the abscissa of uniform convergence defined as the lower bound of those abscissae σ_0 for which the series converges uniformly in the half-plane $\sigma > \sigma_0$, and let $\sigma_B \leqq \sigma_L$ denote the abscissa of boundedness defined as the lower bound of those abscissae σ_0 for which the analytic function $f(s)$ is regular and bounded in the half-plane $\sigma > \sigma_0$.

2. As shown by the author, for certain important types of Dirichlet series, including the ordinary Dirichlet series $f(s) = \sum a_n n^{-s} = \sum a_n e^{-s \log n}$, one has $\sigma_L = \sigma_B$. In a paper to appear elsewhere the types of Dirichlet series with $\sigma_L = \sigma_B$ will be studied more closely. The present paper gives an example of a Dirichlet series for which this relation is as far from being true as possible, in the sense that

$$\sigma_B = -\infty, \quad \sigma_L = +\infty .$$

3. From the inequality

$$\left| \frac{\sin x}{1} + \frac{\sin 2x}{2} + \cdots + \frac{\sin nx}{n} \right| \leqq 2$$

for all real x and all positive integers n it follows that the polynomial

$$P_n(z) = \frac{1}{4}\left[-\frac{1}{n} - \frac{z}{n-1} - \cdots - \frac{z^{n-1}}{1} + \frac{z^{n+1}}{1} + \frac{z^{n+2}}{2} + \cdots + \frac{z^{2n}}{n} \right] = \sum_{\nu=0}^{2n} b_\nu^{(n)} z^\nu$$

is numerically $\leqq 1$ in the unit circle $|z| \leqq 1$ while the numerical value of the sum of the first n coefficients tends to ∞ as $n \to \infty$. An elementary consideration shows

that on the radius ending at $z = -1$ the partial sums of $P_n(z)$ are uniformly bounded. More precisely

$$\left|\sum_{\nu=0}^{n'} b_\nu^{(n)} x^\nu\right| < 1 \quad (0 \leqq n' \leqq 2n, \ -1 \leqq x < 0).$$

4. Emphasizing only what is essential, and replacing z by $-z$ for the sake of convenience, we see that for an arbitrarily large positive constant K we can determine a polynomial

$$Q(z) = \sum_{\nu=0}^{N} c_\nu z^\nu$$

of degree $N = N(K)$ such that

$$|Q(z)| \leqq 1 \quad \text{for} \quad |z| \leqq 1,$$

$$\left|\sum_{\nu=0}^{N'} c_\nu x^\nu\right| \leqq 1 \quad \text{for all} \quad N' \ (0 \leqq N' \leqq N) \quad \text{and} \quad 0 < x \leqq 1,$$

$$\left|\sum_{\nu=0}^{N''} c_\nu z^\nu\right| > K \quad \text{for some} \quad N'' \ (0 \leqq N'' \leqq N) \quad \text{and} \quad z = -1;$$

it follows (by continuity) that for a sufficiently small $\delta = \delta(K) > 0$

$$\max_{|z| = e^{-\delta}} \left|\sum_{\nu=0}^{N''} c_\nu z^\nu\right| > K.$$

5. We now consider the (rapidly increasing) sequence of K-values

$$K_m = m^2 e^{2m^2} \quad (m = 1, 2, \ldots)$$

and denote the polynomial $Q(z)$ corresponding to $K = K_m$ by

$$Q_m(z) = \sum_{\nu=0}^{N_m} c_\nu^{(m)} z^\nu$$

and the numbers N', N'', δ occurring in the preceding inequalities by N'_m, N''_m, δ_m; we assume that δ_m has been chosen $< \dfrac{2m}{N_m}$.

6. In $Q_m(z)$ we replace z by $e^{-\frac{\delta_m}{2m}(s+m)}$ and obtain an exponential polynomial

$$R_m(s) = \sum_{\nu=0}^{N_m} c_\nu^{(m)} e^{-\nu\frac{\delta_m}{2m}(s+m)} = \sum_{\nu=0}^{N_m} d_\nu^{(m)} e^{-\mu_\nu^{(m)} s} \quad \left(\text{where } \mu_\nu^{(m)} = \nu\frac{\delta_m}{2m}\right)$$

such that

(1) $$|R_m(s)| \leqq 1 \quad \text{for} \quad \sigma \geqq -m\,,$$

(2) $$\left|\sum_{\nu=0}^{N'_m} d_\nu^{(m)} e^{-\mu_\nu^{(m)}\sigma}\right| \leqq 1 \quad \text{for all} \quad N'_m \;(0 \leqq N'_m \leqq N_m) \quad \text{and} \quad -m \leqq \sigma < \infty,\; t = 0\,,$$

(3) $$\max_{\sigma=+m}\left|\sum_{\nu=0}^{N''_m} d_\nu^{(m)} e^{-\mu_\nu^{(m)} s}\right| > m^2 e^{2m^2} \quad \text{for some} \quad N''_m < N_m\,.$$

We notice that for every m the exponents $\mu_\nu^{(m)}$ are all < 1.

7. We now consider the Dirichlet series $\sum_1^\infty a_n e^{-\lambda_n s}$ whose exponents $\lambda_1, \lambda_2, \ldots$ are the numbers in the sequence

$$1+\mu_0^{(1)},\, 1+\mu_1^{(1)},\, \ldots,\, 1+\mu_{N_1}^{(1)},\, 2+\mu_0^{(2)},\, 2+\mu_1^{(2)},\, \ldots,\, 2+\mu_{N_2}^{(2)},\, \ldots$$

and for which the coefficients a_n of the terms corresponding to the m^{th} group of exponents

$$m+\mu_0^{(m)},\, m+\mu_1^{(m)},\, \ldots,\, m+\mu_{N_m}^{(m)}$$

are the numbers

$$a_n = \frac{1}{m^2}\, e^{-m^2} d_\nu^{(m)}$$

obtained by multiplying the coefficients $d_\nu^{(m)}$ of $R_m(s)$ by the factor $\dfrac{1}{m^2}\, e^{-m^2}$. This series will have the desired properties.

8. If we introduce brackets in the series we obtain the series

$$f(s) = \sum_{m=1}^{\infty} \frac{1}{m^2}\, e^{-m^2} e^{-ms} R_m(s)\,,$$

which on account of (1) converges uniformly in every half-plane $\sigma > -\sigma_0$ and therefore represents an integral function $f(s)$ bounded in every half-plane $\sigma > -\sigma_0$.

9. We now consider the Dirichlet series $\sum a_n e^{-\lambda_n s}$ itself, i.e., we remove the brackets. It follows from (2) that the series will converge for every real s. Hence it is everywhere convergent. The sum of the series must be $f(s)$. Hence $\sigma_B = -\infty$.

10. Finally, on account of (3) the series is not uniformly convergent in any half-plane $\sigma > \sigma_0$. Hence $\sigma_L = +\infty$.

S 3. Summary of A 20.

A THEOREM ON THE ζ-FUNCTION

Summary of En Sætning om ζ-Funktionen.
Nyt Tidsskrift for Matematik B 21 (1910), 60—66.

In § 3 of a paper by E. Landau and the author* a theorem has been proved which contains the following theorem:

Theorem I. *Let δ and ε be two arbitrarily small positive numbers; then there exists a point $s = \sigma + it$ for which $1-\delta < \sigma < 1+\delta$ and $|\zeta(s)| < \varepsilon$.*

In the present paper the following more general theorem is proved:

Theorem II. *If δ is an arbitrarily small positive number and k and τ are arbitrarily large positive numbers, then the inequality*

$$|\zeta(s)| < (\log \log t)^{-k}$$

has a solution $s' = \sigma' + it'$ in the domain

$$1-\delta < \sigma < 1+\delta, \quad t > \tau .$$

The proof depends on Carathéodory's inequality and on two known results on the ζ-function, namely:

1. the existence for every $\varepsilon > 0$ of a constant $c_1 = c_1(\varepsilon)$ such that for $\sigma > 1+\varepsilon$

$$|\log \zeta(s)| < c_1 .\dagger$$

2. the existence of a constant $c_2 > 1$ such that for every T the inequality

$$|\zeta(1+it)| > \frac{1}{c_2} \log \log t$$

has a solution $t > T$.‡ On placing $\log c_2 = c_3$, this implies the existence of a sequence

* H. Bohr and E. Landau, Über das Verhalten von $\zeta(s)$ und $\zeta_\varkappa(s)$ in der Nähe der Geraden $\sigma = 1$, *Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse* 1910, 303–330.

† Here and in the sequel $\log \zeta(s)$ denotes the branch of the logarithm in the half-plane $\sigma > 1$ which is real for real $s > 1$.

‡ H. Bohr and E. Landau, l. c., § 5.

of values $0 < t_1 < t_2 < \ldots, t_n \to \infty$ for which

$$|\log \zeta(1+it)| > \log\log\log t - c_3 .$$

Proof of Theorem II. If $\zeta(s)$ has zeros arbitrarily near to the line $\sigma = 1$, the statement is obvious. Otherwise, we may assume that $\zeta(s) \neq 0$ in the domain $\sigma > 1-\delta, t > \tau$, and may then for sufficiently large n apply Carathéodory's inequality to the function $-\log \zeta(s)$ in the circle $|s-s_0| \leqq \frac{\delta}{2}$, where

$$s_0 = 1+\frac{\delta}{8k+2}+it_n .$$

It shows that on the circle $|s-s_0| = \frac{\delta}{8k+2}$ the following estimate holds:

$$|-\log \zeta(s)| \leqq |-\log \zeta(s_0)| \frac{4k+1}{2k}+A\frac{1}{2k},$$

where A denotes the maximum of $\Re(-\log \zeta(s)) = -\log|\zeta(s)|$ on the circle $|s-s_0| = \frac{\delta}{2}$. On placing $s = 1+it_n$ in this inequality and using 1. and 2. we find

$$\log\log\log t_n - c_4 < A\frac{1}{2k},$$

where c_4 is independent of n. Now, $A = -\log|\zeta(s')|$ for a point $s' = \sigma'+it'$ on $|s-s_0| = \frac{\delta}{2}$. Thus for this point

$$-\log|\zeta(\sigma'+it')| > 2k \log\log\log t_n - c_5 ,$$

where c_5 is independent of n. Since t' lies in the interval $\left[t_n-\frac{\delta}{2}, t_n+\frac{\delta}{2}\right]$, this shows for a sufficiently large n the existence of a point $s' = \sigma'+it'$ for which $t' > \tau$, $1-\delta < \sigma' < 1+\delta$, and

$$-\log|\zeta(\sigma'+it')| > k \log\log\log t' ,$$

which is the desired result.

Theorem II does not imply that $\zeta(s)$ assumes arbitrarily small values *to the right* of the line $\sigma = 1$. That this is so has been shown in a note that has recently been sent to the Academy in Paris.

ON THE VALUES TAKEN BY THE RIEMANN FUNCTION $\zeta(\sigma+it)$ IN THE HALF-PLANE $\sigma > 1$.

Summary of Om de Værdier, den Riemann'ske Funktion $\zeta(\sigma+it)$ antager i Halvplanen $\sigma > 1$.
Beretning om den Anden Skandinaviske Matematikerkongres i Kjøbenhavn 1911, 113—121.

The author has recently proved that the zeta-function assumes numerically arbitrarily small values in the half-plane $\sigma > 1$. This lecture gives the main features of the proof of the following much more extensive theorem: *The zeta-function assumes in the half-plane* $\sigma > 1$ *all values except* 0, *and it even assumes every value different from* 0 *infinitely often.*

[The detailed exposition is given in B 6.]

A NEW PROOF THAT THE RIEMANN ZETA-FUNCTION $\zeta(s) = \zeta(\sigma+it)$ HAS INFINITELY MANY ZEROS IN THE PARALLEL STRIP $0 \leqq \sigma \leqq 1$.

Summary of ET NYT BEVIS FOR, AT DEN RIEMANN'SKE ZETAFUNKTION $\zeta(s) = \zeta(\sigma+it)$ HAR UENDELIG MANGE NULPUNKTER INDENFOR PARALLELSTRIMLEN $0 \leqq \sigma \leqq 1$.

Nyt Tidsskrift for Matematik B 23 (1912), 81—85.

The conjecture of Riemann that $\zeta(s)$ has infinitely many zeros in the strip $0 \leqq \sigma \leqq 1$, was proved by Hadamard by means of his theory of integral functions. In the present paper, the existence of the zeros is deduced from a recent result of the author* to the effect that $\zeta(s)$ assumes numerically arbitrarily small values in the half-plane $\sigma > 1$, and from some recent function-theoretic lemmas. The method yields only the existence of the zeros and not the further results obtained by means of the Hadamard theory.

The proof is indirect. If $\zeta(s)$ had only a finite number of zeros in the strip, the branch of $\log \zeta(s)$ which is determined in the half-plane $\sigma > 1$ by being real on the real axis would be regular in a quarter-plane $\sigma \geqq -1, t \geqq \tau > 3$. Since† for $\sigma \geqq -1$, $t \geqq 3$

$$|\zeta(s)| = |\zeta(\sigma+it)| < t^{k_1},$$

where k_1 is a constant, we would for $\sigma \geqq -1, t \geqq \tau$ have

$$\Re \log \zeta(\sigma+it) < k_1 \log t .$$

Since $\log \zeta(2+it)$ is bounded, this would imply by Carathéodory's inequality the existence of a constant k_2 such that for $2 \geqq \sigma \geqq -\frac{1}{2}, t \geqq \tau+3$

$$\Re \log \zeta(\sigma+it) > -k_2 \log t ,$$

or

---

* H. Bohr, Sur l'existence de valeurs arbitrairement petites de la fonction $\zeta(s) = \zeta(\sigma+it)$ de Riemann pour $\sigma > 1$, *Oversigt over Det Kgl. Danske Videnskabernes Selskabs Forhandlinger* 1911, 201–208.

† See e. g. E. Lindelöf, Quelques remarques sur la croissance de la fonction $\zeta(s)$, *Bulletin des Sciences Mathématiques* (2) 32, part 1 (1908), 341–356.

$$\left|\frac{1}{\zeta(s)}\right| < t^{k_2}.$$

But $\frac{1}{\zeta(s)}$ is bounded on the lines $\sigma = -\frac{1}{2}$* and $\sigma = 2$. By a theorem of Phragmén and Lindelöf†, $\frac{1}{\zeta(s)}$ would therefore be bounded in the whole domain $2 \geqq \sigma \geqq -\frac{1}{2}$, $t \geqq \tau+3$. This contradicts the result referred to above, according to which $\zeta(s)$ assumes numerically arbitrarily small values in the domain $1 < \sigma < 2, t > \tau+3$.

In a later note another proof of the existence of the zeros will be given, which, like the present one, makes no use of Hadamard's investigations.

---

* This follows immediately from the functional equation, see e. g. Lindelöf, l. c.

† See e. g. Lindelöf, l. c.

A THEOREM ON FOURIER SERIES OF ALMOST PERIODIC FUNCTIONS

Summary of En Sætning om Fourierrækker for næsten-periodiske Funktioner.
Matematisk Tidsskrift B 1925, 31—37.

Certain powerful tools in the theory of ordinary Fourier series, for example the Riemann method and the Fejér-Cesàro summation method, have not yet been generalized to almost periodic functions. Theorems on periodic functions which are usually proved by these methods may, however, be generalized if we can find other proofs based on tools that are available in the new theory. As an example a proof is given of the following theorem, which for periodic functions is an immediate consequence of Fejér's theorem.

Theorem. *If the coefficients a_n in the Fourier series $\sum_1^\infty a_n e^{i\lambda_n x}$ of an almost periodic function $f(x)$ are all positive, then $\sum a_n$ is convergent.*

As a generalization of a well-known remark on continuous functions in a finite interval, one easily proves that if $f(x)$ is almost periodic and G is the upper bound of $|f(x)|$, then

$$\lim_{q\to\infty} \sqrt[2q]{M\{|f(x)|^{2q}\}} = G\,.$$

This implies that if $P_N(x)$ is a finite sum $\sum_1^N a_n e^{i\lambda_n x}$ with positive coefficients a_n, then

$$\lim_{q\to\infty} \sqrt[2q]{M\{|P_N(x)|^{2q}\}} = \max |P_N(x)| = P_N(0) = \sum_1^N a_n\,.$$

Now, let $f(x)$ be an almost periodic function for which all the coefficients in its Fourier series $\sum_1^\infty a_n e^{i\lambda_n x}$ are positive, and let $P_N(x) = \sum_1^N a_n e^{i\lambda_n x}$ be the N^{th} partial sum of the Fourier series. By the multiplication theorem, the Fourier series $\sum c_n e^{i\nu_n x}$ of $(f(x))^q$ is obtained by formally raising the series $\sum a_n e^{i\lambda_n x}$ to the q^{th} power. The

finite sum $(P_N(x))^q$ must therefore be of the form $\sum_1^M b_m e^{i\nu_m x}$, where $c_m \geqq b_m$ for all $m = 1, 2, \ldots, M$. Hence, by the fundamental theorem,

$$M\{|f(x)|^{2q}\} = \sum_1^\infty c_m^2 > \sum_1^M c_m^2 \geqq \sum_1^M b_m^2 = M\{|P_N(x)|^{2q}\}.$$

Taking the $2q^{\text{th}}$ root on both sides we obtain for $q \to \infty$ the inequality

$$\sum_1^N a_n \leqq G,$$

which shows that the series $\sum a_n$ must be convergent.

A CLASS OF INTEGRAL FUNCTIONS

Summary of
En Klasse hele transcendente Funktioner.
Matematisk Tidsskrift B 1926, 41—45.

Proof of the theorem that if the exponents λ_n of the Fourier series $\sum a_n e^{i\lambda_n x}$ of an almost periodic function $f(x)$ are bounded, then $f(x)$ is an integral function, i.e., there exists an integral function $F(z) = F(x+iy)$ which coincides with $f(x)$ on the real axis.

[The proof is the same as that given in C 10.]

Added in proof. Professor Szegö has found another proof that will be published in *Mathematische Annalen*.

STABILITY AND ALMOST PERIODICITY

Summary of
STABILITET OG NÆSTENPERIODICITET.
Matematisk Tidsskrift B 1933, 21—25.

In recent years various authors, especially Birkhoff, Franklin, and Wintner, have shown that the notion of almost periodicity may be applied in mechanics. Recently, A. Markoff* has discussed almost periodic motions of dynamical systems and has thereby pointed out interesting connections between the almost periodicity of the motions and what he calls 'stability in Liapounoff's sense'.

The present note contains the proof of a theorem on the almost periodicity of a stable motion, which is not directly included in Markoff's results. The note contains nothing new in method; the intention is merely to show in a simple case the nature of Markoff's reasoning.

A point $(x_1, \ldots, x_n)$ in n-dimensional Euclidean space E_n is denoted by x and its distance from the origin by $|x|$, so that $|x-y|$ is the distance between the two points x and y. A motion in E_n is defined by n continuous functions $x_\nu = x_\nu(t)$ or, more briefly, by a continuous function $x = x(t)$, where $-\infty < t < \infty$.

A motion $x = x(t)$ is called *almost periodic* if to every $\varepsilon > 0$ there exists a relatively dense set of translation numbers $\tau = \tau(\varepsilon)$ of $x(t)$, i.e., numbers satisfying the condition

$$|x(t+\tau)-x(t)| \leqq \varepsilon \quad (\text{for } -\infty < t < \infty)\,.$$

A motion is almost periodic if and only if all the coordinates $x_\nu(t)$ are almost periodic. Every almost periodic motion is bounded, $|x(t)| < K$.

A motion $x = x(t)$ is called *strongly stable* if to every $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that the inequality

$$|x(t')-x(t'')| \leqq \delta$$

* A. Markoff, Stabilität im Liapounoffschen Sinne und Fastperiodizität, *Mathematische Zeitschrift* 36 (1933), 708–738.

implies the inequality

$$|x(t'+t)-x(t''+t)| \leqq \varepsilon \quad (\text{for all } -\infty < t < \infty)\,.$$

In other words: If t' and t'' are two values such that the points $x(t')$ and $x(t'')$ have a distance $\leqq \delta$, then the difference $t'-t''$ is a translation number $\tau(\varepsilon)$ of the motion.

The theorem to be proved states:

Every bounded and strongly stable motion $x = x(t)$ is almost periodic.

Let D be a cube in E_n containing the motion. Let $\varepsilon > 0$ be given and let $\delta = \delta(\varepsilon)$ be determined according to the previous definition. We divide D into a finite number, say m, cubes of diameter $< \delta$. If now $t_1, \ldots, t_{m+1}$ are $m+1$ arbitrarily chosen numbers, then two of the points $x(t_1), \ldots, x(t_{m+1})$ will lie in the same small cube. Consequently, among the differences $t_i - t_j$ $(1 \leqq i < j \leqq m+1)$ at least one will be a translation number $\tau(\varepsilon)$ of the motion $x(t)$.

Following Markoff, we shall say that a set T of points on the real t-axis has the *property Δ_m* if for $m+1$ arbitrary numbers $t_1, \ldots, t_{m+1}$ at least one of the differences $t_i - t_j$ $(1 \leqq i < j \leqq m+1)$ belongs to T. The theorem therefore follows from the following lemma (due to Markoff):

Every set T which (for some m) has the property Δ_m is relatively dense.

It is easy to prove this lemma by induction.

THE LATEST DEVELOPMENT OF THE THEORY OF ALMOST PERIODIC FUNCTIONS

Summary of DEN SENESTE UDVIKLING AF LÆREN OM NÆSTENPERIODISKE FUNKTIONER.
Åttonde Skandinaviska Matematikerkongressen i Stockholm 1934, 106.

The lecture gave an account of the then not yet published profound investigations by J. v. Neumann on almost periodic functions defined in an arbitrary group. These investigations have now appeared, and the reader is referred to v. Neumann's paper.*

* J. v. Neumann, Almost periodic functions in a group. I, *Transactions of the American Mathematical Society* 36 (1934), 445–492.

A THEOREM ON FOURIER SERIES

Summary of EN SÆTNING OM FOURIERRÆKKER.
Matematisk Tidsskrift B 1935, 77—81.

In a recent paper* the author has proved the following theorem: Let

$$p(x) = \sum_{m=1}^{M} (A_m \cos \Lambda_m x + B_m \sin \Lambda_m x)$$

be a finite sum, whose frequencies Λ_m are $\geqq 1$ and for which

$$|p(x)| \leqq 1 \quad \text{for} \quad -\infty < x < \infty .$$

Let

$$P(x) = \sum_{m=1}^{M} \frac{1}{\Lambda_m} (A_m \sin \Lambda_m x - B_m \cos \Lambda_m x)$$

denote the indefinite integral of $p(x)$ which has no constant term. Then we have for all values of x the inequality

$$|P(x)| \leqq \frac{\pi}{2},$$

and the constant $\frac{\pi}{2}$ cannot be replaced by any smaller constant, not even if we restrict attention to sums $p(x)$ whose frequencies Λ_m are integers.

In the present note this theorem is used to establish the following theorem on ordinary Fourier series: Let

$$f(x) \sim \sum_{n=N}^{\infty} (a_n \cos nx + b_n \sin nx)$$

be a continuous function of period 2π whose Fourier series contains only terms with frequency $n \geqq$ a given positive integer N and for which

$$|f(x)| \leqq 1 \quad \text{for all } x .$$

Let

* H. Bohr, Ein allgemeiner Satz über die Integration eines trigonometrischen Polynoms, *Prace Matematyczno-Fizyczne* 43 (1935), 273–288.

$$F(x) = \sum_{n=N}^{\infty} \frac{1}{n}(a_n \sin nx - b_n \cos nx)$$

denote the indefinite integral of $f(x)$ whose Fourier series has no constant term. Then

$$|F(x)| \leqq \frac{\pi}{2N},$$

and the constant $\frac{\pi}{2N}$ cannot be replaced by any smaller constant.

The inequality follows at once by application of the preceding result to the sums $s_Q\left(\frac{x}{N}\right)$, where $s_Q(x)$ is the Q^{th} Fejér sum of $f(x)$, and letting $Q \to \infty$. That the constant cannot be replaced by a smaller one is seen by choosing for a given $\delta > 0$ a sum $p(x)$ with integral frequencies, for which $|p(x)| \leqq 1$ for all x and for which the integral $P(x)$ for some x satisfies the inequality $|P(x)| > \frac{\pi}{2} - \delta$, and then putting $f(x) = p(Nx)$.

S 11. Summary of C 37.

AN EXAMPLE OF THE APPLICATION OF THE CALCULUS OF PROBABILITY AS AN AID IN MATHEMATICAL ANALYSIS

Summary of
Et Eksempel paa Anvendelsen af Sandsynlighedsregning som Hjælpemiddel i den matematiske Analyse.
Festskrift til Professor, Dr. phil. J. F. Steffensen fra Kolleger og Elever paa hans 70 *Aars Fødselsdag,*
Den Danske Aktuarforening, København 1943, 29—33.

To elucidate the importance of the methods of the calculus of probability for the treatment of mathematical problems which are themselves far from the applications, a sketch is given of the author's original proof of the following theorem:

If the Fourier series $\sum A_n e^{i\Lambda_n t}$ of an almost periodic function $f(t)$ has linearly independent exponents, then the series $\sum |A_n|$ is convergent.

[The original proof will be found in C 3, ch. III.]

ON S-ALMOST PERIODIC FUNCTIONS WITH LINEARLY INDEPENDENT EXPONENTS

Summary of OM S-NÆSTENPERIODISKE FUNKTIONER MED LINEÆRT UAFHÆNGIGE EXPONENTER.
Norsk Matematisk Tidsskrift 26 (1944), 33—40.

The paper contains two theorems on Stepanoff almost periodic functions, dealing with S-a. p. and S_2-a. p. functions respectively. The corresponding norms are denoted by $||\varphi||_S$ and $||\varphi||_{S_2}$.

Theorem 1. *If the series $\sum A_n e^{i\Lambda_n x}$ with linearly independent and bounded exponents Λ_n is the Fourier series of an S-a. p. function $f(x)$, then $\sum |A_n|$ is convergent.*

From the proof of the approximation theorem for S-a. p. functions (by means of Bochner-Fejér sums) it follows, since the exponents are linearly independent, that

$$||f-s_n||_S \to 0 \text{ as } n \to \infty, \text{ where } s_n(x) = \sum_{\nu=1}^{n} A_\nu e^{i\Lambda_\nu x}.$$

Hence the norms $||s_n||_S$ are bounded, say $\leqq C$, i. e.,

$$\int_0^1 |s_n(x+t)|\,dt \leqq C \quad \text{for all } n \text{ and all } x. \tag{1}$$

On writing $A_\nu = \varrho_\nu e^{i\theta_\nu}$ we determine by Kronecker's theorem a number $x_0 = x_0(n)$ such that

$$|\theta_\nu + \Lambda_\nu x_0| < \frac{\pi}{6} \pmod{2\pi} \quad (\nu = 1, 2, \ldots, n).$$

Since the Λ_ν are bounded, say $|\Lambda_\nu| \leqq c$ for all ν, we find for this x_0

$$\Re\{s_n(x_0+t)\} = \sum_{\nu=1}^{n} \Re\{A_\nu e^{i\Lambda_\nu(x_0+t)}\} > \tfrac{1}{2}\sum_{\nu=1}^{n} \varrho_\nu \quad \text{for} \quad |t| < \frac{\pi}{6c}. \tag{2}$$

Putting $d = \min\left(\frac{\pi}{6c}, 1\right)$, we find from (1) and (2)

$$d \cdot \tfrac{1}{2} \sum_{\nu=1}^{n} \varrho_\nu < \int_0^d \Re\{s_n(x_0+t)\}\,dt \leqq C,$$

which shows that the series $\sum_1^\infty \varrho_\nu$ converges.

Next it is proved that it is not true for every series $\sum A_n e^{i\Lambda_n x}$ with linearly independent exponents which is the Fourier series of an S-a. p. function, that $\sum |A_n|$ is convergent. This is contained in the following theorem.

Theorem 2. *For an arbitrary sequence of complex numbers $A_1, A_2, \ldots$ for which $\sum |A_n|^2$ converges, a sequence of linearly independent real numbers $\Lambda_1, \Lambda_2, \ldots$ may be determined such that the infinite series $\sum A_n e^{i\Lambda_n x}$ is the Fourier series of an S_2-a. p. function $f(x)$.*

Let $\mu_1, \mu_2, \ldots$ be an arbitrary sequence of linearly independent numbers. As exponents the numbers $\Lambda_n = p_n \mu_n$ will be used, where the factors $p_1, p_2, \ldots$ are suitably chosen positive integers. For abbreviation the polynomials

$$\sum_{\nu=1}^{n} A_\nu e^{i\mu_\nu x} \quad \text{and} \quad \sum_{\nu=1}^{n} A_\nu e^{i\Lambda_\nu x} = \sum_{\nu=1}^{n} A_\nu e^{ip_\nu \mu_\nu x}$$

will be denoted by $\sigma_n(x)$ and $s_n(x)$. We determine positive integers $m_1 < m_2 < \ldots$ such that if we put

$$b_1 = \sqrt{|A_1|^2 + \cdots + |A_{m_1}|^2}, \ldots, \; b_q = \sqrt{|A_{m_{q-1}+1}|^2 + \cdots + |A_{m_q}|^2}, \ldots$$

the series $\sum_1^\infty b_q$ converges. For every fixed $q > 1$ the mean value of $|\sigma_{m_q}(x) - \sigma_{m_{q-1}}(x)|^2$ is b_q^2, and we can therefore determine a positive integer N_q such that

$$\frac{1}{N_q} \int_x^{x+N_q} |\sigma_{m_q}(t) - \sigma_{m_{q-1}}(t)|^2 dt < 4b_q^2 \quad \text{for all } x.$$

On placing $p_\nu = N_q$ for $\nu = m_{q-1}+1, \ldots, m_q$ we therefore have

$$\int_x^{x+1} |s_{m_q}(t) - s_{m_{q-1}}(t)|^2 dt < 4b_q^2 \quad \text{for all } x.$$

Thus $\|s_{m_q} - s_{m_{q-1}}\|_{S_2} \leqq 2b_q$, which implies (by the triangle inequality) that the sequence $s_{m_1}(x), s_{m_2}(x), \ldots$ is an S_2-fundamental sequence, and hence is S_2-convergent to a function $f(x)$. This function $f(x)$ is S_2-a. p. and has the Fourier series $\sum_{n=1}^{\infty} A_n e^{i\Lambda_n x}$.

TWO NEW SIMPLE PROOFS OF KRONECKER'S THEOREM

Summary of To nye simple Beviser for Kroneckers Sætning.
By Harald Bohr and Børge Jessen.
Matematisk Tidsskrift B 1932, 53—58.

The paper contains two simple analytical proofs of the 'small' Kronecker theorem on Diophantine inequalities with one variable and linearly independent coefficients.

[The first proof is reproduced in D 5 and in D 6, § 4, the second in D 6, § 5.]

ONCE MORE KRONECKER'S THEOREM

Summary of ENDNU ENGANG KRONECKERS SÆTNING.
Matematisk Tidsskrift B 1933, 59—64.

A new analytical proof of Kronecker's theorem is given, which can hardly be simpler. First, the 'small' Kronecker theorem on Diophantine inequalities with one variable and linearly independent coefficients is proved. Next, the 'big' Kronecker theorem concerning arbitrary linear Diophantine inequalities with several variables is proved by the same method.

[The proof of the 'small' Kronecker theorem is reproduced in D 9. The generalization to the 'big' Kronecker theorem corresponds to similar generalizations of earlier proofs in D 2 and D 6, § 6.]

A FUNCTION-THEORETIC REMARK

Summary of En funktionsteoretisk Bemærkning.
Nyt Tidsskrift for Matematik B 27 (1916), 73—78.

As is well known, if $\varphi(x)$ is any positive, continuous and increasing function defined for $x > 0$, there exists an integral function $f(z) = f(x+iy)$ which for $x > 0$ satisfies the condition

$$|f(x)| > \varphi(x)\,.$$

In connection with this theorem, Professor Nørlund mentioned in a conversation the problem, whether there exists an integral function $f(z)$ which increases arbitrarily rapidly in a whole strip, for example the strip $x > 0$, $-1 < y < 1$. It is shown that this is not the case. Actually a more general result is obtained.

Let Ω denote a domain in the complex plane determined by the inequalities

$$x > 0, \quad -\omega(x) < y < \omega(x)\,,$$

where $\omega(x)$ is a positive, continuous function defined for $x \geqq 0$ which decreases to 0 as $x \to \infty$. Then there exists a positive, continuous function $\psi(x)$ defined for $x > 0$ (depending only on Ω) with the following property: Every analytic function $f(z) = f(x+iy)$ in Ω, whose absolute value $|f(z)|$ in the whole of Ω exceeds a positive constant k, satisfies for sufficiently large values of x the condition

$$|f(x)| < \psi(x)\,.$$

Let $w = \alpha(z)$ be the conformal mapping of Ω on the unit circle $|w| < 1$ in the $w = u+iv$-plane for which $\alpha(1) = 0$ and $\alpha'(1)$ is real and positive. Then $0 < x < \infty$, $y = 0$ is mapped on $-1 < u < 1$, $v = 0$. The condition of the theorem is now satisfied by the function

$$\psi(x) = e^{\frac{1}{(\alpha(x)-1)^2}}.$$

Indeed, let $G(w)$ be the function in $|w| < 1$ determined by $G(\alpha(z)) = F(z)$, where $F(z)$ is a branch of $\log f(z)$. From $|f(z)| > k > 0$ it follows that $-\Re(G(w))$ is smaller than a positive constant K. Putting $|\Re(G(0))| = K_1$ we find from Carathéodory's inequality that

$$\Re\big(G(w)\big) \leqq K_1 \frac{1+|w|}{1-|w|} + 2K \frac{|w|}{1-|w|}.$$

Hence, by a crude estimate,

$$\Re\big(G(u)\big) < \frac{1}{(1-u)^2}$$

for all u sufficiently near to 1. Putting $u = \alpha(x)$ we obtain the desired result.

ON THE HADAMARD 'GAP THEOREM'

Summary of Om den Hadamard'ske »Hulsætning«.
Matematisk Tidsskrift B 1919, 15—21.

The paper begins with Landau's proof of the Vivanti–Dienes theorem on power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

with a finite radius of convergence ϱ, according to which a point z_0 of the circle of convergence $|z| = \varrho$, at which all terms $a_n z_0^n$ of the series from a certain stage onward lie in a sector $Re^{i\Theta}$, $R \geqq 0$, $V < \Theta < W$ with $W - V < \pi$, is a singular point of $f(z)$.

Hadamard's gap theorem states that for every power series

$$F(z) = \sum_{p=1}^{\infty} a_{m_p} z^{m_p} \quad (0 \leqq m_1 < m_2 < \cdots)$$

for which

$$\frac{m_{p+1}}{m_p} > k \quad (p = 1, 2, 3, \ldots),$$

where k is a constant > 1, the circle of convergence $|z| = \varrho$ is a natural boundary of $F(z)$.

By means of the Vivanti–Dienes theorem a simple proof is given of this theorem in the case $k > 3$.

We choose a sector $Re^{i\Theta}$, $R \geqq 0$, $V < \Theta < W$, where $W - V < \pi$ is so near to π that

$$\frac{2\pi + W - V}{W - V} < k.$$

For every p, the points z on $|z| = \varrho$ for which $a_{m_p} z^{m_p}$ lies in the sector will form m_p equally spaced arcs B_p. A simple calculation shows that every arc B_p contains in its interior (at least) one of the arcs B_{p+1}. An arbitrarily small arc B on $|z| = \varrho$ contains for a sufficiently large P one of the arcs B_P; this contains in its interior an arc B_{P+1}, etc. These arcs B_P, B_{P+1}, ... have a point z_0 in common, which by the Vivanti–Dienes theorem will be a singular point of $F(z)$. Thus the singular points of $F(z)$ are everywhere dense on $|z| = \varrho$ and the theorem is proved.

S 17. Summary of E 6.

AN EXAMPLE OF A METHOD OF PROOF

Summary of ET EKSEMPEL PAA EN BEVISMETODE.
Matematisk Tidsskrift B 1920, 46.

By an investigation in the analytic theory of numbers a type of problem was encountered of which the following is an especially simple example.

In the unit circle C in the complex z-plane an arbitrary number of points $z_1, \ldots, z_n$ are given. Denoting by z a variable point in C we consider the mean value of the reciprocal distances of z from $z_1, \ldots, z_n$, i. e., the quantity

$$F(z) = \frac{1}{n}\left(\frac{1}{|z-z_1|} + \cdots + \frac{1}{|z-z_n|}\right),$$

and wish to prove that there exists a constant K (incidentally $K = 4$) such that in all cases there is a point z_0 in C at which this mean value $F(z_0)$ is $< K$.

This little theorem is perhaps most easily proved by the following argument, which is applicable to numerous problems of similar type.

We integrate $F(z)$ over C and find that the integral is smaller than the integral of $1/|z|$ over the circle $|z| < 2$, which is 4π. Hence there must be a point z_0 at which $F(z_0) < 4\pi/\pi = 4$.

ON POWER SERIES WITH GAPS

A PROPERTY OF PSEUDO-CONTINUITY

Summary of
OM POTENSRÆKKER MED HULLER. EN PSEUDO-KONTINUITETSEGENSKAB.
Matematisk Tidsskrift B 1942, 1—11.

After some introductory remarks on power series with gaps and a reference to the author's proof of a simple case of Hadamard's gap theorem* a problem is formulated which was encountered in an investigation of analytic almost periodic functions.

Let $\sum a_n z^n$ be a power series with radius of convergence 1 representing an unbounded function $f(z)$ in $|z| < 1$. A number τ in the interval $0 < \theta < 2\pi$ is called a *rotation number* of $f(z)$ belonging to ε if $|f(ze^{i\tau}) - f(z)| \leqq \varepsilon$ for $|z| < 1$. The function $f(z)$ is called *pseudo-continuous* (or, more precisely, pseudo uniformly continuous by rotation) if it possesses arbitrarily small rotation numbers τ for every $\varepsilon > 0$. The problem is whether there exist pseudo-continuous functions.

One easily sees, that a pseudo-continuous function must have the circle of convergence as a natural boundary. The simplest examples of such power series, namely $\sum z^{q^n}$ (q an integer > 1) and $\sum z^{n!}$, are discussed, and it is shown that the first is not pseudo-continuous whereas the second is pseudo-continuous. Thus the question is answered in the affirmative.

It may be shown by the argument used in the author's proof of Hadamard's theorem that any power series with sufficiently large gaps is pseudo-continuous. For simplicity, only sub-series of the series $\sum z^n$ will be considered.

Theorem. *If the sequence of positive integers $m_1 < m_2 < \dots$ increases so rapidly that $\sum \frac{m_n}{m_{n+1}}$ is convergent, then the unbounded function $f(z)$ represented in $|z| < 1$ by the series $\sum z^{m_n}$ is pseudo-continuous.*

---

* H. Bohr, Om den Hadamard'ske »Hulsætning«, *Matematisk Tidsskrift B* 1919, 15–21.

Without loss of generality we may assume that $m_1 > 3$ and that $\frac{m_n}{m_{n+1}} < \frac{1}{8}$ for all n. We put

$$\delta_n = 2\pi \frac{m_n}{m_{n+1}} \quad \text{and} \quad \eta_n = \frac{4\pi}{m_{n+1}} \quad (n = 1, 2, \ldots).$$

The points $z = e^{i\theta}$ determined by $|m_n\theta| \leqq \delta_n$ (mod. 2π) form m_n equally spaced arcs I_n on the unit circle. One easily sees that every arc I_n contains an arc I_{n+1}, and also that the arc $0 < \theta < \eta_n$ contains an arc I_{n+1}.

We have to prove that for every $d > 0$ and every $\varepsilon > 0$ there exists a rotation number τ of $f(z)$ belonging to ε in the interval $0 < \theta < d$. We determine n so large that $\eta_n < d$ and $4\delta_n + \sum_{n+1}^{\infty} \delta_\nu < \varepsilon$. On the arc $0 < \theta < \eta_n$ we determine an arc I_{n+1}, in it an arc I_{n+2}, etc. These arcs $I_{n+1}, I_{n+2}, \ldots$ determine a point τ in $0 < \theta < d$, and a simple estimate shows that this τ is a rotation number of $f(z)$ belonging to ε.

ON ADDITION OF CONVEX CURVES WITH GIVEN PROBABILITY DISTRIBUTIONS

(A SET-THEORETICAL INVESTIGATION)

Summary of
OM ADDITION AF KONVEKSE KURVER MED GIVNE SANDSYNLIGHEDSFORDELINGER.
(EN MÆNGDETEORETISK UNDERSØGELSE.)
Matematisk Tidsskrift B 1923, 10—15.

The problem of adding convex curves 'spread' with probability is encountered in the study of the functions occurring in the theory of prime numbers.* In this note the addition of two curves is discussed by means of the tools of modern set-theory. Throughout, the notions of 'measurable' sets and 'integrable' functions are to be taken in the Lebesgue sense.

Let K_1 and K_2 be convex curves in the complex z-plane, without vertices or straight segments. We suppose that the curve K_i $(i = 1, 2)$ is given by a parametric representation $z_i = f_i(t_i)$, where $0 \leqq t_i \leqq 1$. By the probability that a point z_i of K_i lies on a given arc B of K_i is meant the length of the corresponding parameter interval. It is assumed that this 'arc probability' is determined by a 'point probability', i.e., that there exists a function $s_i(z_i)$ on K_i whose integral over every arc B is equal to the probability of the arc. For simplicity it is assumed that $s_i(z_i)$ is bounded.

The sum Ω of K_1 and K_2 is defined as the set of all points $z = z_1+z_2$, where z_1 belongs to K_1 and z_2 belongs to K_2. It is a closed set bounded either by one or by two convex curves.† By the probability that a point z of Ω lies in a measurable set M in the z-plane is meant the measure μ of the set M of points (t_1, t_2) in the unit square $E: 0 \leqq t_1 \leqq 1,\ 0 \leqq t_2 \leqq 1$ for which $z_1(t_1)+z_2(t_2)$ belongs to M. The proof of the following theorem is sketched:

---

* A survey of the investigations that have led to problems of this kind has been given by the author in a lecture: 'Über diophantische Approximationen und ihre Anwendungen auf Dirichlet'sche Reihen, besonders auf die Riemann'sche Zetafunktion', given at the Scandinavian Congress of Mathematicians at Helsingfors 1922.

† H. Bohr, Om Addition af uendelig mange konvekse Kurver, *Oversigt over Det Kgl. Danske Videnskabernes Selskabs Forhandlinger* 1913, 325–366.

For every measurable set M in Ω the probability μ exists, i.e., the corresponding set M *in E is also measurable. Moreover, this 'set probability' $\mu = \mu(M)$ is determined by a 'point probability' $S(z)$, i.e., there exists a—generally unbounded—point function $S(z)$ such that the measure of* M *is equal to the integral of $S(z)$ extended over the set M.*

The problem will be treated in detail in a later paper in which also the addition of an arbitrary number of convex curves will be discussed.

ON PROBABILITY DISTRIBUTIONS BY ADDITION OF CONVEX CURVES

Summary of
Om Sandsynlighedsfordelinger ved Addition af konvekse Kurver.
By Harald Bohr and Børge Jessen.
Det Kgl. Danske Videnskabernes Selskabs Skrifter, Naturvidensk. og Mathem. Afd. (8) 12, no. 3 (1929), 325—406 [1—82].

This paper is a preliminary to a detailed study of the distribution of the values of the Riemann zeta-function.

Chapter I. **On convex curves.***

The chapter begins with some remarks on closed convex curves ω in the plane.

In a plane with a given origin O the sum $P = \sum_{n=0}^{N} P_n$ of a finite number of points P_n is defined by the vector equation $OP = \sum_{n=0}^{N} OP_n$. The sum $\sum_{n=0}^{N} M_n$ of a finite number of sets M_n is defined as the set of all points $\sum_{n=0}^{N} P_n$, where P_n belongs to M_n. The symmetrical point of a point P with respect to O is denoted by $-P$ and the symmetrical set of a set M with respect to O is denoted by $-M$.

An infinite series $\sum_{n=0}^{\infty} P_n$, where P_n are points, is called convergent with sum P, if the partial sum $\sum_{n=0}^{N} P_n$ converges to P as $N \to \infty$. An infinite series $\sum_{n=0}^{\infty} M_n$, where M_n are sets, is called convergent, if every series $\sum_{n=0}^{\infty} P_n$, where P_n belongs to M_n, is convergent; the set of points $\sum_{n=0}^{\infty} P_n$ is then the sum of the series $\sum_{n=0}^{\infty} M_n$.

Let $\omega_0, \omega_1, \ldots, \omega_N, \ldots$ be a sequence of convex curves. We consider the sets $\Sigma_N = \sum_{n=0}^{N} \omega_n$; evidently they are bounded. It is proved by induction that every set

* This chapter repeats in a condensed form the results in H. Bohr, Om Addition af uendelig mange konvekse Kurver, *Oversigt over Det Kgl. Danske Videnskabernes Selskabs Forhandlinger* 1913, 325–366.

Σ_N is either a closed set bounded by one convex curve or a closed ring-shaped set bounded by two convex curves. Similarly, the set $\Sigma = \sum_{n=0}^{\infty} \omega_n$, if it exists, is of one of these types. In certain cases a set of the first type is to be considered as a set of the second type whose inner boundary has degenerated into a point or a straight segment. The points of this degenerate inner boundary are then not to be considered as interior points of the set.

Chapter II. Introductory definitions and theorems on probability.

If a convex curve ω is defined by a parametric representation, where the parameter θ describes the interval $0 \leqq \theta < 1$, this representation defines an *arc-probability* $w(b)$ on ω, whose value for an arc b is the length of the corresponding part of the parameter interval.

Let $\omega_0, \omega_1, \ldots, \omega_N, \ldots$ be a sequence of convex curves on which such probabilities $w_n(b)$ are determined through parametric representations with parameter intervals $0 \leqq \theta_n < 1$. These representations determine a mapping of every set $\Sigma_N = \sum_{n=0}^{N} \omega_n$ on the unit cube $Q_N(0 \leqq \theta_n < 1)$ in the $(\theta_0, \theta_1, \ldots, \theta_N)$-space, by which a point $P = \sum_{n=0}^{N} P_n$ of Σ_N corresponds to the point $(\theta_0, \theta_1, \ldots, \theta_N)$, if P_n is the point of ω_n corresponding to θ_n. Every point of Q_N corresponds to one point of Σ_N, whereas a point of Σ_N generally corresponds to more than one point of Q_N. This mapping defines a *set-probability* $W_N(M)$ in the plane, by which a set M is allotted the probability $W_N(M)$ if the set of points in Q_N for which the corresponding point in Σ_N belongs to M is Jordan measurable with the measure $W_N(M)$.

A rectangular coordinate system (XY) is introduced in the plane and for simplicity only rectangles $R(x_0 \leqq x < x_1, y_0 \leqq y < y_1)$ are considered. Under the assumption that the curves ω_n do not contain straight segments it is proved by induction* that for every N the probability $W_N(R)$ exists for all R and is a continuous function of R. Moreover

(a) $$W_{N+1}(R) = \int_0^1 W_N(R - P_{N+1}) d\theta_{N+1},$$

where P_{N+1} denotes the point of ω_{N+1} corresponding to the parameter value θ_{N+1}.

---

* The proof is taken from H. Bohr and R. Courant, Neue Anwendungen der Theorie der Diophantischen Approximationen auf die Riemannsche Zetafunktion, *Journal für die Reine und Angewandte Mathematik* 144 (1914), 249–274.

Chapter III. **Demarcation of the domain of curves.**

Let ω be a convex curve without corners or straight segments. The notion of a circle touching ω at a point P and lying on the same side of the tangent t as ω will be taken in the wide sense, including the point P and the tangent t. The *inner radius* r_i ($\geqq 0$) of ω is defined as the largest number with the property that for every point P of ω the touching circle with radius r_i is enclosed by ω. Similarly, the *outer radius* r_y ($\leqq \infty$) is defined as the smallest number with the property that for every point P of ω the touching circle with radius r_y encloses ω.

These radii may be determined as follows. A circle is called an *osculating circle* of ω at a point P if it is the limit of a sequence of circles.touching ω at P and passing through a sequence of points of ω converging to P. It need not be unique. The radii r_i and r_y are equal to the lower and upper bounds of the radii of all osculating circles.† Another determination is as follows. A circle touching ω at a point P is called a *circle of curvature* if its radius is the limit of the ratio of the length of the chord PP^* to the total curvature of the arc PP^* for a sequence of points P^* of ω converging to P. Every osculating circle is also a circle of curvature‡ but not conversely. Nevertheless, the radii r_i and r_y may also be determined as the lower and upper bounds of the radii of all circles of curvature.

The following investigations are restricted to the class K of curves for which r_i and r_y are *positive and finite.*

Let ω and ω^* be two curves of this class such that the inner radius of ω exceeds the outer radius of ω^*. It is then shown that for every point P in the plane the curves ω and $P+\omega^*$ have at most two points in common. The points P for which they have one point in common form two convex curves ω_i and ω_y of the class K, whose radii are estimated by means of the radii of ω and ω^*. The curve ω_y encloses ω_i. The ring between ω_i and ω_y consists of the points P for which ω and $P+\omega^*$ have two points in common. The sines of the angles of intersection of ω and $P+\omega^*$ are estimated by means of the radii of ω and ω^* and the shortest distance of P from ω_i or ω_y.

By choosing ω^* as a circle one obtains results on the parallel curves of the curve ω.

These results are now applied to a special case of the addition of convex curves. We consider three curves ω_0, ω_1, ω_2 of the class K, whose radii satisfy the inequalities

$$r_{0,i} \geqq 2r_{1,y}; \quad r_{1,i} \geqq 2r_{2,y}.$$

---

† Cf. W. Blaschke, *Kreis und Kugel*, Leipzig 1916, 114–117.

‡ J. Hjelmslev, Über die Grundlagen der kinematischen Geometrie, *Acta Mathematica* 47 (1925), 155.

Since a point P belongs to $\Sigma_1 = \omega_0 + \omega_1$ if and only if ω_0 and $P - \omega_1$ have a point in common, one finds that Σ_1 is a ring-shaped set bounded by two convex curves ω_i and ω_y of the class K. Every point of ω_i or ω_y has exactly one representation $P_0 + P_1$ as a sum of a point of ω_0 and a point of ω_1, whereas every interior point of Σ_1 has exactly two such representations. The sets $\omega_i + \omega_2$ and $\omega_y + \omega_2$ will be ring-shaped sets bounded by convex curves ω_{ii}, ω_{iy} and ω_{yi}, ω_{yy} of the class K. One finds that the curve ω_{yi} encloses ω_{iy} and that the set $\Sigma_2 = \omega_0 + \omega_1 + \omega_2$ is the ring-shaped set bounded by ω_{ii} and ω_{yy}.

Chapter IV. **Point-probability.**

We now return to the considerations of chapter II.

The arc-probability $w(b)$ on a convex curve ω is said to be determined by a continuous *point-probability* $f(P)$ if it is of the form $w(b) = \int_b f(P)db$, where $f(P)$ is a continuous function on ω.

We now suppose that the curves $\omega_0, \omega_1, \ldots, \omega_N, \ldots$ belong to the class K and that their radii converge to 0 as $N \to \infty$. Moreover we suppose that the arc-probabilities $w_n(b)$ on the curves are determined by continuous point-probabilities $f_n(P_n)$. It is then proved that *for all $N \geqq N_0$, where N_0 is a number depending only on the given curves, the set-probability $W_N(M)$ is determined by a continuous point-probability $F_N(P)$, i. e., there exists a continuous function $F_N(P)$ in the plane such that*

$$W_N(M) = \iint_M F_N(P)dM\,. \tag{b}$$

The definition of the integral of a continuous non-negative function to be applied here is an adaptation of the definition of Jordan measure. For an arbitrary set M the interior integral is defined as the upper bound of the integral over all sets contained in M composed of a finite number of rectangles. The exterior integral is defined for a bounded set M as the lower bound of the integral over all sets containing M composed of a finite number of rectangles, and for an unbounded set M as the upper bound of the exterior integral over all bounded subsets of M. The integral over M is defined as the common value of the interior and exterior integrals when they are equal.

The formula (b) expresses that the probability and the integral are defined for exactly the same sets M and are equal. It is easily seen that this amounts to the validity of the formula when M is a rectangle R.

It follows from (a) that if $W_N(M)$ is determined by a continuous point-probability $F_N(P)$, then $W_{N+1}(M)$ will be determined by the continuous point-probability

$$F_{N+1}(P) = \int_0^1 F_N(P-P_{N+1}) d\theta_{N+1}\,.$$

From the existence of a continuous point-probability $F_N(P)$ for one value $N = N_0$ therefore follows its existence for all $N > N_0$.

Without loss of generality we may suppose that the radii of the first four curves $\omega_0, \omega_1, \omega_2, \omega_3$ satisfy the inequalities

$$\text{(c)} \qquad r_{0,i} \geqq 2r_{1,y}; \quad r_{1,i} \geqq 2r_{2,y}; \quad r_{2,i} \geqq 2r_{3,y}\,.$$

Otherwise we choose N_0 so large that among the curves $\omega_0, \ldots, \omega_{N_0}$ there are four that satisfy these inequalities and then rearrange the curves $\omega_0, \ldots, \omega_{N_0}$; this does not affect $W_N(M)$ for $N \geqq N_0$. It is shown that under the assumption (c) there is a continuous point-probability already for $N = 3$. Hence, in the case where a rearrangement is needed, there is a continuous point-probability for $N \geqq N_0$.

Under the first two of the assumptions (c) the sets Σ_1 and Σ_2 were discussed at the end of chapter III. From the discussion of Σ_1 it easily follows that (b) is satisfied for $N = 1$ for all rectangles by a discontinuous function $F_1(P)$. This function vanishes outside Σ_1, and for an interior point P of Σ_1 it is determined from the two representations $P = P_0'+P_1'$ and $P = P_0''+P_1''$ of P as a sum of a point of ω_0 and a point of ω_1 by the expression

$$F_1(P) = \frac{f_0(P_0') \cdot f_1(P_1')}{\sin p'} + \frac{f_0(P_0'') \cdot f_1(P_1'')}{\sin p''},$$

where p' and p'' are the angles of intersection of ω_0 and $P-\omega_1$ at P_0' and P_0''. The function $F_1(P)$ is discontinuous on the boundary curves ω_i and ω_y, but it is shown that it satisfies an inequality

$$F_1(P) \leqq \frac{K_1}{\sqrt{d}},$$

where d denotes the shortest distance from P to ω_i or ω_y, and K_1 is a constant.

From the discussion of Σ_2 it now follows that (b) is satisfied for $N = 2$ for all rectangles by the function

$$F_2(P) = \int_0^1 F_1(P-P_2)\, d\theta_2\,.$$

This function is discontinuous on the curves $\omega_{ii}, \omega_{iy}, \omega_{yi}$, and ω_{yy}, but it is less strongly discontinuous than $F_1(P)$. It is shown that for a sufficiently small a it is bounded for points P having a distance $< a$ from ω_{ii} or ω_{yy}, whereas for points P having a distance $d < a$ from ω_{iy} or ω_{yi} it satisfies an inequality

$$F_2(P) \leqq K_2+L_2 \log \frac{a}{d},$$

where K_2 and L_2 are constants.

Finally, by use also of the last of the conditions (c) it is shown that (b) is satisfied for $N = 3$ for all rectangles by the function

$$F_3(P) = \int_0^1 F_2(P-P_3)\, d\theta_3$$

and this function is continuous everywhere. This establishes the theorem.

It follows from the proof that the point-probability $F_N(P)$ for $N \geqq N_0$ is always strictly positive for all interior points of Σ_N.

Chapter V. **Probability distributions by addition of infinitely many convex curves.**

To the assumptions of chapter IV we now add the assumption that the series $\sum_{n=0}^{\infty} \omega_n$ is *convergent* with sum Σ.

It is proved that in this case the sequence of point-probabilities $F_N(P)$, $N \geqq N_0$, converges uniformly to a function $F(P)$. This continuous function vanishes outside the set Σ, is positive for interior points of Σ, and its integral over Σ is 1. It is called the *point-probability* corresponding to Σ.

We can also define a *set-probability* $W(M)$ corresponding to Σ. Since the set-functions $W_N(M)$ are not defined for the same sets, we consider for an arbitrary set M the interior and exterior probabilities $W_{N,i}(M)$ and $W_{N,y}(M)$ defined by the interior and exterior Jordan measure of the corresponding set in the unit cube $Q_N(0 \leqq \theta_n < 1)$ in the $(\theta_0, \theta_1, \ldots, \theta_N)$-space. For $N \to \infty$ they converge to limits $W_i(M)$ and $W_y(M)$, and $W(M)$ is now defined as the common value of $W_i(M)$ and $W_y(M)$ when they are equal. One finds that

$$W(M) = \iint_M F(P)\, dM,$$

where the probability and the integral are defined for the same sets.

The set-probability $W(M)$ may also be defined directly, in analogy to the set-functions $W_N(M)$, by means of a suitably defined Jordan measure in the unit cube $Q(0 \leqq \theta_n < 1)$ in the infinite-dimensional $(\theta_0, \theta_1, \ldots, \theta_N, \ldots)$-space.

The uniform convergence of the sequence of point-probabilities $F_N(P)$, $N \geqq N_0$, still holds in certain cases where the series $\sum_{n=0}^{\infty} \omega_n$ is *divergent.*

S 21. Summary of F 3.

For every fixed $N \geqq N_0$ and every positive p we consider the set $\Sigma_{N,N+p} = \sum_{n=N+1}^{N+p} \omega_n$. To this set there corresponds a set-probability $W_{N,N+p}(M)$ which for all sufficiently large p, say for $p \geqq p_0 = p_0(N)$, will be determined by a point-probability $F_{N,N+p}(P)$. One easily proves that for such p

$$F_{N+p}(P) = \iint_{\Sigma_{N,N+p}} F_N(P-P_{N,N+p}) F_{N,N+p}(P_{N,N+p})\, d\Sigma_{N,N+p} ,$$

where $P_{N,N+p}$ denotes the variable point in $\Sigma_{N,N+p}$. From this relation it easily follows that the sequence $F_N(P)$, $N \geqq N_0$, will be uniformly convergent if to every $N \geqq N_0$ there exist positive numbers ϱ_N and η_N converging to 0 as $N \to \infty$ and such that for every $p \geqq p_0 = p_0(N)$

$$\text{(d)} \qquad 1 - W_{N,N+p}(\Gamma_N) < \eta_N ,$$

where Γ_N denotes the closed circular disk with center O and radius ϱ_N.

Under this assumption, the limit function $F(P)$ of the sequence $F_N(P)$ will be positive for exactly those points P which together with a neighbourhood belong to the interior of Σ_N for all N from a certain stage onward. It is natural to consider the closure Σ of the set of these points as the sum of the divergent series $\sum_{n=0}^{\infty} \omega_n$. The integral of $F(P)$ over Σ is equal to 1. The function $F(P)$ is called the *point-probability* corresponding to Σ. As in the preceding case we can also by a passage to the limit define a set-probability $W(M)$ corresponding to Σ, which will be determined as the integral of $F(P)$. In the present case, where the set Σ is not the image of the infinite-dimensional unit cube, we have no direct determination of $W(M)$ as an infinite-dimensional measure.

The chapter ends with an *example*. The curve ω in the complex $Y = U + iV$-plane determined for a given $r < 1$ by the parametric representation

$$Y = \operatorname{Log}(1 - re^{2\pi i\theta}), \quad 0 \leqq \theta < 1 ,$$

is easily shown to be a convex curve. It has the symmetry axes $V = 0$ and $U = \log \sqrt{1-r^2}$. By calculation of its radius of curvature one finds for its inner and outer radii the values $r_i = r$ and $r_y = \frac{r}{\sqrt{1-r^2}}$. The parametric representation evidently determines a continuous point-probability on ω.

By giving r a sequence of values $r_0, r_1, \ldots, r_N, \ldots$ converging to 0 we therefore obtain a sequence of curves $\omega_0, \omega_1, \ldots, \omega_N, \ldots$ satisfying the conditions of chapter IV.

S 21. Summary of F 3.

The series $\sum_{n=0}^{\infty}\omega_n$ will be convergent when the series $\sum_{n=0}^{\infty}r_n$ converges. If the series $\sum_{n=0}^{\infty}r_n$ diverges, but the series $\sum_{n=0}^{\infty}r_n^2$ converges, then condition (d) will be satisfied, as a standard argument shows, so that here we have a case of a divergent series $\sum_{n=0}^{\infty}\omega_n$ for which the point-probability exists.*

* A study of the sets Σ_N shows that in this case the point-probability $F(P)$ is positive in the whole plane.

ON AN EXTENSION OF A KNOWN CONVERGENCE THEOREM

Summary of Om en Udvidelse af en kendt Konvergenssætning.
Nyt Tidsskrift for Matematik B 20 (1909), 1—4.

P. du Bois-Reymond has proved the following convergence theorem:

The series $\sum_0^\infty a_n \cdot b_n$ is convergent, if $\sum_0^\infty a_n$ and $\sum_1^\infty |b_n - b_{n-1}|$ are convergent.

It is shown that this theorem can be extended as follows:

If $\sum_0^\infty a_n$ *is convergent and if*

$$\begin{array}{llll} b_{0,0} & & & \\ b_{1,0} & b_{1,1} & & \\ \cdots & & & \\ b_{n,0} & b_{n,1} & \cdots & b_{n,n} \\ \cdots & & & \end{array}$$

are complex numbers satisfying the two conditions

$$1) \lim_{\substack{m \text{ const. } n=\infty}} b_{n,m} = b_m \qquad 2) \sum_{r=1}^{r=n} |b_{n,r} - b_{n,r-1}| < B \text{ (for all } n),$$

then $a_0 b_{n,0} + a_1 b_{n,1} + \cdots + a_n b_{n,n}$ *has, for* $n = \infty$, *a limit equal to* $\sum_0^\infty a_n \cdot b_n$.

By means of this theorem simple proofs are given for the two main theorems on multiplication of two infinite series, due to Mertens and Cesàro respectively:

If $U = \sum_0^\infty u_n$ is convergent and $V = \sum_0^\infty v_n$ is absolutely convergent, then the series $\sum_0^\infty w_n$, where

$$w_n = u_0 \cdot v_n + u_1 \cdot v_{n-1} + \cdots + u_n \cdot v_0,$$

is convergent with the sum $U \cdot V$.

If $U = \sum_0^\infty u_n$ and $V = \sum_0^\infty v_n$ are convergent, then the series $\sum_0^\infty w_n$ is summable with the sum $U \cdot V$, i. e., on putting

$$W_n = w_0 + w_1 + \cdots + w_n$$

we have

$$\lim \frac{1}{n}(W_0 + W_1 + \cdots + W_n) = U \cdot V.$$

SOME REMARKS ON FORMAL CALCULATION

Summary of NOGLE BEMÆRKNINGER OM FORMEL REGNING.
Matematisk Tidsskrift B 1928, 7—18.

This paper originates in a discussion with A. F. Andersen, in connection with Andersen's doctoral dissertation*, concerning the advantages of formal calculation with infinite series. After some orientating remarks, illustrated by a proof of Möbius' inversion formula based on Dirichlet series, a modified proof is given of one of Andersen's theorems on differences of fractional order.

For an arbitrary sequence of numbers $\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots$ the differences $\Delta^r\varepsilon_\nu$ of arbitrary real order r may be defined by the formula

$$\text{(I)}\qquad (1-x)^r\left(\varepsilon_0+\frac{\varepsilon_1}{x}+\frac{\varepsilon_2}{x^2}+\cdots\right) = \cdots+\Delta^r\varepsilon_0+\frac{\Delta^r\varepsilon_1}{x}+\frac{\Delta^r\varepsilon_2}{x^2}+\cdots,$$

where the dots before the constant term on the right indicate terms with positive exponents in which we are not interested. If we denote the binomial series for $(1-x)^r$ by $r_0+r_1x+r_2x^2+\ldots$, this means that we put

$$\text{(II)}\qquad \Delta^r\varepsilon_\nu = \sum_{p=\nu}^{\infty}\varepsilon_p r_{p-\nu}.$$

A meaning is attached to the symbol $\Delta^r\varepsilon_\nu$ only when this series converges. Equation (II) is the definition used by Andersen. Equation (I) is formal in the sense that though the series occurring in the formation of the coefficients on the right are supposed to be convergent, it is not assumed that the power series with coefficients ε_ν or $\Delta^r\varepsilon_\nu$ are convergent for any x. The differences $\Delta^r\varepsilon_\nu$ $(\nu = 0, 1, 2, \ldots)$ certainly exist when the ε_ν are bounded and $r > 0$. For in this case the series $\sum r_\nu$ (the binomial series for $x = 1$) is convergent and all but a finite number of its terms have the same sign; the series (II) is therefore convergent. Andersen also introduced the broken differences of order r defined by

$$\text{(III)}\qquad \Delta^r_n\varepsilon_\nu = \sum_{p=\nu}^{n}\varepsilon_\nu r_{p-\nu}.$$

* A. F. Andersen, *Studier over Cesàro's Summabilitetsmetode*, Copenhagen 1921.

The theorem of Andersen to be considered is as follows.

If $|\varepsilon_\nu| < k$ $(\nu = 0, 1, 2, \dots)$, *then the equation*

$$\Delta^s \Delta^r \varepsilon_\nu = \Delta^{r+s} \varepsilon_\nu$$

holds if $r > 0$, $s > -1$, *and* $r+s > 0$, *i.e., the indicated differences exist and are equal.*

[It adds to the interest of this theorem that, as shown by Andersen, it is 'best possible' in a certain sense.]

Whereas Andersen in his proof works directly with the definition (II) [and with the definition (III) of the broken differences]*, the modified proof is based on the expansion (I). In this way any use of the order of magnitude of the binomial coefficients is avoided.

We put (formally)

$$\varepsilon_0 + \frac{\varepsilon_1}{x} + \frac{\varepsilon_2}{x^2} + \cdots = E(x)$$

$$\begin{aligned}
(1-x)^r &= r_0 + r_1 x + r_2 x^2 + \cdots = R(x) \\
(1-x)^s &= s_0 + s_1 x + s_2 x^2 + \dots = S(x) \\
(1-x)^t &= t_0 + t_1 x + t_2 x^2 + \dots = T(x) \quad (t = r+s);
\end{aligned}$$

we also introduce the notation

$$S_n(x) = s_0 + s_1 x + \cdots + s_n x^n .$$

All that is needed about the binomial coefficients r_ν, s_ν, t_ν is: 1) In each of the three series the coefficients have the same sign from a certain step onward, say for $\nu > c$. 2) Since r and $t = r+s$ are both > 0, the series for $(1-x)^r$ and $(1-x)^t$ are both convergent at the point 1 with sum 0. 3) Since $s > -1$, the coefficients s_ν in the, series for $(1-x)^s$ converge to 0 as $\nu \to \infty$.

It is sufficient to prove that $\Delta^s \Delta^r \varepsilon_0 = \Delta^t \varepsilon_0$. Since $r > 0$, $t > 0$, and $|\varepsilon_\nu| < k$ the differences $\Delta^r \varepsilon_\nu$ and $\Delta^t \varepsilon_\nu$ exist and are the coefficients of $\frac{1}{x^\nu}$ in the expansions of $E(x)R(x)$ and $E(x)T(x)$. The broken difference $\Delta^s_n \Delta^r \varepsilon_0$ will be the constant term in the expansion of $E(x)R(x)S_n(x)$. Hence $\Delta^s_n \Delta^r \varepsilon_0 - \Delta^t \varepsilon_0$ is the constant term in the expansion of $E(x)\big(R(x)S_n(x) - T(x)\big)$. We have to prove that it tends to 0 as $n \to \infty$. Let the expansion of $R(x)S_n(x)$ be

$$R(x)S_n(x) = \tau_0 + \tau_1 x + \tau_2 x^2 + \cdots .$$

* In a recent paper: Comparison theorems in the theory of Cesàro summability, *Proceedings of the London Mathematical Society* (2) 27 (1928), 39–71 (58–60), Andersen has given a simpler formulation of his original proof of a similar theorem, but uses also there a direct method of proof.

Evidently $\tau_\nu = t_\nu$ for $0 \leqq \nu \leqq n$. Hence

(IV) $$R(x)S_n(x) - T(x) = \sum_{\nu=n+1}^{\infty} (\tau_\nu - t_\nu) x^\nu .$$

Since the series for $(1-x)^r$ and $(1-x)^t$ are convergent at $x = 1$ with sum 0, the series (IV) is also convergent at $x = 1$ with sum 0, i. e. $\sum\limits_{\nu=n+1}^{\infty}(\tau_\nu - t_\nu) = 0$. But $\sum\limits_{\nu=n+1}^{\infty} t_\nu \to 0$ as $n \to \infty$; hence $\sum\limits_{\nu=n+1}^{\infty} \tau_\nu \to 0$ as $n \to \infty$. [Notice that τ_ν depends on ν and n.] The difference $\Delta_n^s \Delta^r \varepsilon_0 - \Delta^t \varepsilon_0$ is by (IV) equal to $\sum\limits_{\nu=n+1}^{\infty} \varepsilon_\nu(\tau_\nu - t_\nu)$. Since $|\varepsilon_\nu| < k$ and $\sum\limits_{\nu=n+1}^{\infty} |t_\nu| \to 0$, we have $\sum\limits_{\nu=n+1}^{\infty} \varepsilon_\nu t_\nu \to 0$. It remains to prove that $\sum\limits_{\nu=n+1}^{\infty} \varepsilon_\nu \tau_\nu \to 0$ and for this it is sufficient to prove that

(V) $$\sum_{\nu=n+1}^{\infty} |\tau_\nu| \to 0 \quad \text{as} \quad n \to \infty .$$

This relation (V) follows from $\sum\limits_{\nu=n+1}^{\infty} \tau_\nu \to 0$ since the quantity

(VI) $$\sum_{\nu=n+1}^{\infty} |\tau_\nu| - \left| \sum_{\nu=n+1}^{\infty} \tau_\nu \right|$$

tends to 0 as $n \to \infty$. To see this we observe that $\sum\limits_{\nu=n+1}^{\infty} \tau_\nu$ is formed from the products $r_\alpha s_\beta$ for which $\alpha + \beta > n$ and $\beta \leqq n$. Since r_α and s_β have constant signs for $\alpha > c$ and $\beta > c$, the total contribution of the terms $r_\alpha s_\beta$ of 'irregular' sign—which is at least half of the quantity (VI)—is smaller than

$$\sum_{\beta=1}^{c} |s_\beta| \cdot \sum_{\alpha=n-c}^{\infty} |r_\alpha| + \sum_{\alpha=1}^{c} |r_\alpha| \cdot \sum_{\beta=n-c}^{n} |s_\beta| ,$$

and this quantity tends to 0 as $n \to \infty$.

ON GENERAL CONVERGENCE CRITERIA FOR SERIES WITH POSITIVE TERMS

Summary of
OM ALMINDELIGE KONVERGENSKRITERIER FOR RÆKKER MED POSITIVE LED.
Matematisk Tidsskrift B 1945, 1—9.

This paper is extracted from a rather extensive unpublished manuscript 'On series with arbitrary indices' from the author's very early years.

For an arbitrary index sequence $x_0 < x_1 < x_2 < \cdots$ and an associated function u_{x_ν} the sum of u from x_0 to x_n is defined by

$$\underset{(x_0, x_1, \cdots, x_n)}{S} u = u_{x_1}(x_1 - x_0) + u_{x_2}(x_2 - x_1) + \cdots + u_{x_n}(x_n - x_{n-1}) \, .$$

It is related to the difference quotient

$$u^{(1)}_{x_\nu} = \frac{u_{x_\nu} - u_{x_{\nu-1}}}{x_\nu - x_{\nu-1}}$$

by the formula

$$\underset{(x_0, x_1, \cdots, x_n)}{S} u^{(1)} = u_{x_n} - u_{x_1} \, .$$

The limit

$$\underset{(x_0, x_1, \cdots)}{S} u = \lim_{n \to \infty} \underset{(x_0, x_1, \cdots, x_n)}{S} u \, ,$$

when it exists, is called an *infinite series with arbitrary indices* $x_0, x_1, \ldots$. For a positive function u an infinite product with arbitrary indices is defined similarly.

It is shown that these notions lead to a simple deduction of so-called 'general' criteria for convergence or divergence of ordinary infinite series $\sum_1^\infty v_n$ with positive terms, i.e., criteria involving 'arbitrary' functions, as developed by various authors, in particular by Pringsheim.*

For series with arbitrary indices $(0 <)\, x_1 < x_2 < \ldots$ and positive terms one easily obtains by comparison with certain simple series the following special criteria:

* A. Pringsheim, Allgemeine Theorie der Divergenz und Konvergenz von Reihen mit positiven Gliedern, *Mathematische Annalen* 35 (1890), 297—394.

$$\textit{First kind.}\quad \sqrt[x_n]{u_{x_n}} \begin{cases} \leqq a < 1 & \text{(convergence)} \\ \geqq 1 \ (x_n \to \infty) & \text{(divergence)}, \end{cases}$$

$$\textit{Second kind.}\quad \sqrt[x_n - x_{n-1}]{\frac{u_{x_n}}{u_{x_{n-1}}}} \begin{cases} \leqq a < 1 & \text{(convergence)} \\ \geqq 1 \ (x_n \to \infty) & \text{(divergence)}. \end{cases}$$

Now, an ordinary infinite series $\sum_1^\infty v_n$, i.e., a series corresponding to the indices $0, 1, 2, \ldots,$ may be transformed into a series $S_{(x_0, x_1, \cdots)}^{u}$ with arbitrary indices, where $u_{x_n} = \dfrac{v_n}{x_n - x_{n-1}}$. The preceding criteria therefore give the following *general criteria* for the series $\sum v_n$:

$$\textit{First kind.}\quad \sqrt[x_n]{\frac{v_n}{x_n - x_{n-1}}} \begin{cases} \leqq a < 1 & \text{(convergence)} \\ \geqq 1 \ (x_n \to \infty) & \text{(divergence)}, \end{cases}$$

$$\textit{Second kind.}\quad \sqrt[x_n - x_{n-1}]{\frac{v_n}{v_{n-1}} \, \frac{x_{n-1} - x_{n-2}}{x_n - x_{n-1}}} \begin{cases} \leqq a < 1 & \text{(convergence)} \\ \geqq 1 \ (x_n \to \infty) & \text{(divergence)}. \end{cases}$$

These criteria (in a somewhat different form) are well known and are also given by Pringsheim. The first he calls the 'most general criterion of the first kind', but the second does not appear in his exposition as the corresponding criterion of the second kind. As such appears Kummer's criterion, according to which the series $\sum v_n$ converges if

$$\frac{v_{n-1}}{v_n} \varphi_{n-1} - \varphi_n > K > 0$$

for some positive function φ_n. His proof of this criterion is rather unsystematic. By the above method, Kummer's criterion is immediately obtained from the following special convergence criterion for series with arbitrary indices and positive terms:

$$u_{x_n} < k(-u_{x_n}^{(1)}) \quad \text{or} \quad \frac{-u_{x_n}^{(1)}}{u_{x_n}} > K > 0\,.$$

This criterion is obvious since $S_{(x_0, x_1, \cdots, x_n)}^{-u^{(1)}} = u_{x_0} - u_{x_n} < u_{x_0}$. If we insert $u_{x_n} = \dfrac{v_n}{x_n - x_{n-1}}$, it takes the form

$$\frac{v_{n-1}}{v_n} \left(\frac{1}{x_{n-1} - x_{n-2}} \right) - \frac{1}{x_n - x_{n-1}} > K > 0\,,$$

and this is Kummer's criterion, since $\dfrac{1}{x_\nu - x_{\nu-1}} = \varphi_\nu$ is an arbitrary positive function.

HARALD BOHR

22 April 1887 — 22 January 1951.

BY

BØRGE JESSEN

Memorial adress given at a meeting of Danish mathematicians on 6 April 1951 at the University of Copenhagen.*

The Mathematical Society, the Society of Mathematics Teachers, and the club Parentesen have desired that we should meet here today in memory of Harald Bohr, through whose passing we have lost the one who more than anybody else tied us together.

We should have liked to hold this meeting at the Institute of Mathematics, where he had his work; however, space would not permit. But this hall too has its associations. It was here that he, not yet 23 years old, defended his doctoral dissertation. If we can believe the newspapers, it was, however, not mathematicians who filled the hall, but for the most part football enthusiasts, who had come to see one of their favourites in a role so unusual for a star athlete. There was a special atmosphere over the proceedings through Zeuthen's opposition. This was his last official act, and those present understood that it was an heir to his place in Danish mathematics who was now embarking on his academic career. And it was in this hall that four years ago we celebrated Harald Bohr's 60th birthday. His friends and pupils rejoiced on that day in saying what one does not say in words every day. But even on that day it was he who contributed most through an unforgettable lecture, in which he looked back over his life and work and also gave a vivid description of mathematical life at home and elsewhere, especially in his youth. Through kindly, yet well-considered characterizations of Danish and foreign mathematicians whom he had met, he gave at the same time a picture of himself. This lecture, which was printed in *Matematisk Tidsskrift*, will be often read.

I have been asked today to speak about Harald Bohr and his life work, and

* The Danish text has been printed in *Matematisk Tidsskrift A* 1951, 1–18. Certain omissions and minor changes have been made in the present translation, which is reprinted from *Acta Mathematica* 86 (1951), i–xiv.

I feel it as the dearest duty that could be assigned to me, in this way to contribute to preserving his memory. But I have found the task a difficult one. I understand so well the reaction of another of his closest friends whom I had asked if he would write me something about a time which I did not know myself. This friend answered that he had tried again and again, but had found nothing that could be mentioned in a larger circle. And he added: 'He was so fundamentally human that one cannot abstract from the most personal elements without destroying the essential.' Nor have I found it easy to describe Bohr's work. His own radiant exposition has been constantly in my mind. This must be my excuse for not going very deeply into his works. I have restricted myself to pointing out the most important results and their mutual connection.

Harald Bohr was born in Copenhagen in 1887, the son of the distinguished physiologist Christian Bohr; his mother was a daughter of the prominent financier, politician, and philanthropist, D. B. Adler, and Bohr throughout his life maintained the closest ties with the Adler family circle. In his family home, which many of the most distinguished Danish men of science and letters of the time frequented as friends, he imbibed, together with his elder brother Niels Bohr, a deep love for science; and he learned also helpfulness and sympathy for others, as well as the uncommon thoughtfulness which was so strong a trait in his character, the more to be admired since it was coupled with an impulsive temperament.

His elder cousin, Miss Rigmor Adler, has told me that he was the most lovable and attractive child one can imagine, full of bubbling life, thoroughly kind and helpful, quick at everything, intelligent and full of information, musical, and highly amusing. He was everyone's favourite, but even then he was quite unspoiled by so much admiration. He loved to tease those around him, but never in a malicious way. To be sure, he knew also how to be impertinent. His parents said that he reminded them of the irrepressible Lavinia in *Our Mutual Friend*. Even as a child, he read widely and matured rapidly. Then, as throughout his later life, his relationship with his brother was a central influence. Each was the other's closest friend; and they shared everything, thoughts, interests, and worldly goods.

When only 17 years of age, he entered the University of Copenhagen and chose mathematics as his field of study. Richly gifted as he was, there can be no doubt that, even had he chosen a different subject, he would have distinguished himself in it. For example, anyone who knew him could hardly doubt that his sure judgment and profound human understanding would have made him an outstanding jurist. We mathematicians may be grateful that he chose mathematics.

Nevertheless, it was in another field that he won his first laurels. From early boyhood, he had been a keen football player. At the age of eleven, he joined the Copenhagen Football Club, and the next year he transferred to the Academic Football Club, in whose team he played in matches in Denmark and abroad throughout his student days, and occasionally even later, the last time as a young professor. He also played in the national team. Denmark was among the first continental countries where the game became popular, and 'little Bohr', as he was called in football circles, played in the team which won second place for Denmark at the Olympic Games in London in 1908; Britain, of course, held an unassailable first place. The following story will illustrate his great popularity. One day, he had taken his mother to the streetcar. A boy with a large bundle of newspapers under his arm followed her into the car and took a seat next to her. He nudged her with his elbow to attract her attention: 'Do you know who it was that helped you onto the car?' Mrs. Bohr, who did not wish to betray herself, replied, 'What do you mean, my boy?' 'That was Harald Bohr, our greatest football player.' Throughout his life, Bohr maintained a close interest in the game, and was often a spectator, especially when his old club was playing. One would hardly be in error if one connected his great ease in meeting men of all classes with his athlete's life as a young man.

As a student, Bohr attended lectures by Zeuthen, Thiele, and Niels Nielsen, among others. Among these very different persons, Bohr felt the closest kinship to Zeuthen. The strong impression which his teachers made on him showed itself in the frequency and pleasure with which he would talk about them. The most decisive factor, however, in his development as a mathematician was the excellent works which he studied; among these, he himself emphasized Jordan's *Cours d'Analyse* and Dirichlet's *Vorlesungen über Zahlentheorie* with Dedekind's supplements.

In his latter student years, his interests centered on analysis, and he was led to the study of divergent series. His first comprehensive investigation was concerned with the application of Cesàro summability to Dirichlet series of the ordinary type

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

where the coefficients a_n are complex numbers and $s = \sigma + it$ is a complex variable.

Generalizing the well-known theorem of Jensen, according to which every such series possesses an abscissa of convergence γ_0 (which may be $+\infty$ or $-\infty$) such that the series converges in the half-plane $\sigma > \gamma_0$ and diverges in the half-plane $\sigma < \gamma_0$, Bohr showed that, corresponding to every integral order of summability

$n(\geqq 0)$, the series possesses an abscissa of summability γ_n such that the series is summable of the n^{th} order in the half-plane $\sigma > \gamma_n$ but not at any point of the half-plane $\sigma < \gamma_n$. These abscissae form a decreasing sequence, and thus the vertical lines through the points $\gamma_0, \gamma_1, \gamma_2, \ldots$ form the boundaries of a sequence of vertical strips. Bohr showed that the abscissae of summability satisfy certain inequalities which mean, in geometrical terms, that the widths of these strips are all $\leqq 1$ and form a decreasing sequence. He further showed that this is all that can be said regarding the distribution of the abscissae of summability, since for every sequence $\gamma_0, \gamma_1, \gamma_2, \ldots$ satisfying these conditions, he was able to construct a Dirichlet series having exactly these numbers as its abscissae of summability. Still another noteworthy result was obtained by Bohr in this study, by considering the limit γ of the sequence of abscissae of summability. Unlike the abscissa of convergence γ_0, which has no simple function-theoretic significance, the number γ is intimately connected with the function represented by the series. Indeed, the half-plane $\sigma > \gamma$ is the largest half-plane into which the function can be continued analytically while remaining of finite order with respect to the ordinate t.

Almost at the same time as Bohr, and independently of him, Marcel Riesz also studied Cesàro summability of Dirichlet series, which he generalized so as to apply also to Dirichlet series of the general type

$$\sum_{n=1}^{\infty} a_n e^{\lambda_n s}.$$

Such a generalization was also envisaged by Bohr, but since Riesz's method was technically simpler (e. g., it was immediately applicable to summability of non-integral order), Bohr limited himself to working out his theory for ordinary Dirichlet series, as set forth here. Meanwhile, in the spring of 1909, he had received his master's degree; and he used the work described above for his doctoral dissertation, which he defended the following winter.

The problems treated in his doctoral dissertation are not comparable in difficulty to his later work. I have reviewed the dissertation in such detail because in this, the first of his longer papers, we already see the main characteristics of Bohr as a mathematician: a clearly defined and attractive problem and the execution of the investigation leading to conclusive results. The dissertation also shows the mastery of style which distinguishes all of his writings: his rare gift for bringing out the large lines of the investigations while at the same time exposing every detail; the gift of making difficult matters appear simple through explanations inserted at the right places.

Having completed the dissertation, Bohr left this subject and returned to it only in his latter years, when he considered, among other things, the connection between the order of magnitude of the function and the summability abscissae, now for arbitrary non-negative orders of summability. In this topic, he obtained some complete results and also pointed out interesting open problems.

Bohr's investigations as a student had led him into correspondence with Edmund Landau, with the result that Landau had proposed that Bohr should come to study with him. Immediately after his master's examination, therefore, Bohr set off for Göttingen, where Landau had just been appointed. This center of mathematics, to which the best young men of all countries were attracted, became for Bohr almost like a second home; and he returned there again and again in after years for longer or shorter visits. He loved to talk of the rich life that flourished there, and many a young mathematician has thereby formed a lively impression of the glorious period in the history of mathematics which is centered foremost on Hilbert's name. With a number of the mathematicians whom he met in Göttingen, Bohr formed warm friendships. With Landau, he entered into an extremely fruitful collaboration, primarily on the theory of the Riemann zeta-function.

As is well known, the zeta-function $\zeta(s)$ is defined in the half-plane $\sigma > 1$ by the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{1}{n^s};$$

it can be extended analytically over the whole complex plane and is regular except for a pole at the point 1. In the half-plane $\sigma > 1$ it can also be represented by the Euler product

$$\prod_{n=1}^{\infty} \frac{1}{1-\frac{1}{p_n^s}},$$

where the p_n run through the prime numbers. In view of this representation, the function vanishes nowhere in the half-plane $\sigma > 1$. The zeros, so important for the distribution of the prime numbers, are found in the critical strip $0 \leqq \sigma \leqq 1$. The celebrated Riemann hypothesis asserts that they all lie on the vertical center line $\sigma = \frac{1}{2}$.

Among the results which came out of the collaboration with Landau, I shall content myself with describing the theorem in which it culminated, namely the so-called Bohr–Landau theorem, dating from 1914, regarding the distribution of the zeros. It is known that the zeros in the critical strip lie symmetrically both with

respect to the real axis and with respect to the center line of the strip, and further that the number $N(T)$ of zeros with ordinates between 0 and T is asymptotically $\frac{1}{2\pi} T \log T$. The first result they obtained was that in every strip bounded by vertical lines to the right of the center line, the number $n(T)$ of zeros with ordinates between 0 and T is smaller than a constant times T and hence is of lower order than the total number of zeros $N(T)$. Together with the symmetry mentioned above, this showed that in any case the overwhelming majority of the zeros in the critical strip lie in the immediate neighbourhood of or on the center line. Their proof showed that, more generally, for every value of a the number $n_a(T)$ of a-points of $\zeta(s)$ lying in a vertical strip to the right of the center line and having ordinates between 0 and T is smaller than a constant times T. For the number of zeros they succeeded shortly afterwards in proving the stronger result that

$$\lim_{T\to\infty} \frac{n_0(T)}{T} = 0 .$$

At about the same time, Hardy proved that actually infinitely many of the zeros are situated on the center line. Both results have since been sharpened by other authors.

Side by side with the collaboration with Landau, Bohr also carried through, in these years before the first war, a number of investigations on Dirichlet series. Most of these investigations are concerned with a method for the treatment of the distribution of the values of functions represented by Dirichlet series, in particular the zeta-function. This method consists in a combination of arithmetic, geometric, and function-theoretic considerations. In the original form of the method, its arithmetical part depended on Kronecker's theorem on Diophantine approximations, through which, with the use of the Euler product, the zeta-function was brought into connection with functions of infinitely many variables. The method showed that when the point s in the complex plane traces out a vertical line to the right of the point 1, then $\zeta(s)$ will trace out a curve whose closure is a certain figure which can be described. Moreover, this closure is identical with the set of values attained by $\zeta(s)$ at points lying arbitrarily near to the vertical line. A discussion of how the figure changes when the line varies led in particular to the remarkable result that the zeta-function assumes every value except 0 in the half-plane $\sigma > 1$, and in fact infinitely often.

It seemed natural to try to apply this method in the critical strip also. Though

the Euler product is divergent here, one might hope to succeed because of a certain convergence in mean of the product to the right of the center line of the strip. But at first the method failed. Then, as Bohr has told me, he happened to be in Göttingen when Hermann Weyl presented his famous generalization of Kronecker's theorem to the mathematical society. Bohr saw immediately that this refinement was just what was needed to make his method work. In a preliminary exposition of the method, written in collaboration with Richard Courant, it was shown that the values of the zeta-function on a vertical line in the right half of the critical strip are everywhere dense in the whole plane. The final exposition was given by Bohr in a paper in *Acta Mathematica* in 1915. The main result is a counterpart of the Bohr–Landau theorem, to the effect that for every substrip, however thin, of the right half of the critical strip, the above mentioned number $n_a(T)$ of a-points in the strip having ordinates between 0 and T exceeds a positive constant times T, for all sufficiently large T, when $a \neq 0$. Through this result, it was proved for the first time that whether or not the value 0 is assumed by the zeta-function to the right of the center line, this value plays an exceptional role. Indeed, according to the Bohr–Landau theorem, the number of zeros is infinitely small compared with T, whereas for every $a \neq 0$ the number of a-points is exactly of the order of magnitude of T.

This result, however, did not exhaust the possibilities of the method. By elaborating it further, Bohr proved some years later that the zeta-function possesses an asymptotic distribution function on every vertical line to the right of the center line and also that the limit

$$\lim_{T\to\infty} \frac{n_a(T)}{T}$$

exists for every strip to the right of the center line, not only for $a = 0$ but for every a. He was interrupted in the exposition of these results by his discovery of almost periodic functions. When he returned to the subject at the end of the 1920's, he invited me, who was then among his students, to help him with it. From this collaboration, continued through the years, developed our close friendship which has been so decisive a factor in my life.

During the years before the first war, Bohr also came into scientific contact with Hardy and Littlewood and formed close friendships with them. He often went to Cambridge and Oxford to study. In collaboration with Littlewood, he wrote a book on the theory of the zeta-function, which, however, was never sent to a printer. Later, when the theory had been developed further, their manuscript became the basis of the two excellent Cambridge tracts by Ingham and Titchmarsh. Bohr felt

deeply attracted by life in the old English universities, especially by the spirit of freedom and tolerance prevalent there. Just before the outbreak of the war, he also spent a few months in Paris, where in particular he came into contact with Lebesgue.

Bohr's investigations on Dirichlet series and the zeta-function won him an early reputation, which found expression when he was invited to write, together with Cramér, the article on the recent development in the analytic theory of numbers in the *Encyklopädie der mathematischen Wissenschaften.*

Immediately after obtaining his doctor's degree, Bohr had joined the faculty of the University of Copenhagen. In 1915, he was appointed professor at the College of Technology (Polyteknisk Læreanstalt), where at that time the university students also received their introductory courses in mathematics. He retained this position until returning in 1930 to the University of Copenhagen, where through a gift of the Carlsberg Foundation, an Institute of Mathematics was founded with him as leader. It was a real joy to him that it became possible to erect this institute directly adjacent to his brother's Institute for Theoretical Physics. Bohr's work as a teacher has left a deep imprint on Danish mathematics. Together with Mollerup, his colleague at the College of Technology, he wrote a comprehensive textbook on analysis, which has contributed greatly to raising the level of mathematics in Denmark. This occupied him for several years, during which period his own scientific work lay almost dormant.

After the completion of the textbook, Bohr returned to his scientific work; and it was in the following years, in the beginning of the 1920's, that he performed his main achievement, the establishment of the theory of almost periodic functions. The starting point was the problem of characterizing those functions $f(s)$ which can be represented by a Dirichlet series

$$\sum_{n=1}^{\infty}{}' a_n e^{\lambda_n s}.$$

If the series is considered on a vertical line, e. g. the imaginary axis, it reduces to a trigonometric series

$$\sum_{n=1}^{\infty}{}' a_n e^{i\lambda_n t}.$$

It was therefore natural to consider more generally the problem of which functions $F(t)$ of a real variable can be represented by such a series, i. e., can be formed by superposition of pure oscillations. In the special case where the frequencies λ_n are integers, the answer is given in the classical theory of Fourier series. The functions represented by such series are in essence all periodic functions of period 2π. Whereas hitherto in the theory of Dirichlet series one had always worked with frequencies

forming a monotonic sequence, Bohr discovered that, in order to obtain an answer to the problem, one would have to consider series with quite arbitrary frequencies λ_n. The answer was obtained by introducing the notion of almost periodicity.

Restricting myself for the present to functions $F(t)$ of a real variable, I shall briefly state the main results of the theory. The number τ is called a translation number of $F(t)$ belonging to $\varepsilon > 0$ if

$$|F(t+\tau)-F(t)| \leqq \varepsilon \text{ for all } t.$$

The function $F(t)$ is called almost periodic if for every $\varepsilon > 0$ such translation numbers τ exist and form a relatively dense set, i. e., a set with the property that any sufficiently long interval on the real axis contains at least one number of the set. This relative density is the crucial point of the definition.

Having got the idea that this was the desired definition, Bohr developed the theory of these functions systematically. Without undue trouble, it was proved that functions obtained from almost periodic functions by simple operations are again almost periodic. Further it was proved that every almost periodic function possesses a mean value $M\{F(t)\}$ obtained as the limit of the mean value over an interval when the length of the interval tends to infinity. This made it possible to copy the classical theory of Fourier series. For an arbitrary frequency λ, the corresponding Fourier constant was defined as the mean value

$$a(\lambda) = M\{F(t)e^{-i\lambda t}\}.$$

The use of a classical argument showed that this Fourier constant differs from 0 for only a countable set of frequencies $\lambda_1, \lambda_2, \ldots$, and Bohr now attached to the function $F(t)$ the series

$$\sum_{n=1}^{\infty} a(\lambda_n)e^{i\lambda_n t},$$

which he called its Fourier series.

So far everything had gone smoothly. Now the essential difficulty was encountered: namely, to prove that this series actually represents the function in a certain sense, more precisely, that it converges in the mean to $F(t)$, which amounts to the validity of the Parseval equation

$$M\{|F(t)|^2\} = \sum_{n=1}^{\infty} |a(\lambda_n)|^2.$$

This was the decisive criterion, that the class of functions considered was actually the right one. It was during a summer vacation, spent in idyllic surroundings in the country near Copenhagen, that Bohr overcame this difficulty. His proof of this

fundamental theorem is a climax in his work. In the printed version, it fills nearly forty pages. Bohr has often said that he worked with his bare hands. That holds true for this proof more than for anything else he has done. The doctoral dissertation showed that when necessary he could work with a formal apparatus; but in his later work, formal manipulations played for the most part only a subordinate role, and he worked best when he could tackle problems directly. The fact that he nevertheless always reached conclusive results shows his strength as a mathematician. Simpler proofs have since been found for his fundamental theorem and few, unhappily, will now read his own long proof.

Having established the Parseval equation, Bohr, by a skilful extension of his old method of passing to functions of infinitely many variables by use of Kronecker's theorem, arrived at another theorem, which may be considered the main result of the theory. It generalizes the classical theorem of Weierstrass on trigonometric approximation of periodic functions by stating that the class of almost periodic functions is identical with the class of those functions which can be uniformly approximated by means of trigonometric polynomials with arbitrary frequencies.

On the basis of the theory of almost periodic functions of a real variable, it was easy to develop a corresponding theory of almost periodic functions of a complex variable and their representation by Dirichlet series with quite arbitrary exponents λ_n. The further development of this theory led to interesting problems of a function-theoretic nature.

Bohr published the theory in 1924–26 in three extensive papers in *Acta Mathematica*, dedicated to his teacher and friend Edmund Landau, and it created a great sensation. Numerous mathematicians joined in the work on its extension, and pupils from many countries found their way to Copenhagen in the following years to study with him. Soon there appeared new treatments of the fundamental results of the theory. Thus, Weyl and Wiener connected it with the classical theories of integral equations and Fourier integrals. De la Vallée Poussin gave a simpler proof for the Parseval equation. Bochner developed a summation method for Fourier series of almost periodic functions, generalizing Fejér's theorem, and also gave a new definition of almost periodicity. Stepanoff, Wiener, and Besicovitch studied generalizations depending on the Lebesgue integral. Favard considered differential equations with almost periodic coefficients, and Wintner introduced statistical methods into the study of the asymptotic distribution of the values of almost periodic functions. Many more could be mentioned. Bohr rejoiced at every advance, and not the least at those investigations which simplified his own exposition. When in the 1930's von Neumann,

starting from Weyl's treatment and using Bochner's way of defining almost periodicity, succeeded in extending the theory to functions on arbitrary groups, it found a central place in contemporary mathematics, as a step in the unification of different mathematical theories which is such an essential feature in the modern development of our science. I shall not here go into the further development of the theory, in which Bohr always maintained a leading part. His Danish pupils have found in this theory a rich field of study.

As a visiting professor in Göttingen and later in America, where he made new friends, Bohr gave a revised exposition of the fundamental parts of the theory, making use of the simplifications which had been obtained. This he published as a little book in the series *Ergebnisse der Mathematik*. The popularity which this work attained was a source of much pleasure to him, and he used it often in his teaching, primarily as a basis for seminars.

As an academic teacher, Bohr was greatly loved. He always prepared his lectures in minute detail; but this was hardly noticeable to the audience, spellbound as it was by his dynamic delivery. On special occasions when he wanted to cover a great deal of material in a short time, he filled the blackboard in advance, making lavish use of coloured chalk, so that it shone in all the hues of the rainbow. His lectures made an extraordinary impression on all who attended them. One of his students from Göttingen once said to me almost reproachfully: 'Er zerreisst sich ja für die Studenten'. His courses for advanced students alternated between function-theory and number-theory. He did not place so much emphasis on covering a large amount of material as he did on having the students really understand the subject. Early in a course, he liked to point to some great theorem as a goal to be aimed at in order that the students might feel from hour to hour their progress towards this goal.

His collaborators and students in the Institute of Mathematics have enjoyed fruitful and happy years under his leadership. Certain days stand out, among them in particular his 60th birthday, when the students at the morning chocolate served on this occasion presented a cantata in his praise. How he delighted in its facetious words! In the afternoon, he gave here at the University the lecture to which I have already referred. Reluctantly he permitted us on this occasion to set up a plaque portraying him, in the library of the Institute. For the youth who work there in the future, it will be a modest outward expression of the unique place he holds in Danish mathematics.

Nobody who met Harald Bohr could help coming under the influence of his rich and many-sided character. With his deep humanity and radiant spirit, he made

every encounter with him an event. In 1919 he had married Ulla Borregaard, and in their hospitable home Danish mathematicians have spent many happy hours, often with guests from other lands. When their home is mentioned, a smaller circle will also think of many summers spent on the island of Als in the mathematical colony founded by Jakob Nielsen and Bohr.

Harald Bohr had the rare gift of being for many the closest confidant, the first person to whom they would come with their troubles. One never turned to him in vain. Indeed, so active was his helpfulness that one almost felt it was doing him a service to call on him.

To those who had the good fortune to become his close collaborators, scientific or not, Bohr's warm interest was a unique encouragement; and he knew how to stimulate his collaborators to achieve their best. He liked to say that he 'sponged on youth', and indeed he allowed his associates to do a great deal; but his influence was so strong that a joint work always bore his stamp. 'I have long ago bitten off the head of all shame', was a Danish saying which he often used when he had a task in mind for an associate.

Bohr's enthusiastic interest in their work has been a great encouragement for numerous mathematicians, among them many whose achievements are of such excellence that one would not expect them to need it, and has helped to give them confidence that their work was worth the effort. He also loved to praise his friends to others. Once, when his children had reached the age when one can begin to tease in an amiable way, he told me that one of them had asked him: 'Father, why is it that your friends in other countries are all among the very foremost mathematicians?' He smiled at being teased about his tendency to praise, rejoicing at the same time in its being so.

From his childhood, Bohr was well acquainted with the works of Goethe and Schiller, and he often quoted them. Two quotations, both from *Die Wahlverwandtschaften*, he used so often that perhaps I may repeat them here: 'Die angenehmsten Gesellschaften sind die, in welchen eine heitere Ehrerbietung der Glieder gegeneinander obwaltet'—a characterization which applies so aptly to his own circle—and 'Gegen grosse Vorzüge eines andern gibt es kein Rettungsmittel als die Liebe'. Another favourite author of his was Dickens, whose works he had devoured as a child. He read much, in recent as well as classic literature, and liked to make his associates participate in what thus moved him.

When disaster struck Germany in 1933 and hit academic circles, among others, so hard, Bohr was among the first to offer help. His close personal relations with

colleagues in many countries enabled him to help in finding new homes for those scientists who were either forced to leave Germany or who chose to do so, and he turned all his energies to this task. In the summer of 1933, he made several visits to Germany, which were a great encouragement to his friends in their distress. His correspondence on their behalf was enormous; and he and Mrs. Bohr never tired of inviting German mathematicians to come and stay with them to talk things over, since the exchange of letters was difficult. The problem was further complicated by the fact that American universities had been severely hit by the coincident economic crisis. But through his efforts, and those of his friends in other countries, all obstacles were overcome. Many prominent mathematicians, to whom his help in these years was decisive, now hold positions throughout the world, and in particular at American universities.

During these years he also participated energetically in the endeavours of a Danish committee to alleviate the conditions of people in other academic fields who took refuge in our country, and through this work he formed new friendships with others who had dedicated themselves to this task.

He himself did not escape the experience of exile in the latter years of the last war, being compelled to take refuge in Sweden. Here he was warmly received by his Swedish colleagues. Soon he became a member of the university and school committees formed by Danish refugees, and through his gift for encouragement and his resourcefulness, he was able to do much for the Danish youth in Sweden.

After the war, as chairman of a committee for help to Poland, he again undertook a great humanitarian work.

In international mathematical circles and in the academic life of his own country, wherever he moved, Harald Bohr exerted an extraordinary influence. When he had something close to his heart he would bring his whole force to bear on it, and he was irresistible in effecting it. He did not wish to take on any regular administrative duties, though with his sharp eye for essentials and deep understanding of the human factor in every question, he would have been excellently suited for such a task. However, when after the war the position of Provost of Regensen (an old student collegium of the university) became vacant, he felt a wish to assume this post. He made his home here among the students and his warm interest in their lives brought the students very close to him.

When one came into the old courtyard to see him, one found him in his study, worthy of the traditions of the place, puffing on his eternal cigar and often in dressing-gown late in the day. The reason for the latter was, to be sure, a sad one, namely that

he often did not feel well. From his youth, he suffered intermittently from an internal malady, for which he sought a cure through the years in vain. When he said, 'I have been a bit tired and indisposed', one knew what that meant. Yet he always returned refreshed and strengthened from his stays in hospitals and convalescent homes. His spirits were not at all affected by his illness, and only those nearest to him actually understood it.

In the last year or so, it was clear to him that his illness was becoming critical; but even last year when he took part in the International Congress of Mathematicians in America, everyone had the impression that he was in full vigour. However, just after NewYear's Day he had to enter a hospital to undergo an operation, which he did not survive. He remained himself to the last, and those who came to see him in the hospital went away cheered as always. Through his passing we have suffered a great and irreparable loss.

THE PUBLICATIONS OF HARALD BOHR CHRONOLOGICALLY ARRANGED

Reviews, abstracts, problems set and solved, and newspaper articles, have not been included in this list. Translations of Danish titles are given in square brackets.

1908. Recherches sur la multiplication de deux intégrales définies prises entre des limites infinies. *Oversigt over Det Kgl. Danske Videnskabernes Selskabs Forhandlinger* 1908, 213–232 [1–20]. — G 1, volume III.

1909. Sur la série de Dirichlet. *Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences, Paris* 148 (1909), 75–80. — A 1, volume I.

1909. Ueber die Summabilität Dirichletscher Reihen. *Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse* 1909, 247–262. — A 2, volume I.

1909. Om en Udvidelse af en kendt Konvergenssætning. [On an extension of a known convergence theorem.] *Nyt Tidsskrift for Matematik B* 20 (1909), 1–4. — G 2, volume III. English summary, S 22, volume III.

1910. Bidrag til de Dirichlet'ske Rækkers Theori. Afhandling for den filosofiske Doktorgrad. [Contributions to the theory of Dirichlet series. Dissertation for the degree of Doctor of Philosophy.] I Kommission hos G. E. C. Gad, København 1910, i–xii, 1–136. — A 3, volume I. English translation, S 1, volume III.

1910. Sur la convergence des séries de Dirichlet. *Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences, Paris* 151 (1910), 375–377. — A 4, volume I.

1910. Über die Summabilitätsgrenzgerade der Dirichlet'schen Reihen. *Sitzungsberichte der Kaiserlichen Akademie der Wissenschaften, Wien, Mathematisch-Naturwissenschaftliche Klasse* 119 II a (1910), 1391–1397. — A 5, volume I.

1910. Über das Verhalten von $\zeta(s)$ und $\zeta_\kappa(s)$ in der Nähe der Geraden $\sigma = 1$. — With E. Landau. *Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse* 1910, 303–330. — B 1, volume I.

1910. En Sætning om ζ-Funktionen. [A theorem on the ζ-function.] *Nyt Tidsskrift for Matematik B* 21 (1910), 60–66. — B 2, volume I. English summary, S 4, volume III.

1911. Beweis der Existenz Dirichletscher Reihen, die Nullstellen mit beliebig grosser Abszisse besitzen. *Rendiconti del Circolo Matematico di Palermo* 31 (1911), 235–243. — A 6, volume I.

1911. Sur l'existence de valeurs arbitrairement petites de la fonction $\zeta(s) = \zeta(\sigma+it)$ de Riemann pour $\sigma > 1$. *Oversigt over Det Kgl. Danske Videnskabernes Selskabs Forhandlinger* 1911, 201–208 [1–8]. — B 3, volume I.

1911. Über die Zetafunktion. — With E. Landau. *Rendiconti del Circolo Matematico di Palermo* 32 (1911), 278–285. — B 4, volume I.

1911. Om de Værdier, den Riemann'ske Funktion $\zeta(\sigma+it)$ antager i Halvplanen $\sigma > 1$. [On the values taken by the Riemann function $\zeta(\sigma+it)$ in the half-plane $\sigma > 1$.] *Beretning om den Anden Skandinaviske Matematikerkongres i Kjøbenhavn* 1911, 113–121. — B 5, volume I. English summary, S 5, volume III.

1911. Über das Verhalten von $\zeta(s)$ in der Halbebene $\sigma > 1$. *Nachrichten von der Königlichen Gesellschaft*

der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse 1911, 409–428. — B 6, volume I.

1912. Lösung des absoluten Konvergenzproblems einer allgemeinen Klasse Dirichletscher Reihen. *Acta Mathematica* 36 (1912), 197-240. — A 7, volume I.

1912. Sur la fonction $\zeta(s)$ dans le demi-plan $\sigma > 1$. *Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences, Paris* 154 (1912), 1078–1081. — B 7, volume I.

1912. Über die Funktion $\frac{\zeta'}{\zeta}(s)$. *Journal fur die Reine und Angewandte Mathematik* 141 (1912), 217–234. — B 8, volume I.

1912. Et nyt Bevis for, at den Riemann'ske Zetafunktion $\zeta(s) = \zeta(\sigma+it)$ har uendelig mange Nulpunkter indenfor Parallelstrimlen $0 \leqq \sigma \leqq 1$. [A new proof that the Riemann zeta-function $\zeta(s) = \zeta(\sigma+it)$ has infinitely many zeros in the parallel strip $0 \leqq \sigma \leqq 1$.] *Nyt Tidsskrift for Matematik B* 23 (1912), 81–85. — B 9, volume I. English summary, S 6, volume III.

1913. Über die gleichmässige Konvergenz Dirichletscher Reihen. *Journal für die Reine und Angewandte Mathematik* 143 (1913), 203–211. — A 8, volume I.

1913. Über die Bedeutung der Potenzreihen unendlich vieler Variabeln in der Theorie der Dirichletschen Reihen $\sum \frac{a_n}{n^s}$. *Nachrichten von der Koniglichen Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse* 1913, 441–488. — A 9, volume I.

1913. Darstellung der gleichmässigen Konvergenzabszisse einer Dirichletschen Reihe $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ als Funktion der Koeffizienten der Reihe. *Archiv der Mathematik und Physik* (3) 21 (1913), 326–330. — A 10, volume I.

1913. Ein Satz über Dirichletsche Reihen. *Sitzungsberichte der Koniglich Bayerischen Akademie der Wissenschaften, Mathematisch-Physikalische Klasse* 1913, 557–562. — A 11, volume I.

1913. Note sur la fonction zéta de Riemann $\zeta(s) = \zeta(\sigma+it)$ sur la droite $\sigma = 1$. *Oversigt over Det Kgl. Danske Videnskabernes Selskabs Forhandlinger* 1913, 3–11 [1–9]. — B 10, volume I.

1913. Beiträge zur Theorie der Riemannschen Zetafunktion. — With E. Landau. *Mathematische Annalen* 74 (1913), 3–30. — B 11, volume I.

1913. Sur la fonction $\zeta(s)$ dans le voisinage de la droite $\sigma = \frac{1}{2}$. — With E. Landau and J. E. Littlewood. *Académie Royale de Belgique, Bulletin de la Classe des Sciences* 1913, 1144–1175. — B 12, volume I.

1913. Om Addition af uendelig mange konvekse Kurver. With French summary: Sur l'addition d'un nombre infini de courbes convexes. *Oversigt over Det Kgl. Danske Videnskabernes Selskabs Forhandlinger* 1913, 325–366 [1–42]. — F 1, volume III.

1914. Einige Bemerkungen uber das Konvergenzproblem Dirichletscher Reihen. *Rendiconti del Circolo Matematico di Palermo* 37 (1914), 1–16. — A 12, volume I.

1914. Ein Satz uber Dirichletsche Reihen mit Anwendung auf die ζ-Funktion und die L-Funktionen. — With E. Landau. *Rendiconti del Circolo Matematico di Palermo* 37 (1914), 269—272. — B 13, volume I.

1914. Sur les zéros de la fonction $\zeta(s)$ de Riemann. — With E. Landau. *Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences, Paris* 158 (1914), 106–110. — B 14, volume I.

1914. Neue Anwendungen der Theorie der Diophantischen Approximationen auf die Riemannsche Zetafunktion. — With R. Courant. *Journal für die Reine und Angewandte Mathematik* 144 (1914), 249–274. — B 15, volume I.

1914. Sur la fonction $\zeta(s)$ de Riemann. *Société Mathématique de France, Comptes Rendus des Séances* 1914, 52–66. — B 16, volume I.

1914. Sur la fonction $\zeta(s)$ de Riemann. *Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences, Paris* 158 (1914), 1986–1988. — B 17, volume I.

1914. A theorem concerning power series. *Proceedings of the London Mathematical Society* (2) 13 (1914), 1-5. — E 1, volume III.

1915. Die Riemannsche Zetafunktion. (Ein Bericht über den gegenwärtigen Standpunkt der Theorie von funktionentheoretischem Gesichtspunkte aus.) *Jahresbericht der Deutschen Mathematiker-Vereinigung* 24 (1915), 1–17. — B 18, volume I.

1915. Zur Theorie der Riemann'schen Zetafunktion im kritischen Streifen. *Acta Mathematica* 40 (1915), 67–100. — B 19, volume I.

1915. Lærebog i matematisk Analyse. (En foreløbig Udarbejdelse til Brug ved Forelæsninger paa Polyteknisk Læreanstalt.) [A textbook of mathematical analysis. (A preliminary exposition for use in lectures at the College of Technology.)] — With J. Mollerup. I. Elementær-algebraiske Undersøgelser med Anvendelse paa den analytiske Plangeometri og Rumgeometri. [Elementary algebraic investigations with application to plane and solid analytic geometry.] Jul. Gjellerup, København 1915, i–iv, 1–493. — Omitted.

1916. En funktionsteoretisk Bemærkning. [A function-theoretic remark.] *Nyt Tidsskrift for Matematik B* 27 (1916), 73–78. — E 2, volume III. English summary, S 16, volume III.

1916. Über die Koeffizientensumme einer beschränkten Potenzreihe. *Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse* 1916, 276–291. — E 3, volume III.

1916. Lærebog i matematisk Analyse. (En foreløbig Udarbejdelse til Brug ved Forelæsninger paa Polyteknisk Læreanstalt.) [A textbook of mathematical analysis. (A preliminary exposition for use in lectures at the College of Technology.)] — With J. Mollerup. II. Læren om de reelle Funktioner med Anvendelse paa den analytiske Plangeometri og Rumgeometri. [The theory of real functions with application to plane and solid analytic geometry.] Jul. Gjellerup, København 1916, i–ii, 1–908. — Omitted.

1917. Über die Koeffizientensumme einer beschränkten Potenzreihe. (Zweite Mitteilung.) *Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse* 1917, 119–128. — E 4, volume III.

1917. Nogle Mindeord om Professor P. C. V. Hansen. [Professor P.C.V.Hansen in memoriam.] *Ingeniøren* 26 (1917), 167. — Omitted.

1917. Nyere Undersøgelser over Integralregningen. Tre Foredrag holdt i Matematisk Forening. [Recent investigations on the theory of integration. Three lectures given to the Mathematical Society.] — With A.F.Andersen and J.Mollerup. København 1917, i–ii, 1–63. — Omitted.

1918. Zur Theorie der allgemeinen Dirichletschen Reihen. *Mathematische Annalen* 79 (1918), 136–156. — A 13, volume I.

1918. Über streckentreue und konforme Abbildung. *Mathematische Zeitschrift* 1 (1918), 403–420. — E 5, volume III.

1918. Lærebog i matematisk Analyse. (En foreløbig Udarbejdelse til Brug ved Forelæsninger paa Polyteknisk Læreanstalt.) [A textbook of mathematical analysis. (A preliminary exposition for use in lectures at the College of Technology.)] — With J. Mollerup. III. Grænseprocesser. [Limit processes.] Jul. Gjellerup, København 1918, i–ii, 1–182. — Omitted.

1918. Lærebog i matematisk Analyse. (En foreløbig Udarbejdelse til Brug ved Forelæsninger paa Polyteknisk Læreanstalt.) [A textbook of mathematical analysis. (A preliminary exposition for use in lectures at the College of Technology.)] — With J. Mollerup. IV. Læren om de komplekse Tal. [The theory of complex numbers.] Jul. Gjellerup, København 1918, i, 1–158. — Omitted.

1919. Om den Hadamard'ske »Hulsætning«. [On the Hadamard 'gap theorem'.] *Matematisk Tidsskrift B* 1919, 15–21. — E 6, volume III. English summary, S 17, volume III.

1919. Cesàro's Summabilitetsmetode. Tre Foredrag holdt i Matematisk Forening. [Cesàro's method of summability. Three lectures given to the Mathematical Society.] — With A.F.Andersen and J.Mollerup. Jul. Gjellerup, København 1919, i–ii, 1–51, 1–20, 1–25. — Omitted.

1920. Et Eksempel paa en Bevismetode. [An example of a method of proof.] *Matematisk Tidsskrift B* 1920, 46. — E 7, volume III. English summary, S 18, volume III.

1920. Elementerne af de algebraiske Tals Teori. Fire Foredrag holdt i Matematisk Forening. [The ele-

ments of the theory of algebraic numbers. Four lectures given to the Mathematical Society.] — With T. Bonnesen, J. Mollerup, and J. Pál. Jul. Gjellerup, København 1920, i–ii, 1–115. — Omitted.

1920. Lærebog i matematisk Analyse. [A textbook of mathematical analysis.] — With J. Mollerup. I. Elementær-algebraiske Undersøgelser med Anvendelse paa den analytiske Plangeometri og Rumgeometri. [Elementary algebraic investigations with application to plane and solid analytic geometry.] Jul. Gjellerup, København 1920, 1–328, i–iv. — Omitted.

1921. Nogle Bemærkninger om de Dirichletske Rækkers ligelige Konvergens. [Some remarks on the uniform convergence of Dirichlet series.] *Matematisk Tidsskrift B* 1921, 51–55. — A 14, volume I. English summary, S 2, volume III.

1921. Lærebog i matematisk Analyse. [A textbook of mathematical analysis.] — With J. Mollerup. II. Læren om de reelle Funktioner med Anvendelse paa den analytiske Plangeometri og Rumgeometri. [The theory of real functions with application to plane and solid analytic geometry.] Jul. Gjellerup, København 1921, 1–682, i–vi. — Omitted.

1922. Über eine quasi-periodische Eigenschaft Dirichletscher Reihen mit Anwendung auf die Dirichletschen L-Funktionen. *Mathematische Annalen* 85 (1922), 115–122. Also in *Festschrift David Hilbert zu seinem sechzigsten Geburtstag gewidmet*, Julius Springer, Berlin 1922, 115–122. — A 15, volume I.

1922. Über diophantische Approximationen und ihre Anwendungen auf Dirichlet'sche Reihen, besonders auf die Riemann'sche Zetafunktion. *Matematikerkongressen i Helsingfors* 1922. *Den Femte Skandinaviska Matematikerkongressen*, 131–154. — B 20, volume I.

1922. Another proof of Kronecker's theorem. *Proceedings of the London Mathematical Society* (2) 21 (1922), 315–316. — D 1, volume III.

1922. Axel Thue. 1863–1922. *Matematisk Tidsskrift B* 1922, 33 [in Danish]. — Omitted.

1922. Lærebog i matematisk Analyse. [A textbook of mathematical analysis.] — With J. Mollerup. III. Grænseprocesser. [Limit processes.] Jul. Gjellerup, København 1922, 1–178, i–ii. — Omitted.

1923. Über das Verhalten von $\frac{1}{\zeta(s)}$ auf der Geraden $\sigma = 1$. — With E. Landau. *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse* 1923, 71–80. — B 21, volume I.

1923. Sur les fonctions presque périodiques. *Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences, Paris* 177 (1923), 737–739. — C 1, volume II.

1923. Sur l'approximation des fonctions presque périodiques par des sommes trigonométriques. *Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences, Paris* 177 (1923), 1090–1092. — C 2, volume II.

1923. Über einen Satz von Edmund Landau. *Scripta Universitatis atque Bibliothecae Hierosolymitanarum, Mathematica et Physica* 1, no. 2 (1923), 1–5. — E 8, volume III.

1923. Om Addition af konvekse Kurver med givne Sandsynlighedsfordelinger. (En mængdeteoretisk Undersøgelse). [On addition of convex curves with given probability distributions. (A set-theoretical investigation.)] *Matematisk Tidsskrift B* 1923, 10–15. — F 2, volume III. English summary, S 20, volume III.

1923. Die neuere Entwicklung der analytischen Zahlentheorie. — With H. Cramér. *Encyklopädie der Mathematischen Wissenschaften* II 3 (1923), 722–849. — H, volume III.

1923. Lærebog i matematisk Analyse. [A textbook of mathematical analysis.] — With J. Mollerup. IV. Grundtræk af Læren om de komplekse Tal. [Elements of the theory of complex numbers.] Jul. Gjellerup, København 1923, 1–128, i–ii. — Omitted.

1924. Nachtrag zu unseren Abhandlungen aus den Jahrgängen 1910 und 1923. — With E. Landau. *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse* 1924, 168–172. — B 22, volume I.

1924. Zur Theorie der fast periodischen Funktionen. I. Eine Verallgemeinerung der Theorie der Fourierreihen. *Acta Mathematica* 45 (1924), 29–127. — C 3, volume II.

1924. Neuer Beweis eines allgemeinen Kronecker'schen Approximationssatzes. *Det Kgl. Danske Videnskabernes Selskab, Mathematisk-Fysiske Meddelelser* 6, no. 8 (1924), 1–8. — D 2, volume III.

1925. En Sætning om Fourierrækker for næsten-periodiske Funktioner. [A theorem on Fourier series of almost periodic functions.] *Matematisk Tidsskrift B* 1925, 31–37. — C 4, volume II. English summary, S 7, volume III.

1925. Einige Sätze über Fourierreihen fastperiodischer Funktionen. *Mathematische Zeitschrift* 23 (1925), 38–44. — C 5, volume II.

1925. Fastperiodische Funktionen. *Jahresbericht der Deutschen Mathematiker-Vereinigung* 34 (1925), 25–40. — C 6, volume II.

1925. Zur Theorie der fastperiodischen Funktionen. II. Zusammenhang der fastperiodischen Funktionen mit Funktionen von unendlich vielen Variabeln; gleichmässige Approximation durch trigonometrische Summen. *Acta Mathematica* 46 (1925), 101–214. — C 7, volume II.

1925. Sur les fonctions presque périodiques d'une variable complexe. *Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences, Paris* 180 (1925), 645–647. — C 8, volume II.

1925. Über allgemeine Fourier- und Dirichletentwicklungen. *Matematikerkongressen i København* 1925, *Den Sjette Skandinaviske Matematikerkongres*, 173–190. — C 9, volume II.

1925. Sur une classe de transcendantes entières. *Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences, Paris* 181 (1925), 766–768. — C 10, volume II.

1925. Unendlich viele lineare Kongruenzen mit unendlich vielen Unbekannten. *Det Kgl. Danske Videnskabernes Selskab, Mathematisk-Fysiske Meddelelser* 7, no. 1 (1925), 1–42. — D 3, volume III.

1925. En Bemærkning om Ligninger af første Grad. [A remark on equations of the first degree.] *Matematisk Tidsskrift A* 1925, 29–32. — Omitted.

1926. En Klasse hele transcendente Funktioner. [A class of integral functions.] *Matematisk Tidsskrift B* 1926, 41–45. — C 11, volume II. English summary, S 8, volume III.

1926. Zur Theorie der fastperiodischen Funktionen. III. Dirichletentwicklung analytischer Funktionen. *Acta Mathematica* 47 (1926), 237–281. — C 12, volume II.

1926. On the explicit determination of the upper limit of an almost-periodic function. *The Journal of the London Mathematical Society* 1 (1926), 109–112. — C 13, volume II.

1926. Ein Satz uber analytische Fortsetzung fastperiodischer Funktionen. *Journal fur die Reine und Angewandte Mathematik* 157 (1926), 61–65. — C 14, volume II.

1926. Allgemeine Fourier- und Dirichletentwicklungen. *Abhandlungen aus dem Mathematischen Seminar der Hamburgischen Universität* 4 (1926), 366—374. — C 15, volume II.

1926. Sur le théorème d'unicité dans la théorie des fonctions presque périodiques. *Bulletin des Sciences Mathématiques* (2) 50, part 1 (1926), 135-139. — C 16, volume II.

1926. Über lineare Differentialgleichungen mit konstanten Koeffizienten und fastperiodischer rechter Seite. — With O. Neugebauer. *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse* 1926, 8–22. — C 17, volume II.

1927. Some remarks on generalisations of almost periodic functions. — With A. Besicovitch. *Det Kgl. Danske Videnskabernes Selskab, Mathematisk-Fysiske Meddelelser* 8, no. 5 (1927), 1–31. — C 18, volume II.

1927. On the limit values of analytic functions. *The Journal of the London Mathematical Society* 2 (1927), 180–181. — E 9, volume III.

1928. On almost periodic properties of translation numbers. — With A. Besicovitch. *The Journal of the London Mathematical Society* 3 (1928), 172–176. — C 19, volume II.

1928. Grenzperiodische Funktionen. *Acta Mathematica* 52 (1928), 127–133. — C 20, volume II.

1928. Über die Verallgemeinerungen fastperiodischer Funktionen. *Mathematische Annalen* 100 (1928), 357–366. — C 21, volume II.

1928. Bericht über die Theorie der fastperiodischen Funktionen. *Atti del Congresso Internazionale dei Matematici, Bologna* 1928, II, 283–288. — C 22, volume II.

1928. Nogle Bemærkninger om formel Regning. [Some remarks on formal calculation.] *Matematisk Tidsskrift B* 1928, 7–18. — G 3, volume III. English summary, S 23, volume III.

1929. Sur un problème de M. Borel. *Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences, Paris* 189 (1929), 826–827. — E 10, volume III.

1929. Über ganze transzendente Funktionen von einem besonderen Typus. (Beispiel einer allgemeinen Konstruktionsmethode.) *Sitzungsberichte der Preussischen Akademie der Wissenschaften, Physikalisch-Mathematische Klasse* 1929, 565–571. — E 11, volume III.

1929. Om Sandsynlighedsfordelinger ved Addition af konvekse Kurver. [On probability distributions by addition of convex curves.] — With B. Jessen. *Det Kgl. Danske Videnskabernes Selskabs Skrifter, Naturvidensk. og Mathem. Afd.* (8) 12, no. 3 (1929), 325–406 [1–82]. — F 3, volume III. English summary, S 21, volume III.

1930. Über die Werteverteilung der Riemannschen Zetafunktion. — With B. Jessen. (Erste Mitteilung. Das Verhalten der Funktion in der Halbebene $\sigma > 1$.) *Acta Mathematica* 54 (1930), 1–35. — B 23, volume I.

1930. Über analytische fastperiodische Funktionen. *Mathematische Annalen* 103 (1930), 1–14. — C 23, volume II.

1930. Kleinere Beiträge zur Theorie der fastperiodischen Funktionen. I–II. I. Über die Argumentvariation einer fastperiodischen Funktion. II. Eine Bemerkung uber die Wertverteilung fastperiodischer Funktionen. *Det Kgl. Danske Videnskabernes Selskab, Mathematisk-Fysiske Meddelelser* 10, no. 10 (1930), 1–17. — C 24, volume II.

1930. Über ganze transzendente Funktionen, die auf jeder durch den Nullpunkt gehenden Geraden beschränkt sind. *Opuscula Mathematica Andreae Wiman Dedicata, Festskrift tillägnad Anders Wiman*, Lundequist, Upsala 1930, 39–46. — E 12, volume III.

1930. Matematikens ideale Elementer. (Foredrag holdt ved den polytekniske Læreanstalts Aarsfest 1930.) [The ideal elements of mathematics. (A lecture delivered at the anniversary celebration of the College of Technology 1930.)] *Matematisk Tidsskrift A* 1930, 31–45. — Omitted.

1931. Kleinere Beitrage zur Theorie der fastperiodischen Funktionen. III–IV. III. Dirichletsche Reihen und fastperiodische Funktionen. IV. Über die Funktionalgleichung $t = f(t) + \varphi(f(t))$ bei einer gegebenen fastperiodischen Funktion $\varphi(x)$. *Det Kgl. Danske Videnskabernes Selskab, Mathematisk-Fysiske Meddelelser* 10, no. 12 (1931), 1–15. — C 25, volume II.

1931. On the inverse function of an analytic almost periodic function. *Annals of Mathematics* (2) 32 (1931), 247–260. — C 26, volume II.

1931. Almost periodicity and general trigonometric series. — With A. S. Besicovitch. *Acta Mathematica* 57 (1931), 203–292. — C 27, volume II.

1931. Niels Nielsen. 1865–1931. *Matematisk Tidsskrift B* 1931, 41–45 [in Danish]. — Omitted.

1932. Über die Werteverteilung der Riemannschen Zetafunktion. — With B. Jessen. (Zweite Mitteilung. Das Verhalten der Funktion im Streifen $\frac{1}{2} < \sigma \leqq 1$.) *Acta Mathematica* 58 (1932), 1–55. — B 24, volume I.

1932. Fastperiodische Funktionen. *Ergebnisse der Mathematik und ihrer Grenzgebiete* I 5, Julius Springer, Berlin 1932, i–iv, 1–96 [417–516]. — C 28, volume II.

1932. Ueber fastperiodische ebene Bewegungen. *Commentarii Mathematici Helvetici* 4 (1932), 51–64. — C 29, volume II.

1932. Über fastperiodische Bewegungen auf einem Kreis. — With B. Jessen. *Annali della R. Scuola Normale Superiore di Pisa, Scienze Fisiche e Matematiche* (2) 1 (1932), 385–398. — C 30, volume II.

1932. Fastperiodische Funktionen einer komplexen Veränderlichen. *Verhandlungen des Internationalen Mathematiker-Kongresses Zurich* 1932, I, 195–208. — C 31, volume II.

1932. To nye simple Beviser for Kroneckers Sætning. [Two new simple proofs of Kronecker's theorem.] — With B. Jessen. *Matematisk Tidsskrift B* 1932, 53–58. — D 4, volume III. English summary, S 14, volume III.

1932. One more proof of Kronecker's theorem. — With B. Jessen. *The Journal of the London Mathematical Society* 7 (1932), 274–275. — D 5, volume III.

1932. Niels Nielsen. 1865–1931. *Oversigt over Det Kongelige Danske Videnskabernes Selskabs Virksomhed* 1931-32, 137–145 [in Danish]. — Omitted.

1932. Niels Nielsen, 1865–1931. *Festskrift udgivet af Københavns Universitet i Anledning af Universitetets Aarsfest* 1932, 123–128 [in Danish]. — Omitted.

1933. Stabilitet og Næstenperiodicitet. [Stability and almost periodicity.] *Matematisk Tidsskrift B* 1933, 21–25. — C 32, volume II. English summary, S 9, volume III.

1933. Zum Kroneckerschen Satz. — With B. Jessen. *Rendiconti del Circolo Matematico di Palermo* 57 (1933), 123–129. — D 6, volume III.

1933. Endnu engang Kroneckers Sætning. [Once more Kronecker's theorem.] *Matematisk Tidsskrift B* 1933, 59–64. — D 7, volume III. English summary, S 15, volume III.

1934. Mean-value theorems for the Riemann zeta-function. — With B. Jessen. *Quarterly Journal of Mathematics, Oxford Series* 5 (1934), 43–47. — B 25, volume I.

1934. Den seneste Udvikling af Læren om næstenperiodiske Funktioner. [The latest development of the theory of almost periodic functions.] *Åttonde Skandinaviska Matematikerkongressen i Stockholm* 1934, 106. — C 33, volume II. English summary, S 10, volume III.

1934. Anwendung einer Landauschen Beweismethode auf den Kroneckerschen Approximationssatz. *Mathematische Zeitschrift* 38 (1934), 312–314. — D 8, volume III.

1934. Again the Kronecker theorem. *The Journal of the London Mathematical Society* 9 (1934), 5–6. — D 9, volume III.

1935. Kleinere Beiträge zur Theorie der fastperiodischen Funktionen. V. Über den Quotienten zweier analytischer fastperiodischer Funktionen. *Det Kgl. Danske Videnskabernes Selskab, Mathematisk-Fysiske Meddelelser* 13, no. 8 (1935), 1–13. — C 34, volume II.

1935. Un théorème général sur l'intégration d'un polynome trigonométrique. *Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences, Paris* 200 (1935), 1276–1277. — C 35, volume II.

1935. Ein allgemeiner Satz über die Integration eines trigonometrischen Polynoms. *Prace Matematyczno-Fizyczne* 43 (1935), 273–288 [1–16]. — C 36, volume II.

1935. En Sætning om Fourierrækker. [A theorem on Fourier series.] *Matematisk Tidsskrift B* 1935, 77–81. — C 37, volume II. English summary, S 11, volume III.

1935. Über einen Satz von J. Pál. *Acta Litterarum ac Scientiarum Regiae Universitatis Hungaricae Francisco-Josephinae, Sectio Scientiarum Mathematicarum, Szeged* 7 (1935), 129–135. — E 13, volume III.

1935. The arithmetic and geometric means. *The Journal of the London Mathematical Society* 10 (1935), 114. — G 4, volume III.

1936. On the distribution of the values of the Riemann zeta function. — With B. Jessen. *American Journal of Mathematics* 58 (1936), 35–44. — B 26, volume I.

1936. Ein Satz über stabile Bewegungen in der Ebene. — With W. Fenchel. *Det Kgl. Danske Videnskabernes Selskab, Mathematisk-Fysiske Meddelelser* 14, no. 1 (1936), 1–15. — C 38, volume II.

1936. Kleinere Beiträge zur Theorie der fastperiodischen Funktionen. VI. Der Picardsche Satz. *Det Kgl. Danske Videnskabernes Selskab, Mathematisk-Fysiske Meddelelser* 14, no. 2 (1936), 1–8. — C 39, volume II.

1936. Kleinere Beiträge zur Theorie der fastperiodischen Funktionen. VII–VIII. VII. Stark stationäre und schwach stationäre Funktionen. VIII. Über den Logarithmus einer positiven fastperiodischen Funktion. *Det Kgl. Danske Videnskabernes Selskab, Mathematisk-Fysiske Meddelelser* 14, no. 7 (1936), 1–24. — C 40, volume II.

1936. Über die Umkehrung von analytischen fastperiodischen Funktionen. — With B. Jessen. *Mathematische Annalen* 113 (1936), 461–488. — C 41, volume II.

1936. Nogle Bemærkninger vedrørende Talteoriens Elementer. [Some remarks on the elements of the theory of numbers.] *Matematisk Tidsskrift A* 1936, 57–65. — Omitted.

1936. T. Bonnesen. 1873–1935. *Oversigt over Det Kongelige Danske Videnskabernes Selskabs Virksomhed* 1935–36, 87–102 [in Danish]. — Omitted.

1937. Algebraic equations with almost-periodic coefficients. — With D. A. Flanders. *Det Kgl. Danske Videnskabernes Selskab, Mathematisk-Fysiske Meddelelser* 15, no. 12 (1937), 1–49. — C 42, volume II.

1938. Algebraic functions of analytic almost periodic functions. — With D. A. Flanders. *Duke Mathematical Journal* 4 (1938), 779–787. — C 43, volume II.

1938. Über fastperiodische Bewegungen. *Yhdeksäs Skandinaavinen Matemaatikkokongressi Helsingissä* 1938. *Nionde Skandinaviska Matematikerkongressen i Helsingfors* 1938, 39–61. — C 44, volume II.

1938. Lærebog i matematisk Analyse. [A textbook of mathematical analysis.] — With J. Mollerup. Second revised edition by A. F. Andersen and R. Petersen. I. Lineær Algebra. [Linear algebra.] Jul. Gjellerup, København 1938, 1–148. — Omitted.

1939. Om Logaritme- og Eksponentialfunktion. To Foredrag holdt ved Matematiklærerforeningens Efteraarsmøde 1939. [On the logarithmic and exponential functions. Two lectures given at the autumn meeting of the Society of Mathematics Teachers 1939.] *Matematisk Tidsskrift A* 1939, 46–81. — Omitted.

1940. Zum Picardschen Satz. *Академия Наук УССР, Сборник Посвященный Памяти Академика Дмитрия Александровича Граве, Москва Ленинград* 1940, 29–33. Also printed in *Matematisk Tidsskrift B* 1940, 1–6. — E 14, volume III.

1940. Om det matematiske Grundlag for Sandsynlighedsregningen. Udarbejdet efter et Foredrag holdt i Aktuarforeningen. [On the mathematical foundations of the calculus of probability. Composed after a lecture given to the Society of Actuaries.] *Matematisk Tidsskrift A* 1940, 1–29. — Omitted.

1940. Lærebog i matematisk Analyse. [A textbook of mathematical analysis.] — With J. Mollerup. Second revised edition by A. F. Andersen and R. Petersen. II. Funktioner af een reel Variabel. [Functions of one real variable.] Jul. Gjellerup, København 1940, i-x, 1–420. — Omitted.

1942. Über das Koeffizientendarstellungsproblem Dirichletscher Reihen. *Det Kgl. Danske Videnskabernes Selskab, Matematisk-Fysiske Meddelelser* 20, no. 2 (1942), 1–12. — A 16, volume I.

1942. Om Potensrækker med Huller. En Pseudo-Kontinuitetsegenskab. [On power series with gaps. A property of pseudo-continuity.] *Matematisk Tidsskrift B* 1942, 1–11. — E 15, volume III. English summary, S 19, volume III.

1942. A. S. Bang. 1866–1942. *Matematisk Tidsskrift A* 1942, 77–78 [in Danish]. — Omitted.

1942. Lærebog i matematisk Analyse. [A textbook of mathematical analysis.] — With J. Mollerup. Second revised edition by A. F. Andersen and R. Petersen. III. Funktioner af flere reelle Variable. Uendelige Rækker. [Functions of several real variables. Infinite series.] Jul. Gjellerup, København 1942, i-viii, 1–340. — Omitted.

1942. Lærebog i matematisk Analyse. [A textbook of mathematical analysis.] — With J. Mollerup. Second revised edition by A. F. Andersen and R. Petersen. IV. Funktioner af en kompleks Variabel. Specielle Emner. [Functions of a complex variable. Special topics.] Jul. Gjellerup, København 1942, i-viii, 1–197. — Omitted.

1943. Et Eksempel paa Anvendelsen af Sandsynlighedsregning som Hjælpemiddel i den matematiske Analyse. [An example of the application of the calculus of probability as an aid in mathematical analysis.] *Festskrift til Professor, Dr. phil. J. F. Steffensen fra Kolleger og Elever paa hans 70 Aars Fødselsdag*, Den Danske Aktuarforening, København 1943, 29–33. — C 45, volume II. English summary, S. 12, volume III.

1943. Contribution to the theory of analytic almost periodic functions. On the behaviour of an analytic almost periodic function in the neighbourhood of a boundary for its almost periodicity. *Det Kgl. Danske Videnskabernes Selskab, Matematisk-Fysiske Meddelelser* 20, no. 18 (1943), 1–37. — C 46, volume II.

1944. On some types of functional spaces. A contribution to the theory of almost periodic functions. — With E. Følner. *Acta Mathematica* 76 (1944), 31–155. — C 47, volume III.

1944. Om *S*-næstenperiodiske Funktioner med lineært uafhængige Exponenter. [On *S*-almost periodic functions with linearly independent exponents.] *Norsk Matematisk Tidsskrift* 26 (1944), 33–40. — C 48, volume III. English summary, S 13, volume III.

1945. Om almindelige Konvergenskriterier for Rækker med positive Led. [On general convergence criteria for series with positive terms.] *Matematisk Tidsskrift B* 1945, 1–9. — G 5, volume III. English summary, S 24, volume III.

1945. Lærebog i matematisk Analyse. [A textbook of mathematical analysis.] — With A. F. Andersen and R. Petersen. I. Lineær Algebra. [Linear algebra.] Jul. Gjellerup, København 1945, 1–154. — Omitted.

1945. Lærebog i matematisk Analyse. [A textbook of mathematical analysis.] — With A. F. Andersen and R. Petersen. II. Funktioner af een reel Variabel. [Functions of one real variable.] Jul. Gjellerup, København 1945, 1–433. — Omitted.

1946. On some functional spaces. *Den* 10. *Skandinaviske Matematikerkongres i København* 1946, 313–319. — C 49, volume III.

1946. Indledningstale ved Aabningsmødet. [Introductory address at the opening meeting.] *Den* 10. *Skandinaviske Matematikerkongres i København* 1946, xviii–xxv. — Omitted.

1947. Mean motions and almost periodic functions. — With B. Jessen. *Colloques Internationaux du Centre National de la Recherche Scientifique*, XV, *Analyse harmonique, Nancy* 1947, 75–84. — C 50, volume III.

1947. Et Tilbageblik. [Looking backward.] *Matematisk Tidsskrift A* 1947, 1–27. — Danish text omitted. English translation in volume I (as Preface).

1947. Lærebog i matematisk Analyse. [A textbook of mathematical analysis.] — With A. F. Andersen and R. Petersen. III. Funktioner af flere reelle Variable. Uendelige Rækker. [Functions of several real variables. Infinite series.] Jul. Gjellerup, København 1947, 1–335. — Omitted.

1948. Infinite systems of linear congruences with infinitely many variables. — With E. Følner. *Det Kgl. Danske Videnskabernes Selskab, Matematisk-Fysiske Meddelelser* 24, no. 12 (1948), 1–35. — D 10, volume III.

1948. On a structure theorem for closed modules in an infinite-dimensional space. — With E. Følner. *Studies and Essays presented to R. Courant on his 60th birthday*, Interscience, New York 1948, 45–62. — D 11, volume III.

1949. On the convergence problem for Dirichlet series. *Det Kgl. Danske Videnskabernes Selskab, Matematisk-Fysiske Meddelelser* 25, no. 6 (1949), 1–18. — A 17, volume I.

1949. On almost periodic functions and the theory of groups. *American Mathematical Monthly* 56 (1949), 595–609. — C 51, volume III.

1949. Træk af den matematiske Videnskabs Udvikling. [Features of the development of mathematics.] *Matematisk Tidsskrift A* 1949, 49–59. — Omitted.

1949. Forelæggelse af en ny Udgave af Zeuthens Matematikens Historie. [On a new edition of Zeuthen's Matematikens Historie.] *Den 11te Skandinaviske Matematikerkongress i Trondheim* 1949, 195–200. Also printed in *Matematisk Tidsskrift A* 1949, 60–66. — Omitted.

1949. Lærebog i matematisk Analyse. [A textbook of mathematical analysis.] — With A. F. Andersen and R. Petersen. IV. Funktioner af en kompleks Variabel. Specielle Emner. [Functions of a complex variable. Special topics.] Jul. Gjellerup, København 1949, 1–264. — Omitted.

1950. Zur Theorie der Dirichletschen Reihen. *Mathematische Zeitschrift* 52 (1950), 709–722. — A 18, volume I.

1950. On multiplication of summable Dirichlet series. *Matematisk Tidsskrift B* 1950, 71–75. — A 19, volume I.

1950. On limit periodic functions of infinitely many variables. *Acta Scientiarum Mathematicarum, Szeged* 12 B (1950), 145–149. — C 52, volume III.

1950. A survey of the different proofs of the main theorems in the theory of almost periodic functions. *Proceedings of the International Congress of Mathematicians, Cambridge, Massachusetts, U.S.A.* 1950, I, 339–348. — C 53, volume III.

1950. Johannes Hjelmslev in memoriam. 1873–1950. *Acta Mathematica* 83 (1950), vii–ix [in English]. — Omitted.

1950. Address delivered at the presentation of the Fields medals. *Proceedings of the International Congress of Mathematicians, Cambridge, Massachusetts, U.S.A.* 1950, I, 127–134. — Omitted.

1951. En bemærkning om Dirichletske rækkers ligelige konvergens. [A remark on the uniform convergence of Dirichlet series.] *Matematisk Tidsskrift B* 1951, 1–8. — A 20, volume I. English summary, S 3, volume III.

1951. A study on the uniform convergence of Dirichlet series and its connection with a problem concerning ordinary polynomials. *Kungl. Fysiografiska Sällskapets i Lund Förhandlingar* 21, no. 12

(1951), 105-118 [1–14]. Also in *Meddelanden från Lunds Universitets Matematiska Seminarium, Supplementband tillägnat Marcel Riesz* 1952, 21–34. — A 21, volume I.

1951. On the definition of almost periodicity. *Journal d'Analyse Mathématique* 1 (1951), 11–27. — C 54, volume III.

1952. On the summability function and the order function of Dirichlet series. *Det Kgl. Danske Videnskabernes Selskab, Matematisk-Fysiske Meddelelser* 27, no. 4 (1952), 1–39. — A 22, volume I.

NOTES

The references to other authors have been largely confined to contributions directly connected with specific results of the various papers.

Alterations, other than typographical rectifications, made in a paper before reproduction are indicated at the end of the notes to the paper.

VOLUME I

A 1. — The results communicated in this paper are presented in detail in A 2 and, in particular, in A 3 (S 1).

A 2. — For a detailed exposition, see A 3 (S 1).

p. 260. The summability of factorial series was treated by Nørlund; see *Encykl. Math. Wiss.* II 3 (1923), 684, where references are given.

pp. 261–262. The investigations on Dirichlet series of type $\sum a_n e^{-\lambda_n s}$ and on a class of definite integrals, referred to as forthcoming, have not been published; cf. A 18, 710.

A 3. (S 1.) — p. viii. (p. 2.) For the theory of summability of arbitrary (integral or non-integral) order for Dirichlet series of type $\sum a_n e^{-\lambda_n s}$, developed by M. Riesz, see H, 753–759, where references are given. For the later development, see Chandrasekharan and Minakshisundaram, *Typical means*, Oxford University Press 1952.

pp. 23–37. (pp. 23–36.) For a further discussion of the convergence problem for Dirichlet series, see A 17. Examples of Dirichlet series generalizing those of Theorems XVI and XVII are given in A 16, A 17, A 18, and A 22.

p. 61. (p. 53.) Lemma 1a is known as the Bohr-Hardy theorem. For the extensive literature on this theorem and its generalizations, including investigations by Fekete, Anderson, Schur, Bosanquet, and others, see Hardy, *Divergent series*, Oxford University Press 1949, 146, and Bosanquet, *Proc. London Math. Soc.* (2) 50 (1949), 482–496.

p. 94. (p. 82.) The series considered in Theorem II has been further investigated in A 22, 7–8.

pp. 99–110. (pp. 86–97.) The extensions of Theorems I and II to general r, due to Riesz, state that $\lambda_r - \lambda_{r+k} \leqq k$ for $r \geqq 0$ and $k \geqq 0$, and that λ_r is a convex function of r for $r \geqq 0$. As a generalization of Theorem V it is shown in A 22 that these conditions are also sufficient.

p. 114. (p. 100.) Theorem I was extended to general r by Riesz, who further proved that if $f(s) = O(|t|^k)$ for $\sigma > \eta$, then $\lambda_k \leqq \eta$.

As pointed out (orally) by Professor Riesz, the Landau–Schnee theorem follows from this result and the convexity of λ_r. One has to observe first that λ_r is convex even for $r \geqq -1$, where λ_{-1} may be interpreted as the greatest lower bound of all σ for which $a_n n^{-\sigma} = O(n^{-1})$; cf. Bosanquet and Chow, *J. London Math. Soc.* 16 (1941), 46. Now, if $a_n = O(n^\delta)$ for every $\delta > 0$, and if $f(s) = O(|t|^k)$ for $\sigma > \eta$, we find $\lambda_{-1} \leqq 1$ and $\lambda_k \leqq \eta$; hence, by the convexity, $\lambda_0 \leqq \dfrac{\lambda_k + k\lambda_{-1}}{1+k} \leqq \dfrac{\eta+k}{1+k}$, which is the Landau–Schnee theorem.

p. 118. (p. 103.) Theorem I is included in the following theorem, valid for arbitrary $r \geqq 0$: If $f(s) = O(|t|^{r+1+k})$ for $\sigma > \eta$, then $\lambda_{r+1} \leqq \dfrac{\eta + k\lambda_r}{1+k}$. By the preceding argument this theorem follows

immediately from Riesz's results. From the example in A 17, or from A 22, 7–8, it may be seen that the estimate is the best possible.

p. 127. (p. 111.) For the behaviour of the sum-function $f(s)$ in the neighbourhood of the line $\sigma = \Lambda$ in the case considered in Theorem V b, see A 5.

p. 131. (p. 115.) The series $\sum \mu_n n^{-s}$ is actually convergent for $\sigma > \theta$; cf. Littlewood, *C. R. Acad. Sci. Paris* 154 (1912), 263–266. Thus for this series $\lambda_0 = \Lambda$.

pp. 131–132. (pp. 115–116.) For further results on the multiplication of Dirichlet series, see A 18, 720–722, and A 19.

pp. 133–134. (pp. 116–117.) The theory of series with arbitrary indices, mentioned in Thesis 3, is presented in detail in G 5.

A 4. — Theorem I is included in Theorem I of A 8. For the proof of Theorem II, cf. Theorem III of B 1. Theorem III and Theorem IV for $\tau = -\infty$ are proved in A 7; Theorem IV for an arbitrary τ follows by use of the result of A 11.

A 7. — p. 219. The result mentioned at the end of § 3 is contained in Theorem 2 of A 13. For an extension of Theorem 5, in which the condition on the exponents is weakened, see Ostrowski, *Jber. Deutsch. Math. Ver.* 42 (1933), 160–165.

pp. 231 and 237. In connection with Theorems A and B, cf. the general result of Neder, *Ark. Mat. Astr. Fys.* 16, no. 20 (1922), 1–15, according to which the inequalities $-\infty \leqq \alpha \leqq \gamma \leqq \beta \leqq \infty$ and $-\infty \leqq \delta \leqq \gamma \leqq \infty$ satisfied by the abscissae $\alpha, \beta, \gamma, \delta$ of convergence, absolute convergence, uniform convergence, and boundedness, respectively, of a Dirichlet series $\sum a_n e^{-\lambda_n s}$ are the best possible.

Concerning Dirichlet series with linearly independent exponents, see also Mandelbrojt, *Bull. Soc. Math. France* 60 (1932), 208–220, and Aronszajn, *C. R. Acad. Sci. Paris* 199 (1934), 335–337.

A 8. — For further results on the uniform convergence of Dirichlet series, see A 10, A 14, and A 21; see also H, 726–727, 735, 739.

A 9. — Cf. H, 739–743.

p. 446. The problem of determining T was solved by Bohnenblust and Hille, *Ann. of Math.* (2) 32 (1931), 600–622, 33 (1932), 785–786, who proved that $T = \frac{1}{2}$. They further proved the following result: If σ_u is the abscissa of uniform convergence of the series $\sum a_n n^{-s}$, then the subseries consisting of those terms for which n is a product of at most m primes has an abscissa of absolute convergence $\leqq \sigma_u + \dfrac{m-1}{2\,m}$, and these inequalities are the best possible. An entirely different proof that $T = \frac{1}{2}$ was given by Hartman, *Amer. J. Math.* 61 (1939), 955–964.

p. 480. Theorem XI was extended by Kloosterman, *Dan. Mat. Fys. Medd.* 5, no. 6 (1923), 1–29, to Dirichlet series of the type $f(s) = \varphi\left(\sum \mathfrak{P}_n(p_n^{-s})\right)$, where $\mathfrak{P}_1, \mathfrak{P}_2, \ldots$ are power series without constant term, and φ is an integral function.

The method of this paper is applied to Dirichlet series of type $\sum a_n e^{-\lambda_n s}$ in A 13 and to the Riemann zeta-function in B 7 and B 8, and, in an extended form, in B 15, B 19, B 23, and B 24. For an application to mean motions, see C 50, 83.

A 10. — Cf. H, 727. The lemma on p. 327 may also be obtained as a corollary to the general theorem on exponential polynomials $f(s) = \sum a_n e^{-\lambda_n s}$ with positive λ_n which states that if $|f(s)| \leqq M$ on the line $\sigma = \sigma_0$, then $|f(s)| \leqq M$ in the half-plane $\sigma > \sigma_0$; cf. C 12, 245 (Theorem A), and C 13, § 4.

A 11. — For generalizations see Rogosinski, *Math. Ann.* 92 (1924), 104–114, and C 14.

A 12. — Cf. H, 737–739, and V. Bernstein, *Leçons sur les progrès récents de la théorie des séries de Dirichlet*, Gauthier-Villars, Paris 1933, esp. 27–29, 128–132, 194–202. For uniform ultra-convergence, see A 21, 109–110.

A 13. — The method of this paper is applied to the Riemann zeta-function in B 7 and B 8, and, in an extended form, in B 15, B 19, B 23, and B 24.

A 15. — The notion of quasi-periodicity considered in this paper is not identical with the notion of almost periodicity (C 12), since (i) it is not required that the inequalities $|f(s+i\tau_m)-f(s)| < \varepsilon$ are satisfied for all s in a strip, but only that they hold in a bounded set ω, and (ii) the τ_m are subjected

to a weaker condition than that of being relatively dense; (incidentally, in the definition of almost periodicity, this weaker condition is actually sufficient; cf. C 54).

A 17. — See the remark on the Landau-Schnee theorem in the notes to A 3 (S 1).

A 18. — The investigations of this paper are continued in A 19 and A 22.

A 19. — This paper completes the investigations in A 3, 37–39, 131–132 (S 1, 36–38, 115–116), and A 18, 720–722.

A 20. (S 3.) — The result of this paper is a very special case of the theorem of Neder referred to in the notes to A 7. The paper referred to in § 2 as forthcoming is A 21.

B 1. — pp. 305 and 312. For extensions of Theorems I and III, see B 6 and B 7. There is a different proof of Theorem I in B 22, 171.

p. 315. As shown by Littlewood (see B 21, 72), on the Riemann hypothesis Theorem V is best possible. The value of $\limsup \frac{|\zeta(1+it)|}{\log\log t}$ was estimated by Littlewood; see Titchmarsh, *The theory of the Riemann zeta-function*, Oxford University Press 1951, 165, 291.

p. 321. For applications of Theorem VI, see A 5, B 11, 28, and E 14.

B 2. (S 4.) — Theorem II is superseded by B 4. The note mentioned at the end of the paper has apparently not been published; the existence of arbitrarily small values of $\zeta(s)$ in the half-plane $\sigma > 1$ is proved in B 3.

B 3. — p. 204. For a simple proof of Kronecker's theorem, see D 9 (or D 1, D 4, D 5, D 6, D 7, D 8).

pp. 207–208. There is a proof of Theorem II, independent of Kronecker's theorem, in B 22, 170, and a refinement of Theorem III in B 10, 7.

B 6. — The result of this paper is superseded by B 7. The fact that $\zeta(s)$ takes all values $\neq 0$ in the half-plane $\sigma > 1$ was proved without the use of Kronecker's theorem by Landau, *Nachr. Ges. Wiss. Göttingen. Math. Phys. Kl.* 1933, 81–91; cf. D 8.

B 7. — For the application of the method of this paper to the function $\zeta'(s)/\zeta(s)$, see B 8. The method is applied to Dirichlet series in A 9 and A 13. For extensions of the method, see B 15, B 19, B 23, and B 24. For a more detailed discussion of the sets $M(\sigma_0)$, see B 26.

Concerning the closure of the range of analytic functions on curves tending to an essential singularity, see Borel, *C. R. Acad. Sci. Paris* 155 (1912), 201 (= *Méthodes et problèmes de théorie des fonctions*, Gauthier-Villars, Paris 1922, 144–145), Valiron, *J. Math. Pure Appl.* (9) 7 (1928), 113–166, Montel, *Publ. Math. Univ. Belgrade* 1 (1932), 157–169, Popoviciu, *Bull. Sci. Math.* (2) 60, part 1 (1936), 196–198.

B 8. — For a treatment of $\log \zeta(s)$ by the same method, see B 7.

The constant D was computed by Burrau, *J. Reine Angew. Math.* 142 (1912), 51–53, who found $D = 2.576076$.

B 9. (S 6.) — The second proof, mentioned at the end of the paper, has apparently not been published.

B 11. — For the literature on consequences of the Riemann hypothesis, see H, 775–777, and Titchmarsh, *The theory of the Riemann zeta-function*, Oxford University Press 1951, 282–328.

p. 27. Theorem V (and much more) is proved without the use of the Riemann hypothesis in B 19.

B 12. — Cf. H, 777.

p. 1154. For the quotation from Jensen's exposition of the theory of the gamma-function, cf. the English translation in *Ann. of Math.* (2) 17 (1916), 136.

B 13. — The theorem of this paper, or the improvement on it in B 14, is known as the Bohr-Landau theorem. For further improvements on it, due to Carlson, Littlewood, Titchmarsh, Ingham, and Selberg, see H, 747, 772, and Titchmarsh, *The theory of the Riemann zeta-function*, Oxford University Press 1951, 196–211.

B 14. — See the note to B 13.

B 15. — For the further development of the method of this paper, see B 19, B 23, and B 24. The method is applied to almost periodic functions in C 3, 103–112.

B 16. — This lecture is almost identical with B 18.

B 17. — The results announced in this paper are presented in detail in B 19.

B 18. — This lecture is almost identical with B 16.

B 19. — For the further development of the method of this paper, see B 23 and B 24.

B 20. — The results on the distribution of the values of the zeta-function announced on pp. 150–154 are presented in detail in B 23 and B 24.

B 21. — The result of this paper is superseded by B 22.

B 22. — The value of $\limsup \dfrac{1/|\zeta(1+it)|}{\log\log t}$ was estimated by Littlewood and Titchmarsh; see Titchmarsh, *The theory of the Riemann zeta-function*, Oxford University Press 1951, 165, 291.

B 23. — The distribution of the values of the zeta-function and, more generally, of almost periodic functions, has been the subject of investigations by Bochner, Buch, Favard, Hartman, Haviland, Jessen, van Kampen, Kershner, Tornehave, Wintner, and others. New methods have led to extensions of the results of the present paper and its continuation B 24. See the comprehensive expositions by Jessen and Wintner, *Trans. Amer. Math. Soc.* 38 (1935), 48–88, Jessen and Tornehave, *Acta Math.* 77 (1945), 137–279, and Borchsenius and Jessen, *Acta Math.* 80 (1948), 97–166. The first two of these papers contain extensive bibliographies. Some of the results of the last two papers are summarized in C 50.

B 24. — See the note to B 23. For an application of the first main theorem, see B 25.

B 26. — The outer boundary $A(\sigma)$ of the set $M(\sigma)$ was shown by Kershner and Wintner, *Amer. J. Math.* 58 (1936), 421–425, to be a regular analytic curve for every $\sigma > 1$. The inner boundary $B(\sigma)$ of $M(\sigma)$ (for $\sigma > C$) was discussed by Kershner, *Amer. J. Math.* 59 (1937), 167–174, who proved, among others, the theorem mentioned on p. 43. Cf. also van Kampen and Wintner, *Amer. J. Math.* 59 (1937), 175–204, and van Kampen, *Amer. J. Math.* 59 (1937), 679–695.

VOLUME II

C 1. — The results communicated in this paper are presented in detail in C 3.

The reprint differs from the original on p. 739, l. 1; instead of 'l'égalité (2)' the original has 'l'inégalité (3)'.

C 2. — The results communicated in this paper are presented in detail in C 7.

C 3. — An Italian summary (by the author) has appeared in *Boll. Un. Mat. Ital.* 3 (1924), 220–224, 4 (1925), 27–29.

p. 30. The existence of the length $l(\varepsilon)$ in the definition of almost periodicity may be replaced by a weaker condition, in consequence of a result of Bogolioùboff; see C 54, 12. Another characterization of almost periodic functions was given by Bochner, *Math. Ann.* 96 (1927), 143; cf. C 9, 189, C 15, 369, and C 51; see also Favard, *Acta Math.* 51 (1928), 37–40, *Leçons sur les fonctions presque-périodiques*, Gauthier-Villars, Paris 1933, 77–80, and Ursell, *J. London Math. Soc.* 4 (1929), 123–127, 5 (1930), 47–50. Bochner's characterization is fundamental for the extension of the theory to groups due to von Neumann, *Trans. Amer. Math. Soc.* 36 (1934), 445–492; cf. C 51. Still another characterization was given by Maak, *Abh. Math. Sem. Hamburg* 11 (1935), 240–244, *Mat. Tidsskr. B* 1938, 7–12, *Fastperiodische Funktionen*, Springer-Verlag, Berlin/Göttingen/Heidelberg 1950. Concerning almost periodic functions on a half-line, see C 14.

p. 58. There is a simpler arrangement of the proof of the multiplication theorem in C 28, 56–57.

p. 63. A number of different proofs of the fundamental theorem have been given; the most important of these proofs are surveyed in C 53. We give here the main references.

A proof by means of Fourier integrals was given by Wiener, *Math. Z.* 24 (1925), 575–616; it has since been much simplified by Wiener himself and by Bochner; see Wiener, *Proc. London Math. Soc.* 27 (1928), 483—496, *Acta Math.* 55 (1930), 117–258, *The Fourier integral and certain of its applications*, Cambridge University Press 1933, 185–199, and Bochner, *Vorlesungen über Fouriersche Integrale*, Akademische Verlagsgesellschaft, Leipzig 1932, 81–82.

A proof based on an extension of E. Schmidt's method in the theory of integral equations and on a theorem on groups of unitary matrices was given by Weyl, *Math. Ann.* 97 (1927), 338–356. Various alternative forms of this proof have been given by other authors. Thus the application of Schmidt's method has been replaced by a variational argument by Hammerstein, *Sber. Preuss. Akad. Wiss. Phys. Math. Kl.* 1928, 17–20, and by the use of a general theorem on operators in a non-separable Hilbert space by Rellich, *Math. Ann.* 110 (1935), 342–356. The group-theoretical argument has been replaced by a use of differential equations by Lüneburg, *Dan. Mat. Fys. Medd.* 12, no. 3 (1932), 1–7. By a modification of Hammerstein's proof, M. Riesz, *Fysiogr. Sällsk. Lund Förh.* 3, no. 10 (1933), 109–117, was also able to avoid the group-theoretical part of the argument. Weyl's method is fundamental for von Neumann's theory of almost periodic functions in groups. Another version of Weyl's theory, based on a remark by Stepanoff and Tychonoff, was obtained by Pontrjagin, and was extended to almost periodic functions in groups by Weil and van Kampen; see the notes to C 7.

The proofs of Wiener and Weyl are based on the convolution process, which shows that the fundamental theorem is an easy consequence of the uniqueness theorem. A proof of the latter theorem, related to Bohr's proof of the fundamental theorem, but much simpler, was given by de la Vallée Poussin, *Ann. Soc. Sci. Bruxelles Ser. A* 47, part 2 (1927), 141–158, 48, part 1 (1928), 56–57. This proof is reproduced in C 28. A simplification has been indicated by M. Riesz, *Mat. Tidsskr. B* 1934, 11–13. A shorter exposition of Bohr's proof and a simplification of de la Vallée Poussin's proof were given by Jessen, *Dan. Mat. Fys. Medd.* 25, no. 8 (1949), 1–12.

Finally, Bogolioùboff's direct proof of the theorem on uniform approximation of an almost periodic function by trigonometric polynomials, mentioned in the notes to C 7, contains a new proof of the fundamental theorem (since uniform approximation implies approximation in mean).

p. 87. The 'Minimalfehler' $e(\tau)$ has been studied in detail by Bochner, *Math. Ann.* 96 (1927), 136–142, under the name of 'Verschiebungsfunktion'.

p. 92. A much more precise result on the set of translation numbers than that contained in Lemma 9 is given in C 19.

p. 103. For other proofs of the convergence theorem, see C 5, 42, C 28, 76–77 (due to Fekete), and Bochner, *Math. Ann.* 96 (1927), 133–134.

pp. 112–119. For further results on periodic-like functions, see C 17, 16–22, and C 54.

p. 123. Another proof of the integration theorem, based on Bochner's characterization of almost periodic functions, was given by Favard, *Acta Math.* 51 (1928), 43–46, *Leçons sur les fonctions presque-périodiques*, Gauthier–Villars, Paris 1933, 82–85. Concerning integration of almost periodic functions, see also C 36 (and the notes to that paper), and C 40, 17–24. For differential equations and almost periodic functions, see C 17 (and the notes to that paper).

C 4. (S 7.) — p. 34. (p. 128.) There is a slightly different proof of the theorem in C 5, 43, and an extension in C 5, 44. The theorem is a corollary to Bochner's extension of Fejér's theorem to almost periodic functions, *Math. Ann.* 96 (1927), 129, 135.

p. 35. (p. 128.) A detailed proof of the formula $\lim \sqrt[2q]{M\{|f(x)|^{2q}\}} = G$ for the upper bound of the absolute value of an almost periodic function is given in C 13.

C 5. — p. 42. For other proofs of Theorem A, see C 28, 76–77 (due to Fekete), and Bochner, *Math. Ann.* 96 (1927), 133–134.

p. 43. There is a slightly different proof of Theorem B in C 4 (S 7); for another proof of this theorem, see Bochner, *Math. Ann.* 96 (1927), 135.

For Fourier series of Stepanoff almost periodic functions with linearly independent exponents, see C 48.

Concerning absolutely convergent Fourier series, see also Linfoot, *J. London Math. Soc.* 4 (1929), 121–123, Bochner, *Jber. Deutsch. Math. Ver.* 39 (1930), 52–54, Cameron, *Duke Math. J.* 3 (1937), 682–688, Pitt, *J. Math. Phys. Mass. Inst. Tech.* 16 (1937), 191–195.

C 7. — An Italian summary (by the author) has appeared in *Boll. Un. Mat. Ital.* 4 (1925), 211–215.

p. 105. Concerning the connection between translation numbers and Fourier exponents, see also C 29, 59–61.

p. 107. For further results on 'ausgezeichnete Mengen', see Bochner, *Math. Ann.* 96 (1927), 141–146.

p. 111. Almost periodic functions with infinite integral basis, and also with arbitrary finite or infinite basis, have been treated, in close analogy to Bohl's investigations, by E. Pedersen, *Dan. Mat. Fys. Medd.* 8, no. 6 (1928), 1–22.

p. 141. For limit periodic functions of one variable, see also C 20.

p. 163. Several other proofs of the theorem on uniform approximation of almost periodic functions by trigonometric polynomials, which avoid the use of functions of infinitely many variables, have been given. There are comments on these proofs in C 53. We give here the main references.

A proof by extension of Fejér's summation theorem was given by Bochner, *Math. Ann.* 96 (1927), 125–130. This proof is reproduced in C 28, 69–75. There is a variant of it in C 18, 10–16. Another proof was given by Weyl, *Math. Ann.* 97 (1927), 348–349. There is a variant of it by Wiener, *Proc. London Math. Soc.* (2) 27 (1928), 494–496. All these proofs depend on the fundamental theorem (or the uniqueness theorem). A proof independent of the theory of Fourier series was given by Bogolioùboff; see Kryloff and Bogolioùboff, *Méthodes nouvelles de la mécanique non linéaire dans leur application à l'étude du fonctionnement de l'oscillateur à lampe* I (Russian with a French preface), Gosudarstvennoe Tehniko-Teoretičeskoe Izdatel'stvo, Moskva 1934, 62–67. Another version, in which the emphasis is on the arithmetical properties of translation numbers, was given by Bogolioùboff, *Ann. Chaire Phys. Math. Kiev* 4 (1939), 195–205 (cf. *Math. Rev.* 8, 512, and C 54). A slightly generalized form of Bogolioùboff's original proof was given by Marčenko, *Zapiski Naučno-Issled. Inst. Mat. Meh. i Har'kov. Mat. Obšč.* (4) 20 (1950), 28–32.

A simple necessary and sufficient condition that a given trigonometric series should be the Fourier series of an almost periodic function was given by Doss, *Ann. of Math.* (2) 46 (1945), 202, by help of Bochner's theorem.

p. 165. As pointed out by Stepanoff and Tychonoff, *Mat. Sbornik* 41 (1934), 166–178, the introduction of the closure $H\{f(x+h)\}$ amounts to the embedding of the group of real numbers h in a compact topological group; cf. also van Dantzig, *C. R. Acad. Sci. Paris* 196 (1933), 1074–1076, *Mat. Sbornik* (N.S.) 1 [43] (1936), 665–675. On the basis of this remark the fundamental theorems on almost periodic functions may be derived from the theory of compact groups, as shown by Pontrjagin, *Topological groups*, Princeton University Press 1939, 124–125. Analogous results concerning almost periodic functions on arbitrary groups were obtained by Weil and van Kampen; see Weil, *L'intégration dans les groupes topologiques et ses applications*, Hermann & Cie., Paris 1940, 124–139.

If the closure $H\{f(x+h)\}$ is identical with the set $\{f(x+h)\}$, then $f(x)$ is periodic (and conversely); see Følner, *Dan. Mat. Fys. Medd.* 25, no. 14 (1950), 1–15, where an analogous theorem for functions of several variables is proved.

p. 184. The generalized approximation theorem follows immediately from Bochner's summation theorem.

p. 207. The theorem on simultaneous approximation was generalized by Bochner, *Math. Ann.* 96 (1927), 130–132. The conclusion holds under the sole assumption that the functions form an 'ausgezeichnete Menge'. Under this assumption, the exponents of the functions must belong to an enumerable set of numbers.

A theory of almost periodic functions of finitely or infinitely many variables was developed by Bochner, *Math. Ann.* 96 (1927), 383–409; functions of two variables were considered by Franklin, *J. Math. Phys. Mass. Inst. Tech.* 5 (1926), 40–54, 201–237. See also Kitagawa, *Proc. Phys. Math. Soc. Japan* (3) 16 (1934), 39–51, Brauers (Brāzma), *Comment. Math. Helv.* 11 (1939), 330–335, *Acta Univ. Latviensis* (3) 7 (1939), 235–263. Like the original theory, this extension is contained in von Neumann's theory of almost periodic functions in a group, as is also the theory of almost periodic sequences developed by Walther, *Abh. Math. Sem. Hamburg* 6 (1928), 217–234, and Seynsche, *Rend. Circ. Mat. Palermo* 55 (1931), 395–421.

The theory was extended to functions with values from an abstract space by Bochner, *Acta Math.* 61 (1933), 149–184. The corresponding extension of von Neumann's theory was performed by Bochner and von Neumann, *Trans. Amer. Math. Soc.* 37 (1935), 21–50.

Concerning the generalizations to discontinuous functions, see in particular C 27 and C 47 (and the notes to these papers).

The reprint differs from the original on p. 174, l. 13, where instead of [2] the original has [5].

C 8. — The results communicated in this paper are presented in detail in C 12.

C 9. — p. 173. The Innsbruck lecture is C 6.

C 10. — The proof of the theorem of this paper is repeated in C 11 (S 8). Another proof was given by Szegö, *Math. Ann.* 96 (1927), 378–382. It follows from the proof, in conjunction with Theorem 5 in C 12, that $f(s)$ is almost periodic in $[-\infty, +\infty]$.

C 12. — An Italian summary (by the author) has appeared in *Boll. Un. Mat. Ital.* 5 (1926), 137–142.

p. 245. For another proof of Theorem A, see C 13, 111–112.

p. 251. An analytic function $f(s) = f(\sigma+it)$ in a strip $\alpha < \sigma < \beta$ can be almost periodic as a function of t for every σ in $\alpha < \sigma < \beta$ without being almost periodic in $[\alpha, \beta]$; see the notes to C 23.

p. 253. For an extension of Theorem 2 to functions defined in a half-strip $\alpha < \sigma < \beta, t > 0$, see C 14, 64.

p. 257. Concerning the quotient of two almost periodic functions in the case where the denominator may have zeros, see C 34.

pp. 270–271. There is a much simpler proof of the extension of Picard's theorem (Theorem 25) in C 39.

p. 275. Theorem 28 was generalized in various directions by Bang, Favard, R. Petersen, Takahashi, and others; see Favard, *Mat. Tidsskr. B* 1936, 71–75, Petersen, 9. *Skand. Matkongr. Helsingfors* 1938, 105–112, Takahashi, *Jap. J. Math.* 16 (1939), 99–133, and Bang, *Mat. Tidsskr. B* 1941, 53–58; these papers contain further references.

p. 277. For a generalization of Theorem 29, see C 16.

The problem of the behaviour of analytic almost periodic functions outside the strip of almost periodicity was studied by Besicovitch, *Acta Math.* 47 (1926), 283–295, *Almost periodic functions*, Cambridge University Press 1932, 163–169; see also Jessen, *Acta Math.* 63 (1934), 300–307, and Borchsenius and Jessen, *Acta Math.* 80 (1948), 100–137.

A theory of harmonic almost periodic functions was developed by Favard, *Thèse*, Paris 1927, 1–108 (= *J. Math. Pure Appl.* (9) 6 (1927), 229–336), *Leçons sur les fonctions presque-périodiques*, Gauthier-Villars, Paris 1933, 140–159.

Meromorphic almost periodic functions were considered by several authors. For a discussion of this notion, see Norgil, *Mat. Tidsskr. B* 1930, 73–91.

Almost periodic functions of several complex variables were considered by Brāzma, *Acta Univ. Latviensis* (3) 20 (1941), 431–455.

C 16. — In the reprints of the paper the following addition was made:

'*Addition après tirage du fascicule.* — Après avoir reçu l'épreuve de cette Note, j'ai reçu une lettre de MM. A. Besicovitch et S. Bochner qui, tous les deux, ont donné des contributions importantes à la théorie des fonctions presque périodiques. J'apprends qu'ils ont aussi, indépendamment de moi, trouvé le théorème d'unicité général donné ci-dessus.'

For a generalization of the theorem, in which the functions on the two lines are only assumed to be almost periodic in Stepanoff's sense, see Linfoot, *J. London Math. Soc.* 3 (1928), 177–182.

C 17. — Concerning the lemma of Esclangon, mentioned on p. 10, see Landau, *Math. Ann.* 102 (1929), 177–188.

There is an extensive literature on differential (or difference) equations and almost periodic functions. Bohr returned to the subject only in C 40. Among the contributions by other authors we mention the following.

Systems of linear differential equations with almost periodic coefficients were studied by Favard, *Acta Math.* 51 (1928), 31–81, *Leçons sur les fonctions presque-périodiques*, Gauthier-Villars, Paris 1933, 85–107, by Bochner, *J. London Math. Soc.* 8 (1933), 283–288, by Cameron, *Duke Math. J.* 1 (1935), 356–360, *Ann. of Math.* (2) 37 (1936), 29–42, *J. Math. Phys. Mass. Inst. Tech.* 15 (1936), 73–81, *Acta*

Math. 69 (1938), 21–56, by Lewitan, *Uspehi Mat. Nauk* (N. S.) 2, no. 6 (22) (1947), 174–214 (*Amer. Math. Soc. Translation* no. 28 (1950), 1–53), and by Hartman and Wintner, *Amer. J. Math.* 71 (1949), 859–864.

Linear difference equations with constant coefficients and almost periodic right-hand side were considered by Walther, *Nachr. Ges. Wiss. Göttingen. Math. Phys. Kl.* 1927, 196–216. A general discussion of difference-differential equations was given by Bochner, *Math. Ann.* 102 (1929), 489–504, 103 (1930), 588–597, 104 (1931), 579–587.

Almost periodic solutions of the wave equation were considered by Muckenhoupt, *J. Math. Phys. Mass. Inst. Tech.* 8 (1929), 163–199, and Bochner, *Acta Math.* 62 (1934), 227–237. These investigations were generalized to operational-differential equations by Bochner and von Neumann, *Ann. of Math.* (2) 36 (1935), 255–291. Operational-difference equations were considered by Kitagawa, *Tôhoku Math. J.* 44 (1938), 139–161. Further results concerning solutions of the wave equation were obtained by Soboleff, *C. R. (Doklady) Acad. Sci. URSS* (N. S.) 48 (1945), 542–545, 618–620, 49 (1945), 12–15.

C 18. — For a detailed exposition of the theory of generalized almost periodic functions, based on the method of the present paper, see C 27.

p. 21. The approximation theorem for S-*a.p.* functions, and for other classes of generalized almost periodic functions, was proved also by Franklin, *Math. Z.* 29 (1928), 70–86.

C 19. — The investigation referred to in the introduction as forthcoming is C 27; see C 27, 243.

C 20. — Concerning limit periodic functions of one variable, see also Kaluza jun., *J. Reine Angew. Math.* 181 (1939), 153–176.

C 21. — The results announced in this paper are presented in detail in C 27.

C 23. — The idea of this paper is also used in E 11. The investigation announced at the end of the paper was carried out and extended by R. Petersen in several papers; see in particular *Acta Math.* 67 (1936), 81–122. His principal result is that a regular function which is almost periodic on every vertical line in a strip must be almost periodic in an everywhere dense set of substrips, and that, conversely, to an arbitrary everywhere dense set of substrips there exists a regular function in the strip, which is almost periodic on every vertical line, and for which the given strips are the maximal strips of almost periodicity. By the same method, Petersen, *Dan. Mat. Fys. Medd.* 15, no. 8 (1938), 1–25, has constructed an integral function $f(z) = f(x+iy)$ which is almost periodic in $[-\infty, +\infty]$ both vertically and horizontally.

C 24. — I. A new treatment of the subject is given in C 29; for further results, see C 30, C 44, and C 50. There is a detailed bibliography in Jessen and Tornehave, *Acta Math.* 77 (1945), 137–279.

II. The method of condensation of singularities was applied to a modification of Bohr's example by van Kampen, *Amer. J. Math.* 61 (1939), 729–732, who thereby obtained a function for which a prescribed bounded sequence of values a are exceptional. A detailed discussion of the possible irregularities in the distribution of the values of an almost periodic sequence or an almost periodic function for a fixed a was given by Jessen and Tornehave, *Acta Math.* 77 (1945), 204–221.

The reprint differs from the original on p. 11, l. 18; instead of 'fastperiodisch wird' the original has ', und somit nach dem obigen Satze auch die Funktion $\varphi(x) = \arg f(x)$ selbst, fastperiodisch wird'.

C 25. — IV. Another treatment of the subject is given in C 30. The inversion of analytic almost periodic functions is treated in C 26 and C 41.

C 26. — Another treatment of the subject is given in C 41. The inversion of almost periodic functions of a real variable is treated in C 25, 10–15, and C 30.

C 27. — Concerning the classes of S^p-, W^p-, and B^p-*a.p.* functions, see also C 47. With small alterations, the present paper is incorporated in Besicovitch, *Almost periodic functions*, Cambridge University Press 1932, 67–129.

p. 204. For Besicovitch's proof of the analogue of the Riesz–Fischer theorem for B^2-*a.p.* functions, see *Proc. London Math. Soc.* (2) 25 (1926), 495–512, or his above mentioned book, 109–112. There is another version of the proof in C 47, 54–58. Other proofs are mentioned in the notes to C 47. Besicovitch further proved that the theorem does not hold for S^2-*a.p.* functions. It follows from Main Example I in C 47, 111–114, that it does not hold for W^2-*a.p.* functions either.

p. 227. A simplification of the definition of *B-a.p.* functions was given by Besicovitch, *Acta Math.* 58 (1932), 217–230, 62 (1934), 317–318; see also Ursell, *Proc. London Math. Soc.* (2) 37 (1934), 535–546. Another structural characterization was given by Følner, *Dan. Mat. Fys. Medd.* 21, no. 11 (1945), 1–30.

p. 262. Theorem II for S^p-*a.p.* and W^p-*a.p.* functions was proved also by Ursell, *Proc. London Math. Soc.* (2) 32 (1931), 422. Examples of *S-a.p.* functions whose Bochner–Fejér sums do not converge almost everywhere were constructed by Ursell, *Proc. London Math. Soc.* (2) 33 (1932), 457–466.

p. 267. A simplification of the definition of $\overline{B}$-*a.p.* functions was given by Ursell, *Proc. London Math. Soc.* (2) 37 (1934), 535–546.

There is an extensive literature on Fourier series of generalized almost periodic functions. Bohr returned to the subject only in C 48. Among the contributions by other authors we mention the following.

It was proved by Wiener, *Math. Z.* 24 (1925), 583, and Haslam-Jones, *J. London Math. Soc.* 8 (1933), 261–265, that a trigonometric series $\sum a_n e^{i\lambda_n x}$ is the Fourier series of an S^2-*a.p.* function, whenever $\sum A_p^2$ converges, where A_p is the sum of those $|a_n|$ for which λ_n belongs to the interval $p \leqq \lambda_n < p+1$ ($p = 0, \pm 1, \pm 2, \ldots$). (The converse does not hold.) It follows from this theorem that the analogue of the Riesz–Fischer theorem holds for the class of S^2-*a.p.* functions with exponents $\lambda_1, \lambda_2, \ldots$ if there exists a constant K such that the number of λ_n in an arbitrary interval of length 1 is $\leqq K$. This was proved also by Stepanoff, *Doklady Akad. Nauk SSSR* (N. S.) 64 (1949), 171–174, 297–300. As pointed out (orally) by Dr Tornehave, the converse of this statement is also valid.

A simple characterization of the class of S^2-*a.p.* functions for which the numerical differences between the Fourier exponents exceed a positive number was given by Paley and Wiener, *Fourier transforms in the complex domain*, American Mathematical Society, New York 1934, 116–123; see also Bochner, *Prace Mat. Fiz.* 43 (1935), 63–79. Simple characterizations of the trigonometric series that are Fourier series of generalized almost periodic functions of various classes were given by Doss, *Ann. of Math.* (2) 46 (1945), 196–219, who also considered the problem of multiplicators for such classes.

C 28. — A Russian translation by D. A. Raĭkov (under the editorship of A. I. Plesner) was published by Gosudarstvennoe Tehniko-Teoretičeskoe Izdatel'stvo, Moskva-Leningrad 1934. An English translation by H. Cohn (appendices translated by F. Steinhardt) was published by Chelsea Publishing Company, New York 1947.

p. 4. The different proofs of the main theorems on almost periodic functions are surveyed in C 53. See the notes to C 3 and C 7.

p. 28. Concerning the definition of almost periodicity, see the notes to C 3.

p. 77. The idea of the proof is used in proofs of Kronecker's theorem in D 4, D 5, and D 6.

C 29. — See the notes to C 24, I.

C 30. — p. 389. Theorem II was generalized to abstract almost periodic functions by Bochner, *Acta Math.* 61 (1933), 159–161.

C 32. (S 9.) — Concerning stability and almost periodicity, see also Bogolioùboff, *Enseignement Math.* 34 (1935), 337–346. For plane stable motions, see C 38.

C 33. — See C 51.

C 34. — p. 12. Concerning algebraic functions of analytic almost periodic functions, see C 43.

C 35. — The result communicated in this paper is presented in detail in C 36.

C 36. — Theorem A was proved also by Bochner, *Proc. London Math. Soc.* (2) 26 (1927), 443–445.

For an application of the main theorem to periodic functions, see C 37.

New proofs and extensions of the main theorem were given by Favard, *Mat. Tidsskr. B* 1936, 81–94, Lewitan, *C. R. (Doklady) Acad. Sci. URSS* (N. S.) 15 (1937), 169–172, von Sz. Nagy and Strausz, *Mat. Természett. Értes.* 57, part 1 (1938), 121–135, von Sz. Nagy, *Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math. Phys. Kl.* 90 (1938), 103–134, 91 (1939), 3–24, Bang, *Dan. Mat. Fys. Medd.* 19, no. 4 (1941), 1–28, *Om quasi-analytiske Funktioner*, Nyt Nordisk Forlag – Arnold Busck, København 1946, 35–41. For further references, and for applications to the theory of approximation, see Favard, *Coll. Internat. XV. Nancy* 1947, 97–110.

C 39. — The paper referred to on p. 6 is E 14.

C 41. — The inversion of analytic almost periodic functions defined in an arbitrary strip was considered by H. Schmidt, *Ber. Verh. Sachs. Akad. Wiss. Leipzig. Math. Phys. Kl.* 90 (1938), 83–96.

C 42. — p. 25. Intransitive abelian almost translation groups of almost periodic functions were considered by Assadourian, *Duke Math. J.* 8 (1941), 518–524.

C 44. — p. 50. A number $c = h_1\mu_1 + h_2\mu_2$ for which (h_1, h_2) is a lattice point of the convex closure of the lattice points $(n_1^{(i)}, n_2^{(i)})$, need not be the mean motion of a trigonometric polynomial with the exponents $\lambda_i = n_1^{(i)}\mu_1 + n_2^{(i)}\mu_2$. This was shown by an example by Jessen and Tornehave, *Acta Math.* 77 (1945), 176; more general examples were considered by Tornehave, 10. *Skand. Matkongr. København* 1946, 325–328.

VOLUME III

C 47. — p. 31. The paper by Følner referred to is *Bidrag til de generaliserede næstenperiodiske Funktioners Teori*, Fr. Bagges Kgl. Hofbogtrykkeri, København 1944.

p. 54. A much simpler proof of the completeness of the B^p-spaces had previously been given by Ursell, *Proc. London Math. Soc.* (2) 32 (1931), 406, 415. Other proofs have been given later by Marcinkiewicz, *C. R. Acad. Sci. Paris* 208 (1939), 157–159, Wecken, *Math. Z.* 45 (1939), 401, and by Hartman and Wintner, *Proc. Nat. Acad. Sci. U.S.A.* 33 (1947), 128–132.

p. 58. The incompleteness of the W^p-*a.p.* spaces had previously been proved by Ursell, l. c., 407–409, 415.

p. 70. An example to $2b\beta$ had previously been given by Ursell, l. c., 431.

p. 131. Another example with the properties of Main Example IV had been given previously by Lewitan and Stepanoff, *C. R. (Doklady) Acad. Sci. URSS* (N. S.) 22 (1939), 220–223.

C 48. (S 13.) — p. 36. Another proof of Theorem 1 is obtained by combining the theorem on absolute convergence of the Fourier series of an ordinary almost periodic function with linearly independent exponents and a theorem of Doss, *Ann. of Math.* (2) 46 (1945), 218–219, to the effect that an *S-a.p.* function with bounded exponents is equivalent to an ordinary almost periodic function.

p. 38. Another proof of Theorem 2 follows from the theorem of Wiener and Haslam-Jones mentioned in the notes to C 27.

It was proved by Sidon, *J. Reine Angew. Math.* 166 (1931), 62–63, that if a series $\sum A_n e^{i\Lambda_n x}$ with linearly independent exponents Λ_n is the Fourier series of an *S-a.p.* function, then $\sum |A_n|^2$ is convergent. As shown by Bochner and Jessen, *Ann. of Math.* (2) 35 (1934), 257, it is sufficient to assume that the series is the Fourier series of a *B-a.p.* function.

C 50. — p. 77. For Tornehave's extension of Kronecker's theorem, see *Dan. Mat. Fys. Medd.* 24, no. 11 (1948), 1–21; cf. also *Dan. Mat. Fys. Medd.* 25, no. 20 (1950), 1–18.

pp. 83–84. See Borchsenius and Jessen, *Acta Math.* 80 (1948), 97–166.

C 51. — For detailed expositions of the theory of almost periodic functions in groups, see Weil, *L'intégration dans les groupes topologiques et ses applications*, Hermann & Cie., Paris 1940, 124–139, and Maak, *Fastperiodische Funktionen*, Springer-Verlag, Berlin/Göttingen/Heidelberg 1950.

C 53. — For references, see the notes to C 3 and C 7.

C 54. — A Hebrew summary by B. Amirà, printed at the end of the paper, has here been omitted.

D 1. — The method of proof of this paper is used also in B 21 and C 5. It is further used in D 2 for a proof of the general Kronecker theorem. The proof is repeated in D 6, § 3. For a simplification, see D 9.

The proof by Lettenmeyer, referred to at the beginning of the paper, is found in the same volume of *Proc. London Math. Soc.*, immediately preceding the present paper.

D 3. — The investigations of this paper are continued in D 10.

D 4. (S 14.) — The first proof is repeated in D 5 and in D 6, § 4, the second in D 6, § 5. The second proof is extended to a proof of the 'big' Kronecker theorem in D 6, § 6.

D 5. — The proof is repeated from D 4, 54–57, and is repeated in D 6, § 4.

D 6. — The proofs in §§ 4 and 5 were also published in D 4.

D 7. (S 15.) — The proof of the 'small' Kronecker theorem is repeated in D 9.

D 9. — The proof is repeated from D 7, where the 'big' Kronecker theorem is proved by the same device. For an extension to abelian groups, see Bundgaard, *Dan. Mat. Fys. Medd.* 14, no. 4 (1936), 12–14.

D 10. — p. 23. See M. Riesz, *C. R. Congr. Internat. Math. Oslo* 1936, II, 36–37. The paper referred to in footnote 2 is D 11.

D 11. — p. 45. The paper referred to is D 10.

pp. 47–48. See M. Riesz, *C. R. Congr. Internat. Math. Oslo* 1936, II, 36–37.

The connection between the spaces R_{∞} and R^{∞} was used by Følner, *Dan. Mat. Fys. Medd.* 25, no. 19 (1950), 1–15, to prove duality theorems for groups.

E 1. — Cf. Landau, *Darstellung und Begrundung einiger neuerer Ergebnisse der Funktionentheorie*, Julius Springer, Berlin 1916, 9, 26–29.

There is another proof of the theorem by Sidon, *Math. Z.* 26 (1927), 731–732, and an extension of it to Dirichlet series by Rogosinski, *Math. Z.* 20 (1924), 308.

E 2. (S 16.) — The theorem that an analytic function cannot increase arbitrarily rapidly in a strip was proved differently by Pólya, *Mat. Tidsskr. B* 1921, 14–16, who showed that no regular analytic function $f(z) = f(x+iy)$ in the strip $x \geqq 0$, $-\frac{1}{2}\pi \leqq y \leqq \frac{1}{2}\pi$ satisfies the inequality $|f(x+iy)| \geqq e^{xe^x}$ in the whole strip.

E 3. — The investigations of this paper are continued in E 4.

The reprint differs from the original on p. 286, l. 5 from below, where instead of 2, 5 the original has 2, 3.

E 4. — It was proved by Schur, *J. Reine Angew. Math.* 148 (1918), 130, that $|s_0|+|s_1|+\ldots+|s_n| \leqq (n+1)$ for every function $f(z)$ with $|f(z)| < 1$ for $|z| < 1$; as pointed out by Schur, this immediately implies a weaker result than that of Theorem B, namely that $\lim\sup\{G_n-|s_n|\} = \infty$.

It was proved by Neder, *Math. Z.* 11 (1921), 115–123, that Theorem B is best possible in the sense that for every sequence L_n tending to infinity (no matter how slowly) there exists a function $f(z)$ with $|f(z)| < 1$ for $|z| < 1$, and such that $G_n-|s_n| < L_n$ for infinitely many n.

E 5. — Another proof of the main theorem of this paper was given by Rademacher, *Math. Z.* 4 (1919), 131–138, who further proved an analogous theorem, in which the assumption that the mapping be 'streckentreu' is replaced by its counterpart that it be 'winkeltreu'. A restrictive condition in Rademacher's theorem was removed by Menchoff, *Math. Ann.* 95 (1926), 641–670. In a later paper by Menchoff, *Mat. Sbornik* (N. S.) 2 [44] (1937), 339–356, Bohr's theorem is generalized; the condition that the mapping be 'streckentreu' is here replaced by the condition that the ratio between the upper and lower bounds of $|f(z)-f(z_0)|$, when z describes a circle with center z_0 and radius r, should converge to 1 as $r \to 0$, for all z_0 with the possible exception of a denumerable set.

E 6. (S 17.) — Concerning gap theorems, see also C 46, 31–37, and E 15.

E 8. — A Hebrew translation by B. Amirà, which accompanies the paper, has here been omitted.

For the investigations mentioned at the end of the paper, see Kloosterman, *Dan. Mat. Fys. Medd.* 5, no. 6 (1923), 1–29. For an application to integral functions, see Pólya, *J. London Math. Soc.* 1 (1926), 12–15.

The theorem of the paper is contained in the general results of Montel, *Ann. Sci. École Norm. Sup.* (3) 46 (1929), 1–23, concerning the range of an analytic function.

E 10. — The result communicated in this paper is presented in detail in E 11.

E 11. — The investigation on analytic almost periodic functions mentioned at the beginning of the paper is C 23.

As shown by Milloux, *Bull. Sci. Math.* 54, part 1 (1930), 302–310, the asymptotic properties of the characteristic function $T(r,f)$ of a function $f(z)$ which remains bounded outside a strip are closely connected with the narrowness of the strip.

E 12. — Analytic functions $f(z)$ for which $\lim f(re^{i\theta})$, $r \to \infty$, exists for every θ were considered by A. Roth, *Comment. Math. Helv.* 11 (1938), 77–125; see also Lauritzen, *Mat. Tidsskr. B* 1950, 42–48.

E 14. — The application to almost periodic functions mentioned on p. 30 is in C 39. The main theorem on p. 31 was generalized by Jørgensen, *Math. Ann.* 115 (1938), 710–719, who proved that, under the assumptions of the theorem, either $F(s)$ is bounded in the half-plane $\sigma > 1$, or there exist positive constants K, M, m such that $me^{K\sigma} \leqq |F(s)| \leqq Me^{K\sigma}$ in some half-plane $\sigma > \sigma_0$.

E 15. (S 19.) — The investigation on analytic almost periodic functions mentioned on p. 3 (p. 145) is C 46.

F 1. — For the application to the zeta-function mentioned at the end of the introduction, see B 7. For further results on the boundary curves, giving detailed information on lengths, areas, curvatures, etc., see Haviland, *Amer. J. Math.* 55 (1933), 332–334, and Kershner, *Amer. J. Math.* 58 (1936), 737–746, 59 (1937), 423–426.

F 2. (S 20.) — p. 10. (p. 147.) The lecture given at the congress at Helsingfors is B 20.

p. 15. (p. 148.) The paper referred to as forthcoming is F 3. The problem is treated in F 3 under more restrictive assumptions on the curves and the probability distributions.

F 3. (S 21.) — For the applications to the zeta-function mentioned at the end of the preface, see B 23 and B 24.

A new treatment of the subject, depending on the methods of probability theory, was given by Jessen and Wintner, *Trans. Amer. Math. Soc.* 38 (1935), 48–88; see also van Kampen and Wintner, *Amer. J. Math.* 59 (1937), 175–204, van Kampen, *Amer. J. Math.* 59 (1937), 679–695.

G 1. — p. 232. Concerning series with arbitrary indices, see A 3, 133–134 (S 1, 116–117), and G 5 (S 24).

G 2. (S 22.) — The theorem of this paper may also be formulated as a theorem on sequences. In this form it was established by Toeplitz, *Prace Mat. Fiz.* 22 (1911), 11, and Kojima, *Tôhoku Math. J.* 12 (1917), 297, who proved that the conditions of the theorem are also necessary for the existence of the limit.

GENERAL INDEX

Papers are referred to by their numbers in the edition; the Preface and A 1 - B 26 are in volume I, C 1 - C 46 are in volume II, and C 47 - S 27 are in volume III. Small figures indicate pages.

Brackets indicate references to results given without proof, and other references of secondary importance.